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G-SPACES: PRODUCTS, ABSOLUTES AND REMOTE POINTS

by

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G-SPACES: PRODUCTS, ABSOLUTES AND REMOTE POINTS

Thomas J. Peters [1]

1. Introduction

The purpose of this paper is to catalogue some of the topological properties of G-spaces. The work presented here is not exhaustive.

The difficulty and complexity of the definition of a G-space has led us to consider more manageable related properties, some weaker and some stronger than the G-space property itself. Fortunately, many of these properties are important and interesting in their own right. Among the matters we discuss relevant to G-spaces are disjoint topological unions, π-bases, strong G-spaces, products which are G, products which are not G, preservation of G-spaces by functions, absolutes of G-spaces and passage to subspaces.

The material included here which most directly contributes to the theory of remote points concerns strong G-spaces. The hypothesis of a strong G-space may be used as an alternative to normality in the Chae and Smith theorem (2.9)

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[1] These results constitute a portion of the author's doctoral dissertation, written under the supervision of W. W. Comfort at Wesleyan University, Middletown, Connecticut. The author gratefully acknowledges Wesleyan University's financial support. Moreover, deep thanks are due to W. W. Comfort for the sharing of his wonderful mathematical insight, intuition and philosophy--a process which helped to make this work possible. The author also acknowledges, with appreciation, many helpful and stimulating discussions and communications with S. B. Chae, A. Dow and A. W. Hager. Thanks are also due to the referee, whose many helpful comments improved both the mathematical content and the exposition of this article.

concerning the existence of remote points. Utilizing this variant of the Chae and Smith theorem, we provide a partial solution to a problem of Woods concerning homeomorphic dense subspaces of various remainders. We generalize a result of Dow concerning remote points of large products. These three results--(1) the variant of the Chae and Smith theorem (8.2), (2) the partial solution of Woods' problem (8.3), and (3) the generalization of Dow's result (8.7)--are closely related and are principal contributions of this work.

The above comments indicate our successes in the study of properties of G-spaces. We achieved only limited success, however, in two important areas: the characterization of strong G-spaces, and the determination of products which are G-spaces. Three challenges calling for further investigation are (1) characterize strong G-spaces, (2) determine if G-spaces are finitely productive, and (3) determine interesting conditions on an infinite family of G-spaces which will ensure that their product is a G-space.

2. Basic Preliminaries

All our spaces are completely regular and Hausdorff. Many of our results, particularly amongst the product theorems, may be proved under milder separation hypotheses, but we leave such tasks as exercises for the interested reader. Also, in order to avoid trivial technicalities, all our spaces are assumed to have at least two points. As is common terminology, we use the phrase "infinite product" to refer to a product whose indexing set is infinite. The phrases "countable product " and "finite product" have corresponding meanings.

The symbol ω denotes the least infinite cardinal; $\omega_1, \omega_2, \cdots$ are its immediate cardinal successors. The Greek letters α, γ, and λ will always denote infinite cardinals. The Greek letter κ will be used to denote a cardinal, either finite or infinite. Cardinals are represented by initial ordinals. The space ω always has the discrete topology. Cardinality of a space X is denoted by $|X|$.

For terminology not specifically defined below, see [CoN].

Let X be a space. The notation bX denotes a compactification of X; βX denotes the Stone-Čech compactification of X; EX denotes the absolute of X (defined below (2.7)); cX denotes the cellularity of X; πX denotes the π-weight of X (defined below (2.5)); TX denotes the set of remote points of X (defined below (2.1)); $\tau(X)$ denotes the family of open subsets of X; $\tau^*(X)$ denotes the family of non-empty open subsets of X. The subspace $\beta X \setminus X$ will be abbreviated as X^*.

The symbol $\mathbb{R}$ denotes the real line with the usual topology. The notation $U(\alpha)$ denotes the space of uniform ultrafilters on the discrete space α. That is,

$$U(\alpha) = \{p \in \beta(\alpha): |A| = \alpha \text{ for all } A \in p\}$$

The symbol $\oplus$ denotes disjoint topological union. If X and X' are homeomorphic spaces, we write $X \approx X'$.

2.1 *Definition.* Let X be a space. A point $p \in bX \setminus X$ is bX-*remote* for X if there is no nowhere dense subset A of X such that $p \in c\ell_{bX} A$. When a point is βX-remote, it will simply be called a remote point (for X).

2.2 *Definition.* A space X is a G-*space* if for each non-empty, open subset U and each $n < \omega$, there exists a family $\mathbf{F}(U,n)$ of non-empty sets with the n-intersection property such that

(i) if $F \in \mathbf{F}(U,n)$, then $F \subset U$ and $F = c\ell_X F$;

(ii) if V is a dense open subset of U, then there is $F \in \mathbf{F}(U,n)$ such that $F \subset V$.

2.3 *Definition.* A family $\mathbf{F}(U,n)$ as in Definition 2.2 is said to be a G-*family* (for U and n).

2.4 *Definition.* A space X is *strong* G if both X and βX are G-spaces.

We note that the properties of being G and of being strong G are topological properties in the sense that if X and Y are homeomorphic spaces and X is a G-space (or, a strong G-space), then Y is also.

2.5 *Definition.* Let X be a space. A π-*base* of X is a collection $\mathbf{B}$ of non-empty open subsets of X such that each open subset of X contains an element of $\mathbf{B}$. The π-*weight* of X (designated πX) is the least cardinality of a π-base of X.

2.6 *Definition.* A family $\mathbf{S}$ which can be represented in the form $\mathbf{S} = \cup_{n<\omega} \mathbf{S}_n$, where each $\mathbf{S}_n$ is locally finite, is said to be σ-*locally finite.* A space with a σ-locally finite π-base will be called a σ-π *space*.

2.7 *Definition.* For a space X, the *absolute* of X (denoted EX) is that unique (up to homeomorphism) extremally disconnected space that can be mapped onto X by a perfect, irreducible, continuous function. (For further details, see the survey [Wo_3].)

The following theorems were crucial to our work and are essential for the understanding of the sequel.

2.8 *Theorem* (van Douwen [vD]). *If* X *is a nonpseudocompact space with countable* π*-weight, then* X *has* 2^c *remote points.*

2.9 *Theorem* (Chae and Smith [CS]). *If* X *is a nonpseudocompact normal* G*-space, then* X *has at least* 2^c *remote points.*

2.10 *Theorem* (Chae and Smith [CS]). *A space with a* σ*-locally finite* π*-base is a* G*-space. In particular, every metric space is a* G*-space.*

2.11 *Theorem* (van Douwen and van Mill [vDvM]). *The space* $\omega \times U(\omega_2)$ *has no remote points.*

2.12 *Theorem* (Dow [D_4]). *If* Y *is a compact space with* $cY > \omega_1$*, then the space* $\omega \times Y^\omega$ *has no remote points.*

3. Elementary Properties

We characterize G-spaces by means of their π-bases. This simple result is quite useful. Other elementary properties of G-spaces are also discussed. For the sake of later reference, we first state a trivial result.

3.1 *Theorem. Let* $\{X_\xi\}_{\xi<\alpha}$ *be a family of spaces. The space* $\oplus_{\xi<\alpha} X_\xi$ *is* G *if and only if* X_ξ *is* G *for each* $\xi < \alpha$.

3.2 *Remarks.* By technical modifications of proofs from the literature [vDvM], [D_4], it can be shown [P_2] that if α is an infinite cardinal and X is a compact space such that (1) X is covered by nowhere dense P_{α^+}-sets, or (2) $X = \Pi_{\xi<\alpha} X_\xi$ such that $cY_\xi > \alpha^+$ for each $\xi < \alpha$, then $\alpha \times X$ has no remote points, when α is discrete. Since $\alpha \times X$ is a disjoint topological union which is nonpseudocompact and normal, its failure to have remote points implies that $\alpha \times X$ is not a G-space (2.9). Hence, X is not a G-space.

We also note that Dow [D_4] gives other examples of compact spaces X such that $\omega \times X$ has no remote points. Each such space X is also not a G-space.

We now consider the importance of π-bases in the definition of a G-space.

3.3 *Theorem. For a space* X, *the following are equivalent:*

(1) *There exists a* π*-base of* X *consisting of* G*-spaces;*

(2) X *is* G*; and*

(3) *every* π*-base of* X *consists of* G*-spaces.*

Proof. (1) $\Rightarrow$ (2). Let $\mathbf{B}$ be a π-base of X such that each $B \in \mathbf{B}$ is a G-space. Let $U \in \tau^*(X)$. There exist B, $B' \in \mathbf{B}$ such that $c\ell_X B' \subset B \subset U$. Since B' is an open subset of the G-space B, we have that $c\ell_B B'$ is a G-space [CS]. Note $c\ell_B B' = c\ell_X B'$.

Now consider B' as an open subset of the G-space $c\ell_B B'$, and let $n < \omega$. There exists a family $\mathbf{F}(B',n)$ of non-empty sets with the n-intersection property such that (i) if $F \in \mathbf{F}(B',n)$, then $F \subset B'$ and $F = c\ell_{c\ell_B B'} F$, and (ii) if V' is a dense open subset of B', then there exists $F \in \mathbf{F}(B',n)$ such that $F \subset V'$.

Now let $\mathbf{F}(U,n) = \mathbf{F}(B',n)$. Since $c\ell_B B' = c\ell_X B'$, and since if V is a dense open subset of U then $V \cap B'$ is a dense open subset of B', it is easy to see that $\mathbf{F}(U,n)$ is a G-family.

(2) $\Rightarrow$ (3). If X is a G-space, then each $U \in \tau^*(X)$ is a G-space [CS].

(3) $\Rightarrow$ (1). Obvious.

3.4 *Corollary. Let* X *be a space and let* Y *be a dense open subspace of* X. *Then* X *is* G *if and only if* Y *is* G.

Later we will see that every dense subspace of a G-space is a G-space (7.17). However, arbitrary subspaces of G-spaces need not be G.

3.5 *Example.* Let $X = \omega_2$, with the discrete topology. Clearly, X is G and βX is G (3.4). But $U(\omega_2) \subset \beta X$ and $U(\omega_2)$ is not G (2.11), (3.2). Thus, arbitrary subspaces of G-spaces need not be G-spaces.

4. Strong G-spaces

We are now able to discuss more fully the class of strong G-spaces. The class of strong G-spaces is of particular interest because it properly contains the class of spaces of countable π-weight and later (8.2) we will see that each nonpseudocompact strong G-space has remote points.

4.1 *Theorem. Every space* X *of countable* π*-weight is a strong* G*-space.*

Proof. It is easy to see that $\pi X = \pi\beta X = \omega$. But every space of countable π-weight is a G-space (2.10).

4.2 *Theorem. If* X *is a locally compact* G*-space or if* X *has a* π*-base of singletons, then* X *is a strong* G*-space.*

Proof. Corollary 3.4.

For other examples of strong G-spaces, see (7.22, 7.23, 8.5, 8.7).

It remains an open problem to "characterize strong G-spaces." (We know that a metrizable space need not be strong G (7.19).) Even though our knowledge of strong G-spaces is not complete, we are able to use the property in the context of our search for remote points (§8).

As the locally compact G-spaces have some particularly nice properties, we present three results which may help one to determine if a particular locally compact space is G. These results are due to W. W. Comfort [Co_3].

4.3 *Theorem. Let* X *be a locally compact space. Let* U *be a non-empty open subset of* X *which contains no isolated points and which has compact closure. Let* $\mathbf{F}(U)$ *be a family of closed subsets of* X *such that* (i) *if* $F \in \mathbf{F}(U)$*, then* $F \subset U$*, and* (ii) *if* V *is a dense open subset of* U*, then there exists* $F \in \mathbf{F}(U)$ *such that* $F \subset V$*. Then* $\mathbf{F}(U)$ *does not have the finite intersection property.*

Proof. Suppose $\mathbf{F}(U)$ has the finite intersection property. Since $c\ell_X U$ is compact, there exists $p \in \cap\, \mathbf{F}(U)$. Since $U\setminus\{p\}$ is dense open in U, there exists $F \in \mathbf{F}(U)$ such that $F \subset U\setminus\{p\}$. We have $p \in F \subset U\setminus\{p\}$, a contradiction.

4.4 *Corollary. Let* X *be a locally compact* G-*space. Let* U *be a non-empty open subset of* X *which contains no isolated points and which has compact closure. Let* $\mathbf{F}(U,n)$ *be a* G-*family. Then, for any subfamily* $\mathbf{F}(U) \subset \mathbf{F}(U,n)$ *such that* $\mathbf{F}(U)$ *has property* (ii) *of the previous theorem,* $\mathbf{F}(U)$ *does not have the finite intersection property. In particular, the* G-*family* $\mathbf{F}(U,n)$ *does not have the finite intersection property.*

4.5 *Corollary. Let* X *be a locally compact space. Let* U *and* $\mathbf{F}(U)$ *be as defined in Theorem* 4.3. *Let* $\mathbf{D} = \{V$: V *is a dense open subset of* $U\}$. *Let* $\phi\colon \mathbf{D} \to \mathbf{F}(U)$ *be an assignment function such that for each* $V \in \mathbf{D}$, $\phi(V) \subset V$. *Then* ϕ *is not monotone--that is,* ϕ *cannot be chosen so that if* $V_1, V_2 \in \mathbf{D}$ *and* $V_1 \subset V_2$ *then* $\phi(V_1) \subset \phi(V_2)$.

Proof. Without loss of generality assume that ϕ is onto $\mathbf{F}(U)$. (If necessary, replace $\mathbf{F}(U)$ by $\{\phi(V)\colon V \in \mathbf{D}\}$.) Assume that ϕ is monotone.

We present a contradiction by showing $\mathbf{F}(U)$ has the finite intersection property. Suppose $m < \omega$ and $\{F_k\}_{k<m}$ is a finite subset of $\mathbf{F}(U)$. For each $k < m$, let $V_k \in \mathbf{D}$ such that $\phi(V_k) = F_k$. Let $V = \cap_{k<m} V_k$. Clearly $V \in \mathbf{D}$ and $V \subset V_k$ for each $k < m$. We have $\phi(V) \neq \emptyset$ and if ϕ is monotone, $\phi(V) \subset \cap_{k<m}\phi(V_k) = \cap_{k<m} F_k$.

But since $\{F_k\}_{k<m}$ was an arbitrary finite subset of $\mathbf{F}(U)$ we have obtained the desired contradiction.

5. Finite Products of G-spaces

We have been able to identify a class of G-spaces which is finitely productive, but we have been unable to answer the following question: "If X and Y are G-spaces, must $X \times Y$ be a G-space?" The proof of the following theorem follows easily from (3.3).

5.1 *Theorem. If* X *and* Y *are* G-*spaces each having a* π-*base whose elements are* σ-π *spaces, then* $X \times Y$ *also has a* π-*base whose elements are* σ-π *spaces, and* $X \times Y$ *is a* G-*space.*

6. Infinite Products and G-spaces

We now consider infinite products of G-spaces. Our first result, based on work of A. Dow [D_4], shows that an infinite product of G-spaces need not be G, even when each finite partial product is strong G. We next display an infinite product which is G even though no finite partial product is G. These examples demonstrate the difficulty of characterizing the G-property with respect to the formation of products.

6.1 *Example.* Let $\omega_2 + 1$ be the one point compactification of the discrete space ω_2. Clearly, $\omega_2 + 1$ has a π-base of singletons; thus, for each $n < \omega$, $(\omega_2 + 1)^n$ is a strong G-space (4.2). However, we know that $(\omega_2 + 1)^\omega$ is not G (3.2). So, strong G-spaces are not countably productive.

Similar arguments show that for any discrete $\alpha > \omega_1$, α^γ is not a strong G-space. (See also Theorem 7.24.)

Before producing an example of an infinite product which is G even though no finite partial product is G, we present a theorem which gives sufficient conditions for an infinite product to be σ-π. The concise proof given here employs techniques utilized by Tkachenko [T]. For an alternative proof, see [P_2], which also contains many other related results.

6.2 *Theorem. Let* $\{X_\xi\}_{\xi<\alpha}$ *be a family of spaces and let* $\lambda = \sup_{\xi<\alpha}\{\pi X_\xi\} + \alpha$. *If there exists a countably infinite subfamily of spaces which are not pseudo-*λ*-compact then* $\prod_{\xi<\alpha} X_\xi$ *is* σ-π.

Proof. Without loss of generality, assume that the countably infinite subfamily of spaces which are not pseudo-λ-compact is $\{X_\xi\}_{\xi<\omega}$. Note that $\pi X_\xi = \lambda$ for each $\xi < \omega$ [Co_1].

For each $n < \omega$, let

$$Y_n = \prod_{\xi<n+1} X_\xi \times \prod_{\omega\leq\xi<\alpha} X_\xi.$$

Note, also, that $\pi Y_n = \lambda$ for each $n < \omega$. For each $n < \omega$, let $\mathbf{U}_n$ be a π-base for Y_n such that $|\mathbf{U}_n| = \lambda$, let $\mathbf{W}_n$ be a subset of $\tau^*(X_n)$ such that $\mathbf{W}_n$ is locally finite and $|\mathbf{W}_n| = \lambda$, and let f_n be a one to one function $f_n\colon \mathbf{U}_n \to \mathbf{W}_{n+1}$.

For each $n < \omega$, let

$$\mathbf{B}_n = \{U \times f_n(U) \times \prod_{n+1<\xi<\omega} X_\xi : U \in \mathbf{U}_n\}.$$

Let $\mathbf{B} = \cup_{n<\omega}\mathbf{B}_n$. It is easy to see that $\mathbf{B}$ is a σ-locally finite π-base for $\prod_{\xi<\alpha} X_\xi$.

We are now in a position to give the promised example. We note that the interested reader could easily construct other such examples.

6.3 *Example.* Let $X = D \times U(\omega_2)$, where D is a discrete space of cardinality 2^{ω_2}. We know that X is not a G-space (3.2). For each $n < \omega$, $X^n \approx D \times (U(\omega_2))^n$. But, since $U(\omega_2)$ can be covered by nowhere dense P-sets, $(U(\omega_2))^n$ can also be covered by nowhere dense P-sets. Hence, $(U(\omega_2))^n$ is not G (3.2), and X^n is not G (3.1). But $X^\omega \approx D^\omega \times (U(\omega_2))^\omega$ which is σ-π (6.2), hence G (2.10).

6.4 *Example.* There exist spaces X and Y such that X is G, Y is not G, but $X \times Y$ is G. Let $X = D^\omega$, where D is a discrete space of cardinality 2^{ω_2}, and let $Y = U(\omega_2)$. The product $X \times Y$ is a σ-π space (6.2), hence it is also a G-space (2.10).

So, even for finite products to be σ-π or G, it is not necessary that each factor be G. For further results on products which are G-spaces, see (7.23, 8.5, 8.6, 8.7).

7. Absolutes and Subspaces

In this section we show that certain functions preserve the G-space property. As a result, we are able to conclude that a space is G if and only if its absolute is G. We prove that every dense subspace of a G-space is a G-space. We show that a space is strong G if and only if it has a π-base whose elements are strong G-spaces. We give a necessary condition for a product to be a strong G-space.

7.1 *Defintion* [vM]. A function $f: X \to Y$ is called *quasi-open* if for each $U \in \tau^*(X)$, $\mathrm{int}_Y f(U) \neq \emptyset$.

A lemma similar to the one we prove below is found in [Wo_1].

7.2 *Lemma. Let* X *and* Y *be spaces. Let* $f: X \to Y$ *be a continuous, closed, quasi-open function from* X *onto* Y. *If* $T \in \tau^*(Y)$ *and* S *is a dense open subset of* T, *then* $f^{-1}(S)$ *is dense open in* $f^{-1}(T)$.

Proof. That $f^{-1}(S)$ is open is clear. Suppose $f^{-1}(S)$ is not dense in $f^{-1}(T)$. Then there exists $W \in \tau^*(X)$ such that $W \subset f^{-1}(T)$ and $W \cap f^{-1}(S) = \emptyset$. But then $f(W) \subset T$ and $f(W) \cap S = \emptyset$. Hence $\mathrm{int}_Y f(W) \cap S = \emptyset$, contradicting the density of S.

7.3 *Theorem. Let* X *and* Y *be spaces. Let* $f: X \to Y$ *be a continuous, closed, quasi-open function from* X *onto* Y. *If* X *is a* G-*space, then* Y *is a* G-*space.*

Proof. Let $T \in \tau^*(Y)$ and $n < \omega$. Since $f^{-1}(T) \in \tau^*(X)$ and X is G, there exists a G-family $\mathbf{F}(f^{-1}(T),n)$. Let $\mathbf{F}(T,n) = \{f(F): F \in \mathbf{F}(f^{-1}(T),n)\}$. Then $\mathbf{F}(T,n)$ is a G-family.

Remark. It is easy to see that continuous open surjections do not, in general, preserve the G-space property. Consider the G-space $X \times Y$, defined in (6.4), and consider the natural projection onto Y.

The following definition and lemmas are from Ponomarev [Po_1, Po_2].

7.4 *Definition* $[Po_1]$. Let $f: X \to Y$ be a function and $A \subset X$. The *small image of* A, denoted $f^{\#}(A)$ is the set $f^{\#}(A) = \{y: y \in Y \text{ and } f^{-1}(y) \subset A\}$.

7.5 *Remark*. Note that $Y \backslash f^{\#}(A) = f(X \backslash A)$ $[Po_1]$.

7.6 *Lemma* $[Po_2]$. *A function* $f: X \to Y$ *from* X *onto* Y *is closed if and only if for each* $U \in \tau(X)$, $f^{\#}(U) \in \tau(Y)$.

7.7 *Lemma* $[Po_2]$. *A continuous function* $f: X \to Y$ *from* X *onto* Y *is irreducible if and only if for each* $U \in \tau^*(X)$, $f^{\#}(U) \neq \emptyset$.

7.8 *Theorem*. *If a continuous, closed function from* X *onto* Y *is irreducible, then it is quasi-open.*

Proof. Lemmas 7.6 and 7.7.

7.9 *Lemma* $[Po_2]$. *Let* $f: X \to Y$ *be a continuous, closed irreducible function from* X *onto* Y. *If* $U \in \tau^*(X)$ *and* V *is a dense open subset of* U, *then* $f^{\#}(V)$ *is a dense open subset of* $f^{\#}(U)$.

Proof. That $f^{\#}(V)$ is open is clear. Let W be a non-empty open subset of $f^{\#}(U)$. Let $B = f^{-1}(W) \cap V$ and note $B \in \tau^*(X)$. So $f^{\#}(B) \neq \emptyset$ and $f^{\#}(B) \subset f^{\#}(V)$. Further, $f^{\#}(B) \subset f(B) \subset W$. So $f^{\#}(V) \cap W \neq \emptyset$ and $f^{\#}(V)$ is dense in $f^{\#}(U)$.

7.10 *Theorem*. *Let* $f: X \to Y$ *be a continuous, closed, irreducible function from* X *onto* Y. *If* Y *is* G, *then* X *is* G.

Proof. Let $U \in \tau^*(X)$ and $n < \omega$. By (7.6) and (7.7), we see that $f^{\#}(U) \in \tau^*(Y)$. Since Y is G, there exists a

G-family $\mathbf{F}(f^{\#}(U),n)$. Let $\mathbf{F}(U,n) = \{f^{-1}(F)\colon F \in \mathbf{F}(f^{\#}(U),n)\}$. Note that if V is a dense open subset of U, then by (7.9), there exists $F \in \mathbf{F}(f^{\#}(U),n)$ such that $F \subset f^{\#}(V)$. It is then immediate that $f^{-1}(F) \subset V$ and that $\mathbf{F}(U,n)$ is a G-family (for U and n). Hence, X is a G-space.

Combining the previous results (7.3), (7.8), and (7.10), we have the following corollaries.

7.11 *Corollary. Let* $f\colon X \to Y$ *be a continuous, closed, irreducible function from* X *onto* Y. *Then* X *is a* G-*space if and only if* Y *is a* G-*space.*

7.12 *Corollary. A space is a* G-*space if and only if its absolute is a* G-*space.*

7.13 *Theorem. If* X *is a locally compact space and* Y *is a space such that* $X \times Y$ *is* G, *then* Y *is a* G-*space.*

Proof. Let $U \in \tau^*(X)$ with $c\ell_X U$ compact. Since $X \times Y$ is G, so also $c\ell_X U \times Y$ is G [CS]. But then the natural projection $\pi\colon c\ell_X U \times Y \to Y$ is a continuous, closed, open surjection, so Y is G (7.3).

We focus on subspaces of G-spaces. Our principal results are that the members of the G-families $\mathbf{F}(U,n)$ may be taken to be regular-closed sets, and that any dense subspace of a G-space is a G-space.

7.14 *Lemma. Let* X *be a* G-*space,* $U \in \tau^*(X)$, $n < \omega$ *and* $\mathbf{F}(U,n+1)$ *a* G-*family. Then* $\{\mathrm{int}_X F\colon F \in \mathbf{F}(U,n+1)\}$ *has the* n-*intersection property.*

Proof. Suppose there exist $F_0, \cdots, F_{n-1} \in \mathbf{F}(U,n+1)$ such that $\mathrm{int}_X \bigcap_{i<n} F_i = \bigcap_{i<n} \mathrm{int}_X F_i = \emptyset$. Then $U \setminus \bigcap_{i<n} F_i$ is a dense open subset of U and, hence, there exists $F \in \mathbf{F}(U,n+1)$ such that $F \subset U \setminus \bigcap_{i<n} F_i$. Then $F \cap \bigcap_{i<n} F_i = \emptyset$, which is a contradiction to the $n+1$ intersection property of $\mathbf{F}(U,n+1)$.

7.15 *Theorem. Let* X *be a* G*-space. Then the members of the* G*-families may be taken to be regular-closed sets.*

Proof. Let $U \in \tau^*(X)$, $n < \omega$ and $\mathbf{F}(U,n+1)$ be a G-family (for U and $n+1$). Let $\mathbf{F}'(U,n) = \{c\ell_X\ \mathrm{int}_X F : F \in \mathbf{F}(U,n+1)\}$. It follows from the previous lemma that $\mathbf{F}'(U,n)$ is a G-family (of regular-closed subsets of X) for U and n.

Before we show that any dense subspace of a G-space is a G-space, we state the following lemma, whose easy proof is left to the interested reader: We use Lemma 7.16 only in the proof of Theorem 7.17.

7.16 *Lemma. Let* X *be a space,* Y *a dense subspace of* X*,* $U \in \tau^*(Y)$ *and* $U' \in \tau^*(X)$ *such that* $U = U' \cap Y$*. Let* V *be a dense open subset of* U *and let* $V' \in \tau^*(X)$ *such that* $V = V' \cap U$ *and* $V' \subset U'$*. Then* V' *is dense in* U'*.*

7.17 *Theorem. If* X *is a* G*-space and* Y *is a dense subspace of* X*, then* Y *is a* G*-space.*

Proof. Given $U \in \tau^*(Y)$ and $n < \omega$, choose $U' \in \tau^*(X)$ such that $U = U' \cap Y$, and $\mathbf{F}(U',n+1)$ is a G-family for U' and $n+1$. By Lemma 7.14, if $F \in \mathbf{F}(U',n+1)$ then $\mathrm{int}_X F \neq \emptyset$. So, the density of Y guarantees that for each $F \in \mathbf{F}(U',n+1)$,

$F \cap Y \neq \emptyset$. Let $\mathbf{F}(U,n) = \{F \cap Y: F \in \mathbf{F}(U',n+1\}$. Note that $\mathbf{F}(U,n)$ has the n-intersection property (7.14). Let V be a dense open subset of U. Then there exists V' such that V' is a dense open subset of U' and $V = V' \cap U$ (7.16). Since $\mathbf{F}(U',n+1)$ is a G-family there exists $F \in \mathbf{F}(U',n+1)$ such that $F \subset V' \subset U'$. But then, $F \cap Y \subset V' \cap U' \cap Y = V' \cap U = V$. That is, there exists $F \cap Y \in \mathbf{F}(U,n)$ such that $F \cap Y \subset V$. Hence, $\mathbf{F}(U,n)$ is a G-family and Y is a G-space.

7.18 *Corollary. The following are equivalent.*

(1) X *is a strong* G-*space* (that is, both X and βX are G-spaces),

(2) βX *is* G,

(3) bX *is* G *for every compactification* bX *of* X,

(4) bX *is* G *for some compactification* bX *of* X.

Proof. That (2) implies (3) follows from (7.11). That (4) implies (1) follows from (7.11) and (7.17). The other implications are obvious.

7.19 *Remark*. The converse of Theorem 7.17 is false. For example, when ω_2 is discrete and $\omega_2 + 1$ is its one-point compactification, the metric space ω_2^ω is a dense subspace of $(\omega_2 + 1)^\omega$ (which is not a G-space (6.1)). Since $(\omega_2 + 1)^\omega$ is a compactification of ω_2^ω, it is clear that the metric space ω_2^ω is not a strong G-space (7.18).

7.20 *Theorem. If* X *has a* π-*base whose elements are strong* G-*spaces, then* X *is a strong* G-*space*.

Proof. Let $\mathbf{B}$ be a π-base for X such that each $B \in \mathbf{B}$ is a strong G-space. It suffices to show βX is G (7.18).

Let $\mathbf{U} = \{U\colon U \in \tau^*(\beta X)$ and $U \cap X = B$ for some $B \in \mathbf{B}\}$.

We show that (a) $\mathbf{U}$ is a collection of strong G-spaces; (b) $\mathbf{U}$ is a π-base for βX.

(a) Let $U \in \mathbf{U}$ and $B \in \mathbf{B}$ such that $B = U \cap X$. Then $c\ell_{\beta X}B = c\ell_{\beta X}(U \cap X) = c\ell_{\beta X}U$, which is a G-space (7.18) (because B is a strong G-space). But $c\ell_{\beta X}B$ is a compactification of U so U is a strong G-space (7.18).

(b) Let $V \in \tau^*(\beta X)$. There exists $B \in \mathbf{B}$ such that $B \subset V \cap X$. Let $U \in \mathbf{U}$ such that $B = U \cap X$. Then $B \subset (U \cap V) \cap X \subset U \cap X = B$, and therefore $B = (U \cap V) \cap X \in \mathbf{B}$. Thus, we have that $U \cap V \in \mathbf{U}$.

Hence, βX has a π-base $\mathbf{U}$ whose elements are (strong) G-spaces. Therefore, βX is a G-space (3.3).

7.21 *Corollary. For a space* X, *the following are equivalent:*

(1) *There exists a* π*-base for* X *consisting of strong* G-*spaces;*

(2) X *is strong* G*; and*

(3) *every* π*-base of* X *consists of strong* G-*spaces.*

Proof. (1) ⇒ (2). Theorem 7.20.

(2) ⇒ (3). Let $U \in \tau^*(X)$. Since βX is a G-space, $c\ell_{\beta X}U$ is a G-space [CS]. Hence, U is a strong G-space (7.18).

(3) ⇒ (1). Obvious.

7.22 *Corollary. Let* $\{X_\xi\}_{\xi<\alpha}$ *be a family of spaces. The space* $\bigoplus_{\xi<\alpha}X_\xi$ *is strong* G *if and only if* X_ξ *is strong* G *for each* $\xi < \alpha$.

We show that certain products are strong G-spaces.

7.23 *Theorem. Let* X *be a strong* G-*space with a* π-*base of singletons. Then* Y *is a strong* G-*space if and only if* $X \times Y$ *is a strong* G-*space.*

Proof. ($\Rightarrow$) Since X has a π-base of singletons and βY is a G-space, we have that $b(X \times Y) = \beta X \times \beta Y$ is a G-space (3.3). Hence, $X \times Y$ is a strong G-space (7.18).

($\Leftarrow$) Since $X \times Y$ is a strong G-space, we have that $\beta X \times \beta Y$ is a G-space (7.18). The natural projection $\pi\colon \beta X \times \beta Y \to \beta Y$ satisfies the hypotheses of Theorem 7.3. So, βY is a G-space, and hence Y is a strong G-space (7.18).

We show that the class of strong G-spaces is better behaved with respect to products than the class of σ-π spaces and the class of G-spaces. Previously, in (6.3) we gave an example of a large product which was σ-π and G even though no finite partial product had the corresponding property. We show now that no such pathology is available in the class of strong G-spaces.

7.24 *Theorem. Let* $\{X_\xi\}_{\xi<\kappa}$ *be a family of spaces such that* $X = \prod_{\xi<\kappa} X_\xi$ *is strong* G. *Then each partial product of* X *must be a strong* G-*space. In particular, each* X_ξ *is a strong* G-*space.*

Proof. Let $K \subset \kappa$. Note that $X \approx \prod_{\xi\in K} X_\xi \times \prod_{\xi\in\kappa\setminus K} X_\xi$. Since X is strong G, the product $\beta(\prod_{\xi\in K} X_\xi) \times \beta(\prod_{\xi\in\kappa\setminus K} X_\xi)$ is G (7.18). The natural projection

$$\pi\colon \beta(\prod_{\xi\in K} X_\xi) \times \beta(\prod_{\xi\in\kappa\setminus K} X_\xi) \to \beta(\prod_{\xi\in K} X_\xi)$$

shows that $\beta(\prod_{\xi\in K} X_\xi)$ is G (7.3). Hence, $\prod_{\xi\in K} X_\xi$ is a strong

G-space (7.18).

8. Strong G-spaces: Remote Points, a Problem of Woods and Generalization of a Theorem of Dow

This section contains a principal contribution of this work, namely, the triad of results: (1) a variant of a theorem of Chae and Smith (8.2), (2) a partial solution to a problem of Woods (8.3), and (3) a generalization of a result of Dow (8.7).

Theorem 8.1 and Corollary 8.3 generalize results of van Douwen [vD, 4.2], [vD, 4.4]. We have modified van Douwen's arguments so that they now apply to strong G-spaces. Corollary 8.2 is a variant of the Chae and Smith Theorem concerning the existence of remote points.

8.1 *Theorem. Let* X *be any strong* G-*space. If* A *is any non-empty* G_δ *subset of* βX *such that* $A \subset X^*$, *then* A *contains at least* 2^c *remote points of* X.

Proof. Let $Y = \beta X - A$. Since $\beta Y = \beta X$, it is clear that Y is nonpseudocompact [W, 1.57]. Further, Y being an F_σ of βX implies that Y is normal. And Y is a G-space (7.17). Hence, Y has at least 2^c remote points (2.9). But every remote point of Y is a remote point of X, since every nowhere dense subset of X is nowhere dense in Y.

8.2 *Corollary. If* X *is a nonpseudocompact strong* G-*space, then* X *has at least* 2^c *remote points.*

Proof. Since X is nonpseudocompact, there exists a non-empty G_δ subset A of βX such that $A \subset X^*$ [W, 1.57], and A contains at least 2^c remote points of X.

8.3 *Corollary. If* X *is a realcompact strong* G*-space, then* TX *is a dense subspace of* X*. *Furthermore,* X* *and* (EX)* *have homeomorphic dense subspaces.*

Proof. Since X is realcompact, each point of X* is contained in a G_δ subset of βX which misses X [W, 1.53]. For a proof of the second statement of the theorem, see [Wo_3].

8.4 *Remark.* Woods [Wo_3] posed the problem "Characterize those (real-compact) spaces X for which X* and (EX)* have dense homeomorphic subspaces." Although Corollary 8.3 does not provide a solution to his problem, it does provide many new examples of spaces having the desired property.

We consider the class of nonpseudocompact strong G-spaces and the class of nonpseudocompact normal G-spaces and we demonstrate that neither class is a subclass of the other.

8.5 *Example.* We present a nonpseudocompact strong G-space which is not normal. The example we give is a variant of a space described by Steen and Seebach [SS], and called (by them) the Thomas Plank.

Let $\omega + 1$ be the one-point compactification of ω and let $\omega_1 + 1$ be the one-point compactification of the discrete space ω_1.

Let $X = ((\omega + 1) \times (\omega_1 + 1)) - \{(\omega,\omega_1)\}$.

The space X is nonpseudocompact since the collection of open sets $\{(n,n): n < \omega\}$ is locally finite. The space X is strong G because it possesses a π-base of singletons (4.2).

We show X is not normal. Let $A = \{(n,\omega_1): n < \omega\}$ and let $B = \{(\omega,\xi): \xi < \omega_1\}$. Then A and B are disjoint closed subsets of X which cannot be contained in disjoint open subsets of X.

By Corollary 8.2, X has remote points. Since X is not normal, Theorem 2.9 is not directly applicable. We remark that X is also locally compact.

8.6 *Example.* A nonpseudocompact normal G-space need not be a strong G-space.

Let $\omega_2 + 1$ be the one-point compactification of the discrete space ω_2. With ω_2 discrete, let $X = \omega_2^\omega \times (\omega_2 + 1)^\omega$. Then X is a nonpseudocompact, normal [St], G-space (2.10, 6.2). However, $(\omega_2 + 1)^\omega$ is not a G-space (6.1) and, thus, X is not strong G (7.24).

Dow $[D_1]$ defined the concept of a good π-base. It follows easily that any space with a good π-base is also a G-space. Hence, we offer the following modest generalization of a theorem of Dow.

8.7 *Theorem. If* $\{X_\xi\}_{\xi<\kappa}$ *is a family of spaces of countable* π*-weight and if* $\{Y_i\}_{i<n}$ *is a finite family of spaces each possessing a* π*-base consisting of elements of countable* π*-weight, then* $X = \prod_{\xi<\kappa} X_\xi \times \prod_{i<n} Y_i$ *is a strong* G*-space. Furthermore, if* X *is nonpseudocompact, then* X *has at least* 2^c *remote points.*

Proof. Let $\mathbf{U}$ be a π-base for X such that if $U \in \mathbf{U}$, then $U = \prod_{\xi<\kappa+n} U_\xi$, where $\pi U_\xi = \omega$, for each $\xi < \kappa + n$. Let

$bU = \prod_{\xi<\kappa+n} \beta U_\xi$. Note that for each $\xi < \kappa + n$, $\pi\beta U_\xi = \pi U_\xi = \omega$. Hence, bU has a good π-base [D_1] and is, therefore, a G-space. But bU is a compactification of U. So, U is a strong G-space (7.18). Thus, X is a strong G-space (7.20). The last statement of the theorem follows from (8.2).

8.8 *Example.* Let $\alpha > \omega$ and let $L(\alpha)$ be the long line on α. Consider the space $\mathbf{R}^\kappa \times L(\alpha)$. Clearly, $L(\alpha)$ has a π-basis of which each element has countable π-weight, so $\mathbf{R}^\kappa \times L(\alpha)$ is a nonpseudocompact, strong G-space, which has at least $2^{\mathfrak{c}}$ remote points. Note, however, that $\pi L(\alpha) = \alpha > \omega$, so that Dow's work [$D_1$] is not directly applicable.

8.9 *Remark.* Many spaces satisfying the hypotheses of Theorem 8.7 are also nowhere locally compact. Note that for any nowhere locally space with remote points, we can exhibit points p and q in the remainder such that $f(p) \neq q$ for any automorphism f of the remainder [VW]. Thus, we are able to demonstrate why such remainders are not homogeneous.

We offer another variant of the Chae and Smith theorem (2.9) concerning the existence of remote points. We have not found any spaces which satisfy the hypotheses of Theorem 8.10 but fail to satisfy the hypotheses of previously stated, more tractable theorems.

8.10 *Theorem. Suppose* X *is a nonpseudocompact* G-*space satisfying the conditions:*

(1) *Every nowhere dense subset of* X *is contained in a nowhere dense zero set of* X.

(2) *If* Z *is a zero set of* X *and* F *is a regular closed subset of* X *such that* Z *and* F *are disjoint, then* Z *and* F *are completely separated.*

Then the space X *has remote points.*

Proof. Since X is a nonpseudocompact G-space, there exists a free ultrafilter $\mathbf{F}$ of closed subsets of X, no member of which is nowhere dense [CS]. Without loss of generality, assume that each $F \in \mathbf{F}$ is a regular closed set [CS], (7.15). If A is a nowhere dense subset of X, then there exists a nowhere dense zero set Z such that $A \subset Z$. For each such Z, there exists $F \in \mathbf{F}$ such that $Z \cap F = \emptyset$. Hence, $c\ell_{\beta X}Z \cap c\ell_{\beta X}F = \emptyset$. Thus, each point of $\cap\{c\ell_{\beta X}F\colon F \in \mathbf{F}\}$ is a remote point of X. We note that $\cap\{c\ell_{\beta X}F\colon F \in \mathbf{F}\}$ is nonempty, because $\{c\ell_{\beta X}F \in \mathbf{F}\}$ has the finite intersection property.

8.11 *Remark.* Every ccc space satisfies condition (1) above. That result is due to A. Hager [H] and we include it as Lemma 8.12 below. Spaces satisfying condition (2) above are objects of study in their own right. J. Mack [M] has investigated such spaces and he terms them *weakly* δ*-normally separated.*

8.12 *Lemma* (Hager [H]). *Let* X *be a space. If* X *is* ccc *and* D *is a nowhere dense subset of* X*, then there exists a nowhere dense zero set* Z *of* X *such that* $D \subset Z$.

Proof. Let $V = X\backslash c\ell_X D$. Since V is an open subspace of X, V is ccc. Let $\mathcal{W}$ be a cover of V by cozero subsets of X. There exists a countable subfamily $\{W_\xi\}_{\xi<\omega} \subset \mathcal{W}$ such that $\cup_{\xi<\omega}W_\xi$ is dense in V [CoH]. Let $Z = X\backslash\cup_{\xi<\omega}W_\xi$.

9. Questions

The following is a summary of what we consider to be the most important unresolved problems deriving from our work.

9.1 Characterize strong G-spaces.

9.2 Is the class of G-spaces finitely productive?

9.3 Determine conditions on an infinite family of G-spaces which will ensure that their product is G. Specifically, if $\{X_\xi\}_{\xi<\alpha}$ is a family of spaces such that each countable partial product is G, then must the full product be G?

9.4 Does there exist a space satisfying the conditions of Theorem 8.10 which does not satisfy the conditions of more tractable theorems?

9.5 Do there exist non-G-spaces X and Y such that $X \times Y$ is G?

9.6 Do there exist spaces X and Y such that neither X nor Y is a σ-π space but $X \times Y$ is a σ-π space?

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