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by

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MAPPINGS AND DIMENSION

James Keesling[1] and David C. Wilson

1. Introduction

In 1936 Eilenberg posed the following problem in the *Scottish Book* [10, Problem #137]. If f maps a compact metric space X (having the same dimension at each of its points) onto Y such that dim X > dim Y > 0, must there be a closed subset K contained in X such that dim K < dim f(K)? Theorem 6.2 of [8] gives a counterexample to this question. Keesling and Wilson [6] proved the following theorem. If $n \geq 2$, then there is an n-dimensional continuum X contained in I^{n+1} and a map $f: X \to I = [0,1]$ such that f is onto and if K is any 0-dimensional compactum in X, then f(K) is 0-dimensional. In addition the map f and the space X have the following properties. (1) The space X is n-dimensional at each of its points. (2) The map f is the restriction of the projection mapping. (3) for each $y \in I$ the set $f^{-1}(y)$ is either an n-cell or chainable. Thus the map f is cell-like and X has trivial shape. (4) If K is a closed subset of X and $\dim K \leq n - 2$, then dim f(K) = 0. (5) There is a closed (n-1)-dimensional subset K such that f(K) = I.

The purpose of this paper is to give sufficient conditions for a positive solution to the problem of Eilenberg. The first result in this direction is the following theorem.

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[1]This work was presented by the first author at the Spring Topology Conference held in March 1982 at the U.S. Naval Academy in Annapolis, Maryland.

Theorem 2.3. *Let* $f(X) = Y$ *be a proper map between separable metric spaces. If* $\dim X \leq n$ *and* $\dim f^{-1}(y) \geq m$ *for all* $y \in Y$, *then there is a closed set* K *in* X *such that* $\dim K \leq n - m$ *and* $\dim f(K) = \dim Y$.

A consequence of Theorem 2.3 is that if each point-inverse set of a map has dimension ≥ 2, then property (4) must be violated.

Theorem 2.7 gives sufficient conditions which ensure that the dimension of K is small even when the dimension of X is large.

Theorem 2.7. *Let* $f(X) = Y$ *be a map between compact metric spaces, where* $\dim Y < \infty$. *If* $1 \leq \dim f^{-1}(y) \leq k$ *for all* $y \in Y$, *then there is a closed set* K *in* X *such that* $\dim K \leq k$ *and* $\dim f(K) = \dim Y$.

The proof of Theorem 2.7 is similar to Kelley's Theorem 7.8 [7]. Keesling ([4] and [5]) has results similar to Theorem 2.3, Theorem 2.7, and their corollaries for open mappings.

Theorem 3.2 and Theorem 4.5 give sufficient conditions for the domain of $f(X) = Y$ to contain a closed set K such that $\dim K \leq \dim X - 1$ and $\dim f(K) = \dim Y$. The point-inverse sets are assumed to be ANR's in 3.2 and perfect in 4.5.

The examples given by Theorem 5.2 show that Theorem 2.3 is best possible. Note also that Theorem 5.2 generalizes the construction given in [6].

If $A \subset X$, then the interior of A will be denoted by

Int A. The boundary of A will be denoted by Fr A. The letters ANR will denote absolute neighborhood retract. If Γ denotes a collection of subsets of a set X and $A \subset X$, then the star of A (relative to Γ), denoted st(A,Γ), is the collection of all members of Γ which meet A. The mesh of Γ, denoted $\mu(\Gamma)$, will be the sup$\{\text{diam } \gamma | \gamma \in \Gamma\}$.

The authors wish to express their appreciation to Alice Mason for her helpful remarks and examples concerning the results of this paper.

2. The Main Theorems

For a discussion of the basic concepts of dimension theory see the classic text by Hurewicz and Wallman [1]. A discussion of the important properties of essential families is given in [8]. A map is *proper* if the inverse image of each compact set is compact.

Proposition 2.1. *Let* $f(X) = Y$ *be a closed mapping between metric spaces. If* X *is finite dimensional and* Y *is infinite dimensional, then there is a closed set* K *in* X *such that dim* $K = 0$ *and dim* $f(K) = \infty$.

Proof. This proposition is a routine corollary of Theorem 1.4 in [3].

Proposition 2.2. *Let* $\{X_i\}_{i=1}^{\infty}$ *be a sequence of closed subsets of a separable metric space* X *such that* $\lim_{i\to\infty} X_i \subset X_\infty$, *where* X_∞ *is compact. Also assume that if* U *is any neighborhood of* X_∞, *then there is an* N *such that for* $i \geq N$, $X_i \subset U$. *If* $(A_1^i,B_1^i),\cdots,(A_m^i,B_m^i)$ *is an essential family for* X_i *such that* $\lim_{i\to\infty} A_k^i = A_k$, $\lim_{i\to\infty} B_k^i = B_k$, *and* $A_k \cap B_k = \phi$ *for all*

$k = 1,\cdots,m$, *then the set* $\{(A_1,B_1,\cdots,(A_m,B_m)\}$ *forms an essential family for* X_∞.

Proof. Let $S_1,\cdots,S_m$ be closed separators in X_∞ for the pairs $(A_1,B_1),\cdots,(A_m,B_m)$, respectively. We must show that $\cap_{k=1}^m S_k \neq \phi$. Since S_k is a separator for A_k and B_k in X_∞ we can find a separator S_k' in X such that $X - S_k' = U_k \cup V_k$, $A_k \subset U_k$, $B_k \subset V_k$, and $S_k' \cap X_\infty = S_k$. Since $\lim_{i\to\infty} A_k^i = A_k$ and $\lim_{i\to\infty} B_k^i = B_k$ we can find an integer N such that $i \geq N$ implies that $A_k^i \subset U_k$ and $B_k^i \subset V_k$. Thus, $S_k' \cap X_i$ is a separator for A_k^i and B_k^i in X_i, which implies that $\cap_{k=1}^m S_k' \cap X_i \neq \phi$. Equivalently $X_i \cap (\cap_{k=1}^m S_k') \neq \phi$. Without loss of generality we can assume that $\lim_{i\to\infty} (X_i \cap (\cap_{k=1}^m S_k'))$ exists. Since $\lim_{i\to\infty} X_i \subset X_\infty$, $\lim_{i\to\infty} (X_i \cap (\cap_{k=1}^m S_k')) \subset X_\infty \cap (\cap_{k=1}^m S_k') = \cap_{k=1}^m (S_k' \cap X_\infty) = \cap_{k=1}^m S_k$. Since X_∞ is compact, $\lim_{i\to\infty} (X_i \cap (\cap_{k=1}^m S_k')) \neq \phi$. Thus $\cap_{k=1}^m S_k \neq \phi$ and the proof is complete.

Theorem 2.3. *Let* $f(X) = Y$ *be a proper map between separable metric spaces. If dim* $X \leq n$ *and dim* $f^{-1}(y) \geq m$ *for all* $y \in Y$, *then there is a closed set* K *in* X *such that dim* $K \leq n - m$ *and dim* $f(K) =$ *dim* Y. *If in addition* Y *is complete, then there is a closed* K_0 *in* X *such that dim* $K_0 \leq n - m$ *and Int* $f(K_0) \neq \phi$.

Proof. If Y is infinite dimensional, then we are done by Proposition 2.1. Thus we may assume dim $Y < \infty$.

Let B be a countable basis for X. Let $\mathcal{P} = \{(U_1,V_1), \cdots,(U_m,V_m)\}$ be a sequence of m pairs of open sets of X with the property that each U_i and V_i is a finite union of members of B and $\overline{U}_i \cap \overline{V}_i = \phi$ for $i = 1,\cdots,m$. In particular, the collection $\mathcal{P}$ is countable. If $P \in \mathcal{P}$, then let $Z(P)$ be the set of all points $y \in Y$ such that $f^{-1}(y)$ has an essential family $\{(A_1(y),B_1(y)),\cdots,(A_m(y),B_m(y))\}$ such that $A_i(y) \subset \overline{U}_i$ and $B_i(y) \subset \overline{V}_i$. Since f is proper and $\dim f^{-1}(y) \geq m$ for all $y \in Y$, $Y = \cup_{P\in\mathcal{P}} Z(P)$. By Proposition 2.2 the set $Z(P)$ is a closed subset of Y. By the Sum Theorem $\dim Z(P) = \dim Y$ for some $P \in \mathcal{P}$.

Let $J = f^{-1}(Z(P))$. Since $\dim J \leq n$, we can apply Theorem B [1, p. 34] m times to find closed sets $S_1,\cdots,S_m$ such that S_i separates $\overline{U}_i$ and $\overline{V}_i$ and $\dim (S_1 \cap S_2 \cap \cdots \cap S_m) \leq n - m$. Let $K = S_1 \cap S_2 \cap \cdots \cap S_m$. An easy check shows that $K \cap f^{-1}(y) \neq \phi$ for all $y \in Z(P)$. Thus $\dim K \leq n - m$ and $f(K) = Z(P)$.

If Y is complete, then by applying the Baire Category Theorem we can find a set $Z(P)$ such that $\text{Int } Z(P) \neq \phi$. The argument now proceeds as before.

Corollary 2.4. *Let* $f(X) = Y$ *be a proper map between separable metric spaces. If* $\dim f^{-1}(y) = \dim X < \infty$ *for all* $y \in Y$, *then there is a closed* 0*-dimensional set* $K \subset X$ *such that* $\dim f(K) = \dim Y$.

Corollary 2.5. *Let* $f(X) = Y$ *be a proper map between separable metric spaces. If* $\dim X \leq n$, $\dim f^{-1}(y) \geq m$ *for all* $y \in Y$, *and* $\dim Y \geq k(n-m+1)$, *then there is a closed set* $K \subset X$ *such that* $\dim K = 0$ *and* $\dim f(K) \geq k$.

Proof. By Theorem 2.3 there is a closed set $K' \subset X$ such that $\dim K' \leq n - m$ and $\dim f(K') = \dim Y$. Since

$\dim Y \geq k(n-m+1)$, Theorem 1.4 of [3] assures the existence of a closed set $K \subset K'$ such that $\dim K = 0$ and $\dim f(K) \geq k$.

Corollary 2.6. *Let* $f(X) = Y$ *be a proper map between separable metric spaces. If* $\dim X \leq n$, $\dim f^{-1}(y) \geq m$ *for all* $y \in Y$, *and* $\dim Y \geq n - m + 1$, *then there is a closed set* $K \subset X$ *such that* $\dim K = 0$ *and* $\dim f(K) > 0$.

Note that Corollaries 2.4 and 2.6 give conditions on a map which ensure a positive answer to the question of Eilenberg.

Theorem 2.7. *Let* $f(X) = Y$ *be a map between compact metric spaces, where* $\dim Y < \infty$. *If* $1 \leq \dim f^{-1}(y) \leq k$ *for all* $y \in Y$, *then there is a closed set* K *in* X *such that* $\dim K \leq k$ *and* $\dim f(K) = \dim Y$. *Also, there is a closed set* K_0 *in* X *such that* $\dim K_0 \leq k$ *and* $\text{Int } f(K_0) \neq \phi$.

Proof. Since the dimension of Y is finite, Y can be embedded in E^m for some integer m. We now follow the proof and notation of Theorem 2.3 to find a closed set $Z(P)$ of Y such that $\dim Z(P) = \dim Y$. In this situation the pair $P = (U,V)$, where $\overline{U} \cap \overline{V} = \phi$ and each $f^{-1}(y)$ has a component meeting both $\overline{U}$ and $\overline{V}$. For convenience assume $f: X \to Y$ where each component of each $f^{-1}(y)$ meets both $\overline{U}$ and $\overline{V}$.

We will prove the theorem by showing the existence of sequences $\{J_i\}_{i=1}^{\infty}$, $\{S_i\}_{i=1}^{\infty}$, $\{W_i\}_{i=1}^{\infty}$, and $\{\mathcal{S}_i\}_{i=1}^{\infty}$ with the following properties. (1) Each $\mathcal{S}_i$ is a finite collection of pairwise disjoint separators of $\overline{U}$ and $\overline{V}$. (2) The set S_i is the union of the members of $\mathcal{S}_i$. (3) Each J_i is a closed subset of S_i and $f(J_i) = Y$. (4) Each W_i is an open subset of X such that $J_i \cup \overline{W}_{i+1} \subset W_i$. (5) If $f^{-1}(y) \cap J_i \cap S_i(\alpha) \neq \phi$, where $S_i(\alpha) \in \mathcal{S}_i$, then $f^{-1}(y) \cap J_i \cap S_i(\alpha) = f^{-1}(y) \cap S_i(\alpha)$. (6) The open set W_i is covered by a finite collection of open sets $\mathcal{W}_i$, where $\mu(\mathcal{W}_i) < \frac{1}{i}$ and the order of $\mathcal{W}_i$ is $\leq k + 1$.

Under the assumption that properties (1) to (6) are satisfied we can define $K = \cap_{i=1}^{\infty}\overline{W}_i$. By properties (3) and (4) $f(K) = Y$. By property (6) dim $K \leq k$.

To begin the induction assume that the diameter of X is less than 1. Let $J_1 = S_1$ be any separator of $\overline{U}$ and $\overline{V}$. Let $\mathcal{S}_1 = \{S_1\}$ and $W_1 = X$.

Assume inductively that there are finite sequences $\{J_i\}_{i=1}^{q}$, $\{S_i\}_{i=1}^{q}$, $\{W_i\}_{i=1}^{q}$, and $\{\mathcal{S}_i\}_{i=1}^{q}$ which satisfy properties (1) to (6).

For each $S_q(\alpha) \in \mathcal{S}_q$ find a collection $\mathcal{S}_{q+1}(\alpha) = \{S_q(\alpha,i) | i = 1,2,\cdots,3^m\}$ of 3^m pairwise disjoint separators of $\overline{U}$ and $\overline{V}$. Each $S_q(\alpha,i)$ is to be chosen so close to $S_q(\alpha)$ that if $f^{-1}(y) \cap J_q \cap S_q(\alpha) \neq \phi$, then $f^{-1}(y) \cap S_q(\alpha,i) \subset W_q$. Let $\mathcal{S}_{q+1} = \cup_\alpha \mathcal{S}_{q+1}(\alpha)$ and S_{q+1} be the union of the members of $\mathcal{S}_{q+1}$. Let $J(\alpha,i) = f^{-1}(f(J_q \cap S_q(\alpha))) \cap S_q(\alpha,i)$. Let $J(\alpha) = \cup_{i=1}^{3^m} J(\alpha,i)$ and let $g = f|J(\alpha)$. For each $y \in g(J(\alpha))$ there is an open set $W(y)$ such that $g^{-1}(y)$ is contained in $W(y)$ and $\overline{W}(y)$ is contained in W_q. Since dim $g^{-1}(y) \leq k$, we can choose $W(y)$ so that it can be covered by a finite collection of open sets $\mathcal{W}(y)$, where $\mu(\mathcal{W}(y)) \leq \frac{1}{q+1}$ and the order of $\mathcal{W}(y) \leq k + 1$. Since g is a closed map there is an open set $V_y \subset Y$ such that $g^{-1}(V_y) \subset W(y)$. Since Y is compact subset of E^m, there is a finite closed cover Γ of

$g(J(\alpha))$ with the property that Γ refines the collection $\{V_y | y \in g(J(\alpha))\}$ and $|St(\gamma,\Gamma)| \leq 3^m$ for all $\gamma \in \Gamma$. Let $Y_1,\cdots,Y_s$ be a listing of all the members of Γ. Let $K_1(\alpha) = S_q(\alpha,i) \cap g^{-1}(Y_1)$. Assume inductively that there are pairwise disjoint closed sets $K_1(\alpha),\cdots,K_s(\alpha)$ where $K_i(\alpha) = S_q(\alpha,t) \cap g^{-1}(Y_i)$ for some choice of t. Since $|St(Y_{r+1},\Gamma)| \leq 3^m$ and since there are 3^m sets $S_q(\alpha,1),\cdots, S_q(\alpha,3^m)$, we can find an integer t' with the property that if $Y_i \in St(Y_{r+1},\Gamma)$ and $i \leq r + 1$, then $S_q(\alpha,t') \cap g^{-1}(Y_{r+1})$ misses $K_i(\alpha)$. Let $K_{r+1}(\alpha) = S_q(\alpha,t') \cap g^{-1}(Y_{r+1})$. If we let $K(\alpha) = \cup_{i=1}^{s} K_i(\alpha)$, then note that $g(K(\alpha)) = g(J(\alpha))$. If we let $J_{q+1} = \cup_{\alpha=1}^{p} K(\alpha)$, then notice that $g(J_{q+1}) = Y$.

The open set W_{q+1} can be found by covering each $K_i(\alpha)$ with one of the open sets $W(y_i)$. Since the members of the collection $\{K_i(\alpha) | \alpha = 1,\cdots,p$ and $i = 1,\cdots,s\}$ are pairwise disjoint we can shrink each $W(y_i)$ slightly to a set W_i' so that the members of the collection $\{W_i' | i = 1,\cdots s\}$ are pairwise disjoint. (That is, open sets associated with different separators are made disjoint.) If we let W_{q+1} be the union of the sets W_i', then W_{q+1} will have the property that it can be covered by a finite collection of open sets whose order is $\leq k + 1$. Clearly we can choose W_{q+1} so that $\overline{W}_{q+1}$ is contained in W_q. Thus we have completed the proof of the theorem.

3. The ANR Case

In this section we consider the case when $f^{-1}(y)$ is an ANR for every $y \in Y$.

Proposition 3.1. *Let* Y *be a set of cardinality* c. *If* N *denotes a positive integer,* $\mathcal{P}(R)$ *denotes the power set of the reals, and* $\Theta: Y \to \mathcal{P}(R)$ *is a function such that* $|\Theta(y)| = c$ *for all* $y \in Y$, *then there is a function* $\Psi: Y \to \mathcal{P}(R)$ *which has the properties that* $\Psi(y) \subset \Theta(y)$, $|\Psi(y)| = N$, *and* $\Psi(y) \cap \Psi(y') = \phi$ *if* $y \neq y'$.

Proof. The proof of this proposition is a straight forward application of the well-ordering principle and the axiom of choice.

Theorem 3.2. *Let* $f(X) = Y$ *be a map between compact metric spaces where dim* $Y < \infty$ *and where* $f^{-1}(y)$ *is an* ANR *for all* $y \in Y$. *If* $1 \leq$ *dim* $f^{-1}(y) \leq k + 1$ *for all* $y \in Y$, *then there is a closed set* K *in* X *such that dim* $K \leq k$ *and dim* $f(K) =$ *dim* Y. *Also, there is a closed set* K_0 *in* X *such that dim* $K_0 \leq k$ *and Int* $f(K_0) \neq \phi$.

Proof. As in the proof of Theorem 2.7 we can assume there is a pair $P = (U,V)$ with the properties that $\overline{U} \cap \overline{V} = \phi$ and each component of $f^{-1}(y)$ meets both $\overline{U}$ and $\overline{V}$.

We will prove the theorem by showing the existence of sequences $\{J_i\}_{i=1}^{\infty}$, $\{S_i\}_{i=1}^{\infty}$, $\{W_i\}_{i=1}^{\infty}$, and $\{\mathcal{S}_i\}_{i=1}^{\infty}$ with properties (1) to (6) listed in the proof of Theorem 2.7.

If $S_\varepsilon(\overline{U}) = \{x \in X | d(x,\overline{U}) = \varepsilon\}$, then a theorem of Sieklucki [9] can be applied to show that for each $y \in Y$, the set $\{\varepsilon > 0 | \dim (S_\varepsilon(\overline{U}) \cap f^{-1}(y) \leq k\}$ has cardinality c. Proposition 3.1 can now be used to find collections $\mathcal{S}_{q+1}(\alpha,y) = \{S_q(\alpha,y,i) | i = 1,2,\cdots 3^m\}$ of pairwise disjoint separators of $\overline{U}$ and $\overline{V}$ such that $\dim (S_q(\alpha,y,i) \cap f^{-1}(y)) \leq k$ and $S_q(\alpha,y,i) \cap S_q(\beta,y',j) = \phi$ whenever $(\alpha,y,i) \neq (\beta,y',j)$.

Clearly, the members of $\mathcal{S}_{q+1}(\alpha,y)$ can be chosen close to $S_q(\alpha)$ and the proof now follows the proof of Theorem 2.7 exactly.

Corollary 3.3. *Let* $f(X) = Y$ *be a map between metric spaces where* $\dim Y < \infty$ *and* $f^{-1}(y)$ *is a* 1-*dimensional* ANR *for all* $y \in Y$. *Then there is a closed* 0-*dimensional set* K *in* X *such that* $\dim f(K) = \dim Y$.

4. The Perfect Point-Inverse Case

A space is said to be *perfect* if it contains no isolated points. We now consider maps with perfect point-inverses. A map $f: X \to Y$ is irreducible if $f(X) = Y$, but $f(K) \neq Y$ for every proper closed subset $K \subset X$.

Proposition 4.1. *If* $f(X) = Y$ *is a proper map between compact metric spaces, then there is a closed set* $K \subset X$ *such that the map* $f|K$ *is an irreducible map from* K *onto* Y.

Proof. The proof of this proposition is a simple application of the Hausdorff Maximal Principle.

A surjective map $f: X \to Y$ will be called *locally open* at $y \in Y$ if for every open set $U \subset X$ which meets $f^{-1}(y)$, the point y is in the interior of $f(U)$. As a point of notation we let Y_f denote the set of all points of Y at which f is locally open and $X_f = f^{-1}(Y_f)$.

Proposition 4.2. *Let* X *be complete and* $f(X) = Y$ *be a proper map. Then* Y_f *is a dense* G_δ *subset of* Y.

Proof. Since X is complete and f is proper, Y is also complete [11]. Since the collection $G = \{f^{-1}(y) | y \in Y\}$

forms an upper semi-continuous decomposition of X, the proposition follows from Jones [2].

Proposition 4.3. *Let* X *be complete and* $f(X) = Y$ *be a proper irreducible map. Then* X_f *is a dense subset of* X. *Moreover,* $f^{-1}(y)$ *is degenerate for all* $y \in Y_f$ *and* $f|X_f$ *is a homeomorphism of* X_f *onto* Y_f.

Proof. Let U be a non-empty open subset of X. Since $f(X-U) \neq Y$ and Y_f is dense in Y, there is a point $y \in Y_f \cap (Y-f(X-U))$. Since $y \in f(U)$, $X_f \cap U \neq \phi$ and X_f is dense.

Let y be a point where f is locally open. If x_1 and x_2 are distinct points in $f^{-1}(y)$, then we can find disjoint open sets U_1 and U_2 such that $x_1 \in U_1$ and $x_2 \in U_2$. Let $V = (\text{Int } f(U_1)) \cap (\text{Int } f(U_2))$ and let $W = U_1 \cap f^{-1}(V)$. If $K = X - W$, then K is a proper closed subset of X with the property that $f(K) = Y$. We now have a contradiction to the irreducibility of f.

Since f is locally open at the points of Y_f, $f|X_f$ is a homeomorphism of X_f onto Y_f.

Let $f(X) = Y$ be an open mapping with complete perfect point-inverses. Then by Theorem 1.1 of [4], there is a closed 0-dimensional set K in X with $f(K) = Y$. The example of [6] outlined in the introduction shows that we cannot find such a closed 0-dimensional set if f is only assumed to be a proper map with perfect point-inverses. However, we now show that under these circumstances we can find a closed nowhere dense set K in X such that $f(K) = Y$.

Theorem 4.4. *If* $f(X) = Y$ *is a proper map where* X *is complete and* $f^{-1}(y)$ *is perfect for all* $y \in Y$, *then there is a closed nowhere dense set* K *in* X *such that* $f(K) = Y$.

Proof. Let K be a closed subset of X such that $g = f|K$ is irreducible. If U is open in X and $U \subset K$, then since X_g is dense in K, there is a $y \in Y_g$ such that $g^{-1}(y) \subset U$. But since $f^{-1}(y)$ is perfect, $f^{-1}(y) \cap U = g^{-1}(y) \cap U$ must contain at least two points. This contradiction shows that K must be nowhere dense.

Theorem 4.5. *If* $f(X) = Y$ *is a proper map from a locally finite* n-*dimensional polyhedron onto a Hausdorff space* Y *with the property that* $f^{-1}(y)$ *is perfect for all* $y \in Y$, *then there is a closed subset* $K \subset P$ *such that* $\dim K \leq n - 1$ *and* $f(K) = Y$.

Proof. Let K be the subset of X guaranteed by Theorem 4.4. Since K is nowhere dense, $\dim K \leq n - 1$.

5. The Examples

Theorem 5.2 shows that Theorem 2.3 cannot be improved without some additional hypotheses.

Proposition 5.1. *Let* $\{(A_1,B_1),\cdots,(A_n,B_n)\}$ *be an essential family for a compact* n-*manifold* N. *Let* $S_{k+1},\cdots,S_n$ *and* $M_{k+1},\cdots,M_n$ *be closed subsets of* N *where* $1 \leq k < n$. *Let* $L_{k+1},\cdots,L_n$ *be open subsets of* N *with the property that* $S_i \subset L_i \subset \overline{L}_i \subset M_i$ *and* M_i *misses* $A_i \cup B_i$. *If* $M = \cap_{i=k+1}^{n} M_i$, $M_i - L_i = U_i \cup V_i$, *and* S_i *separates* $A_i \cup U_i$ *from* $B_i \cup V_i$ *in* N, *then the collection* $\{(M \cap A_1, M \cap B_1), \cdots, (M \cap A_k, M \cap B_k), (M \cap U_{k+1}, M \cap V_{k+1}), \cdots, (M \cap U_n, M \cap V_n)\}$

is an essential family for M.

Proof. For $i = 1,\cdots,k$ let T_i be a closed subset of M which separates $M \cap A_i$ from $M \cap B_i$ in M. For $i = k + 1, \cdots, n$ let T_i be a separator of $M \cap U_i$ from $M \cap V_i$ in M. We must show that $\cap_{i=1}^n T_i \neq \phi$.

Let $k + 1 \leq i \leq n$. Since $T_i \subset L_i$, $M_i - L_i = U_i \cup V_i$, and T_i separates $U_i \cap M$ from $V_i \cap M$, we can find a closed set $T_i' \subset L_i$ such that $T_i' \cap M = T_i$ and T_i' separates U_i from V_i in M_i. Since L_i contains S_i, L_i separates A_i from B_i in N. Since L_i is an open subset of a manifold and $T_i' \subset L_i$, T_i' must also separate A_i from B_i in N. (For if not, then we could run an arc α in N from A_i to B_i missing T_i'. This arc would then contain a subarc β in M_i which would run from U_i to V_i missing T_i'.)

If $1 \leq i \leq k$, then there is a closed set T_i' containing T_i such that $T_i' \cap M = T_i$ and T_i' separates A_i from B_i in N.

Since $\cap_{i=1}^n T_i' \subset \cap_{i=k+1}^n T_i' \subset \cap_{i=k+1}^n L_i \subset M$ and $T_i' \cap M = T_i$, $\cap_{i=1}^n T_i' = \cap_{i=1}^n (T_i' \cap M) = \cap_{i=1}^n T_i$. Thus, $\cap_{i=1}^n T_i' \neq \phi$ implies $\cap_{i=1}^n T_i \neq \phi$.

Theorem 5.2. *If* $n > m \geq 1$, *then there is an* n-*dimensional compactum* $X \subset I^{n+1}$ *and a map* $f: X \to I$ *with the following properties.* (1) *The map* f *is the restriction to* X *of the projection map* $p: I^{n+1} \to I$ *onto the last factor.* (2) *For each* $y \in I$ *dim* $f^{-1}(y) \geq m$. (3) *If* K *is a closed subset of* X *and dim* $K \leq n - m - 1$, *then dim* $f(K) = 0$. (4) *The set* X *is* n-*dimensional at each of its points.*

Proof. Let $\{r_1, r_2, \cdots\}$ be a countable dense subset of I. The space X will be obtained as the nested intersection of compacta $\{X_k | k = 0,1,\cdots\}$. Let $X_0 = I^{n+1}$, $U_0 = \phi$, and let f_k denote the restriction of p to the set X_k. Let $\pi_i: I^n \to I$ be the projection onto the ith factor and let $A_i = \pi_i^{-1}(0)$ and $B_i = \pi_i^{-1}(1)$.

Assume inductively that we have found compacta $X_0,\cdots,X_k$ and open sets $U_0,U_1,\cdots,U_k$ with the following properties. (1) If $j > 0$, then U_j is an open subset of I with diameter less than $\frac{1}{j}$ which is the union of a countable sequence of pairwise disjoint open intervals which converge to r_j. (2) If $i \neq j$, then $r_i \notin Fr(U_j)$ and $Fr(U_i) \cap Fr(U_j) = \phi$. (3) If $i < j$, then either $\overline{U}_i \cap \overline{U}_j = \phi$ or $\overline{U}_j$ is contained in one component of U_i. (4) $X_k \subset X_{k-1} \subset \cdots \subset X_0 = I^{n+1}$. (5) The set $f_j^{-1}(y)$ is a PL n-manifold for all $y \in I$. (6) The set $f_{j-1}^{-1}(I-U_j) = f_j^{-1}(I-U_j)$. (7) If $U_j(i)$ is a component of U_j, then there is a compact PL n-manifold with boundary $N_j(i) \subset I^n$ such that $f_j^{-1}(U_j(i)) = N_j(i) \times U_j(i)$. (8) Each n-manifold $N_j(i)$ has the property that the collection $\mathcal{J} = \{(A_1 \cap N_j(i),B_1 \cap N_j(i)),\cdots,(A_m \cap N_j(i),B_m \cap N_j(i))\}$ forms an essential family for $N_j(i)$. Moreover, each $N_j(i)$ also has the property that there exists a collection $\mathcal{J}' = \{(A'_{m+1},B'_{m+1}),\cdots,(A'_n,B'_n)\}$ such that $\mathcal{J} \cap \mathcal{J}' = \phi$ and $\mathcal{J} \cup \mathcal{J}'$ is an essential family for $N_j(i)$. (9) As the set of components, $\{U_j(i)\}$, of U_j converge to r_j, the set of n-manifolds $N_j(i)$ are dense in the set of all PL n-manifolds which satisfy property (8).

We now want to establish the existence of sets X_{k+1} and U_{k+1} which satisfy properties (1) to (9) above. Proceeding as in [6] we first find points a and b such that

$a < r_{k+1} < b$ and the interval $[a,b]$ misses the set $\{r_1,\cdots,r_k\}$. By properties (2) and (3) we can also assume that if $[a,b]$ meets a set U_j, then it is contained in some component of U_j. Inside the interval (a,b) now find a small open set U_{k+1} which is the union of a countable sequence of pairwise disjoint open intervals which converge to r_{k+1}. By properties (6) and (7) we know that $f_k^{-1}((a,b)) = N \times (a,b)$ for some PL n-manifold N, where N satisfies property (8). Let $\{M_i \mid i = 1,2,\cdots\}$ be a sequence of PL n-manifolds in I^n which satisfy property (8) and are dense (in the hyperspace topology) in the collection of all continua N which satisfy property (8). If $U_{k+1} = \cup_{i=1}^{\infty} U_{k+1}(i)$, where the set $U_{k+1}(i)$ is the ith component of U_{k+1} converging to r_{k+1}, then define the set X_{k+1} above $U_{k+1}(i)$ to be the set $M_i \times U_{k+1}(i)$. Above the set $I - U_{k+1}$ leave the set X_k as it was before. More precisely, let $f_k^{-1}(I-U_{k+1}) \cap X_k = f_{k+1}^{-1}(I-U_{k+1}) \cap X_{k+1}$. We have now shown that the sets X_{k+1} and U_{k+1} satisfy properties (1) to (9).

Let $X = \cap_{k=1}^{\infty} X_k$ and $f = p|X$. By Proposition 2.2 dim $f^{-1}(y) \geq m$ for all $y \in I$.

Let K be a closed subset of X such that dim $K \leq n - m -$ Suppose $f(K)$ contains a nondegenerate interval $[a,b]$. If so, then select a point r_k from the countable dense subset such that $r_k \in \overline{U}_k \subset (a,b)$. By properties (6) and (7) we can assume that $f_k^{-1}((a,b)) = N \times (a,b)$ for some PL n-manifold N. Since dim $K \leq n - m - 1$, dim $(K \cap (N \times \{r_k\}))$ $\leq n - m - 1$. By property (8) we can inductively find closed sets $S_m,\cdots,S_n$ such that S_i separates A_i' from B_i'

and $S = S_{m+1} \cap \cdots \cap S_n$ misses $K \cap (N \times \{r_k\})$. Choose compact PL n-manifold neighborhoods M_i of S_i so that M_i misses $A_i' \cup B_i'$ and $M = M_{m+1} \cap \cdots \cap M_n$ misses K. By Proposition 5.5 [8] the collection $\mathcal{F} = \{(M \cap A_1, M \cap B_1), \cdots, (M \cap A_m, M \cap B_m)\}$ is an essential family for M and thus for N. By Proposition 5.1 there is a collection $\mathcal{F}'$ such that $\mathcal{F} \cap \mathcal{F}' = \phi$ and $\mathcal{F} \cup \mathcal{F}'$ is an essential family so that M also satisfies property (8). By property (9) there is an integer i such that $U_k(i) \subset (a,b)$ and $N_k(i)$ is so close to M that $N_k(i)$ misses $K \cap (N \times \{r_k\})$. If i is chosen large enough, then $N_k(i) \times U_k(i)$ will miss K. Thus $f_k(K)$ does not contain $U_k(i)$ contradicting the assumption that f(K) contains (a,b). Thus we have shown that the image of every closed (n-m-1)-dimensional subset of X is 0-dimensional.

Since the proof that X is n-dimensional at each of its points is virtually the same as the argument given in [6], it will be omitted.

The authors would be interested in answers to the following questions.

Question 1. If f(X) = Y is a mapping between compact metric spaces such that $m \leq \dim f^{-1}(y) \leq n$ for all $y \in Y$, then is there a closed set $K \subset X$ such that $\dim K \leq n - m$ and dim f(K) = dim Y?

Question 2. Can the maps in Theorem 5.2 be made monotone or cell-like?

Added in Proof: Question 1 has been answered in the affirmative by Eiji Kurihara. The proof will appear elsewhere.

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