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## CLASSES OF LOCALLY FINITE COLLECTIONS IN TOPOLOGICAL AND UNIFORM SPACES

by

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## CLASSES OF LOCALLY FINITE COLLECTIONS IN TOPOLOGICAL AND UNIFORM SPACES

H.L. Shapiro and F.A. Smith

In this paper we study several classes of locally finite collections. We show their interrelationships and how they relate to known topological concepts such as paracompact and expandable.

If  $(X, \mathcal{J})$  is a topological space and if  $n \in \underline{\mathbb{N}}$ , we say that a cover  $\mathcal{G}$  of  $X$  is *n-even* if there exist neighborhoods  $W_1, \dots, W_n$  of the diagonal of  $X$  such that  $W_i^2 \subset W_{i-1}$  for  $i = 2, \dots, n$  and  $(W_1(x))_{x \in X}$  refines  $\mathcal{G}$ . (Here  $W(x) = \{y \in X: (x, y) \in W\}$ .)

If  $(X, \mathcal{U})$  is a uniform space we say that a cover  $\mathcal{G}$  of  $X$  is *Lebesgue* if there exists a  $U$  in  $\mathcal{U}$  such that  $(U(x))_{x \in X}$  refines  $\mathcal{G}$ . If every open cover of  $X$  is Lebesgue we say that  $X$  has the *Lebesgue property*.

We write  $\mathcal{U}_0$  for the universal uniformity on  $X$ : i.e. the finest uniformity compatible with the topology on  $X$ . If the universal uniformity is the collection of all neighborhoods of the diagonal of  $X$  we will say that  $X$  is *strongly collectionwise normal*. It is known that if  $X$  is paracompact then  $X$  is strongly collectionwise normal and that strongly collectionwise normal implies collectionwise normal (see [2] and [3]). Furthermore, in general, neither of these implications can be reversed. In [10] we proved the following theorem.

*Theorem 1. A completely regular topological space is strongly collectionwise normal if and only if every even open cover is normal.*

Later we will show that certain classes of covers are equivalent if  $X$  is strongly collectionwise normal. If  $\mathcal{F} = (F_\alpha)_{\alpha \in I}$  and  $\mathcal{G} = (G_\beta)_{\beta \in J}$  are two families of subsets of  $X$  we say that  $\mathcal{F}$  is finite with respect to  $\mathcal{G}$  if for all  $\beta \in J$  there exists a finite subset  $K_\beta$  of  $I$  such that  $F_\alpha \cap G_\beta = \emptyset$  if  $\alpha \notin K_\beta$ . This terminology relates to the following observation which motivates our next definitions.

*Proposition 2. Suppose that  $\mathcal{F}$  is an open cover of a topological space  $X$ . Then the following statements hold.*

(a) *The cover  $\mathcal{F}$  is locally finite if and only if there exists an open cover  $\mathcal{G}$  such that  $\mathcal{F}$  is finite with respect to  $\mathcal{G}$ .*

(b) *The cover  $\mathcal{F}$  is star-finite if and only if  $\mathcal{F}$  is finite with respect to itself. (The cover  $\mathcal{F} = (F_\alpha)_{\alpha \in I}$  is star-finite if for each  $\alpha \in I$ ,  $\{\beta \in I: F_\beta \cap F_\alpha \neq \emptyset\}$  is finite.)*

The previous result shows that for covers, the property of being locally finite is the same as the existence of another open cover such that the original cover is finite with respect to this new cover. Thus we have the natural question of what type of covers  $\mathcal{G}$  can be characterized by the existence of another cover  $\mathcal{H}$  such that  $\mathcal{G}$  is finite with respect to  $\mathcal{H}$ . With this in mind we present the following definitions.

*Definition.* Suppose that  $(X, \mathcal{J})$  is a topological space and that  $\mathcal{F} = (F_\alpha)_{\alpha \in I}$  is a family of subsets of  $X$ . We say that

(a) The family  $\mathcal{F}$  is  *$K$ -locally finite* if there exists a locally finite open cover  $\mathcal{G}$  of  $X$  such that  $\mathcal{F}$  is finite with respect to  $\mathcal{G}$ .

(b) The family  $\mathcal{F}$  is *strongly  $K$ -locally finite* if there exists a star-finite even open cover  $\mathcal{G}$  of  $X$  such that  $\mathcal{F}$  is finite with respect to  $\mathcal{G}$ .

(c) The family  $\mathcal{F}$  is *weakly  $K$ -even* if there exists a neighborhood  $U$  of the diagonal of  $X$  such that  $\mathcal{F}$  is finite with respect to  $(U(x))_{x \in X}$ .

(d) The family  $\mathcal{F}$  is  *$K$ -point-even* if there exists a point-finite even open cover  $\mathcal{G}$  such that  $\mathcal{F}$  is finite with respect to  $\mathcal{G}$ .

(e) The family  $\mathcal{F}$  is  *$K$ -even* if there exists a locally finite even open cover  $\mathcal{G}$  of  $X$  such that  $\mathcal{F}$  is finite with respect to  $\mathcal{G}$ .

*Definition.* If  $(X, \mathcal{U})$  is a uniform space and if  $\mathcal{F}$  is a family of subsets of  $X$ , then we say that

(a) The family  $\mathcal{F}$  is  *$\mathcal{U}$ -locally finite* if there exists a  $U \in \mathcal{U}$  such that  $\mathcal{F}$  is finite with respect to  $(U(x))_{x \in X}$ .

(b) The family  $\mathcal{F}$  is  *$K\mathcal{U}$ -locally finite* if there exists a locally finite Lebesgue cover  $\mathcal{H}$  of  $X$  such that  $\mathcal{F}$  is finite with respect to  $\mathcal{H}$ .

(c) The family  $\mathcal{F}$  is  *$\mathcal{U}$ -discrete* if there exists a  $U \in \mathcal{U}$  such that  $U(x)$  meets at most one element of  $\mathcal{F}$  for all  $x$  in  $X$ .

In [6] Katětov first defined  $K$ -locally finite where he called it "uniformly locally finite." For a discussion of  $K$ -locally finite see [1, p. 134ff]. Katětov proved that a normal space  $X$  is collectionwise normal if and only if, for every closed subspace  $S$  of  $X$ , if  $\mathcal{G}$  is a  $K$ -locally finite open cover of  $S$  then  $\mathcal{G}$  can be extended to a  $K$ -locally finite open cover of  $X$ . In [11] J. C. Smith first defined  $\mathcal{U}$ -locally finite where he called it "uniformly locally finite." In [11] J. C. Smith proved the following.

*Theorem 3.* If  $(X, \mathcal{U})$  is a uniform space and if  $\mathcal{G}$  is a Lebesgue cover then the following statements are equivalent:

- (1) The cover  $\mathcal{G}$  has a locally finite Lebesgue refinement.
- (2) The cover  $\mathcal{G}$  has a  $\mathcal{U}$ -locally finite refinement.

The following results are stated for completeness. The implications are for the most part obvious and the proofs are straightforward and may be used as exercises for those who are not familiar with concepts of this type.

*Proposition 4.* If  $(X, \mathcal{U})$  is a uniform space and if  $\mathcal{F}$  is a collection of subsets of  $X$  then the following statements hold.

- (1) If  $\mathcal{F}$  is  $K\mathcal{U}$ -locally finite then  $\mathcal{F}$  is  $K$ -even.
- (2) If  $\mathcal{F}$  is  $K\mathcal{U}$ -locally finite then  $\mathcal{F}$  is  $\mathcal{U}$ -locally finite.
- (3) If  $\mathcal{F}$  is  $\mathcal{U}$ -locally finite then  $\mathcal{F}$  is weakly  $K$ -even.

*Proposition 5.* If  $X$  is a topological space and if  $\mathcal{F}$  is a collection of subsets of  $X$  then the following statements hold.

- (1) If  $\mathcal{F}$  is  $K$ -even then  $\mathcal{F}$  is  $K$ -locally finite.
- (2) If  $\mathcal{F}$  is  $K$ -even then  $\mathcal{F}$  is  $K$ -point-even.
- (3) If  $\mathcal{F}$  is  $K$ -point-even then  $\mathcal{F}$  is weakly  $K$ -even.

At this point let us also observe the following propositions which are what helped motivate the definitions of  $K$ -even and  $K$ -locally finite.

*Proposition 6.* If  $X$  is a topological space and if  $\mathcal{F} = (F_\alpha)_{\alpha \in I}$  is a family of subsets of  $X$  then the following statements are equivalent:

- (1) The family  $\mathcal{F}$  is weakly  $K$ -even.
- (2) There exists an even cover  $\mathcal{H}$  such that  $\mathcal{F}$  is finite with respect to  $\mathcal{H}$ .

*Proposition 7.* If  $(X, \mathcal{U})$  is a uniform space and if  $\mathcal{F} = (F_\alpha)_{\alpha \in I}$  is a family of subsets of  $X$  then the following statements are equivalent:

- (1) The family  $\mathcal{F}$  is  $\mathcal{U}$ -locally finite.
- (2) There exists a Lebesgue cover  $\mathcal{H}$  of  $X$  such that  $\mathcal{F}$  is finite with respect to  $\mathcal{H}$ .

In order to obtain additional implications we will need additional hypothesis on  $X$ . For example it is not difficult to show that if  $X$  is paracompact then if a collection is weakly  $K$ -even it is  $K$ -locally finite. However we can do better. First let us show the following proposition.

*Proposition 8.* If  $X$  is a normal topological space and if  $\mathcal{F}$  is a  $K$ -locally finite collection of subsets of  $X$  then  $\mathcal{F}$  is  $K$ -even (and therefore weakly  $K$ -even).

*Proof.* In [1, Theorem 11.7] it is shown that every locally finite open cover in a normal space is normal and in [10, Theorem 3.11] it is shown that every normal open cover is even (actually more is shown but this result follows immediately).

We are now ready to state one of our main results which shows the equivalence of several of these families of subsets if the topological space is strongly collectionwise normal.

*Theorem 9.* If  $X$  is a strongly collectionwise normal space and if  $\mathcal{F} = (F_\alpha)_{\alpha \in I}$  is a family of subsets of  $X$  then the following statements are equivalent.

- (1) The family  $\mathcal{F}$  is  $K$ -even.
- (2) The family  $\mathcal{F}$  is  $K$ -point even.
- (3) The family  $\mathcal{F}$  is weakly  $K$ -even.
- (4) The family  $\mathcal{F}$  is  $K$ -locally finite.

*Proof.* (1) implies (2) and (2) implies (3) are true in general. To see that (3) implies (4) suppose that  $\mathcal{F}$  is weakly  $K$ -even. Then there exists a neighborhood  $U$  of the diagonal of  $X$  such that  $\mathcal{F}$  is finite with respect to  $\mathcal{G} = (U(x))_{x \in X}$ . Since  $\mathcal{G}$  is even and  $X$  is strongly collectionwise normal we have, by Theorem 1 that  $\mathcal{G}$  is normal. Hence there exists a locally finite cozero-set cover  $\mathcal{H}$  such that  $\mathcal{H}$  refines  $\mathcal{G}$  ([1, Theorem 10.10]). Then  $\mathcal{F}$  is finite

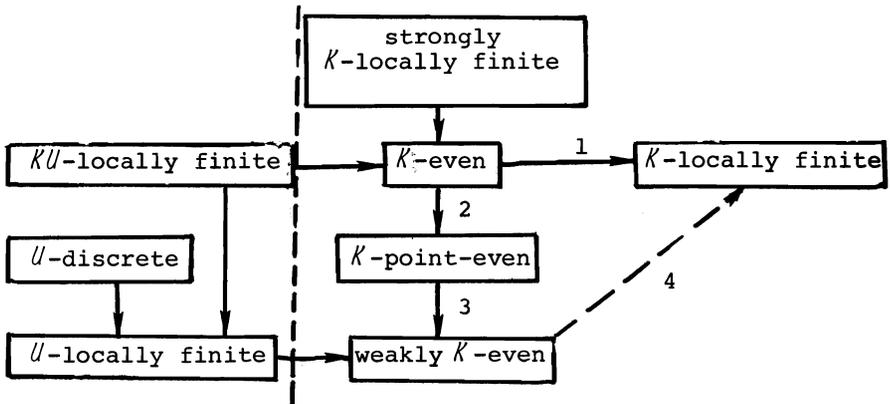
with respect to  $\mathcal{H}$  and therefore (4) holds.

To show that (4) *implies* (1) note that  $X$  strongly collectionwise normal implies that  $X$  is collectionwise normal and thus  $X$  is normal so Proposition 7 applies.

*Example.* A topological space with a collection  $\mathcal{J}$  and two compatible uniformities  $\mathcal{U}_1$  and  $\mathcal{U}_2$  such that  $\mathcal{J}$  is  $\mathcal{U}_1$ -locally finite but is not  $\mathcal{U}_2$ -locally finite.

Let  $X$  be a discrete space, let  $\mathcal{U}_0$  be the universal uniformity. Let  $\mathcal{U}$  be any other uniformity not equal to  $\mathcal{U}_0$  (such a uniformity exists since a discrete space does not have a unique uniformity) and let  $\mathcal{J} = (\{x\})_{x \in X}$ . To see that  $\mathcal{J}$  is  $\mathcal{U}_0$ -locally finite but not  $\mathcal{U}$ -locally finite observe that the diagonal  $U = \{(x,x) : x \in X\}$  is an element of  $\mathcal{U}_0$  but not  $\mathcal{U}$ .

The following diagram helps clarify these concepts: Let  $(X, \mathcal{J})$  be a topological space. The following implications hold. Any dotted arrow holds under additional conditions that will be stated after the diagram. If in addition there is a uniformity  $\mathcal{U}$  on  $X$  that is compatible with the topology  $\mathcal{J}$ , then the left half of the diagram also holds.



If  $X$  is strongly collectionwise normal, then implication 4 and its reverse hold in addition to the reverse of implications 2 and 3. Implication 1 reverses in case  $X$  is normal.

The following proposition follows rather easily from Proposition 2.

*Proposition 10.* If  $\mathcal{J}$  is an open cover of a topological space then (1) implies (2) implies (3).

- (1) The cover  $\mathcal{J}$  is star-finite.
- (2) The cover  $\mathcal{J}$  is  $K$ -locally finite.
- (3) The cover  $\mathcal{J}$  is locally finite.

We say that a space has the *star-finite property* if every open cover has a star-finite open refinement (see [8] for background). In [9] such spaces were called strongly paracompact and in [5] they were called hypocompact.

*Definition.* A topological space  $(X, \mathcal{J})$  has the *Katětov property* if every open cover has a  $K$ -locally finite refinement.

*Theorem 11.* A topological space  $X$  is paracompact if and only if it has the Katětov property.

*Proof.* From Proposition 10 the Katětov property implies that  $X$  is paracompact. Conversely let  $\mathcal{G} = (G_\alpha)_{\alpha \in I}$  be an open cover of the paracompact space  $X$ . Since  $X$  is paracompact there exists a locally finite open cover  $\mathcal{H} = (H_\alpha)_{\alpha \in I}$  such that  $H_\alpha \subset G_\alpha$  for each  $\alpha \in I$ . Also a paracompact space is normal and hence the cover  $\mathcal{H}$  is normal. Thus there exists a closed cover  $\mathcal{F} = (F_\alpha)_{\alpha \in I}$  and an open

cover  $\mathcal{A} = (A_\alpha)_{\alpha \in I}$  such that  $\text{cl } A_\alpha \subset F_\alpha \subset H_\alpha$ . Let  $[I]$  be the set of all finite subsets of  $I$  and for each  $J \in [I]$  let

$$U_J = (\bigcap_{\alpha \in J} G_\alpha) - (\bigcup_{\alpha \notin J} F_\alpha)$$

and set  $\mathcal{U} = (U_J)_{J \in [I]}$ . We assert that  $\mathcal{A}$  is a locally finite open cover of  $X$  and that  $\mathcal{A}$  is finite with respect to  $\mathcal{U}$ .

Clearly  $U_J$  is open. If  $x \in X$  let  $J_x = \{\alpha \in I : x \in H_\alpha\}$ . Then  $J_x$  is finite and  $x \in U_{J_x}$  hence  $\mathcal{U}$  is an open cover.

Since  $\mathcal{H}$  is locally finite,  $(\bigcap_{\alpha \in J} H_\alpha)_{J \in [I]}$  is locally finite and since  $U_J \subset \bigcap_{\alpha \in J} H_\alpha$  it follows that  $\mathcal{U}$  is locally finite.

To see that  $\mathcal{A}$  is finite with respect to  $\mathcal{U}$  observe that if  $J \in [I]$  and if  $\beta \notin J$  then by definition  $F_\beta \cap U_J = \emptyset$ . Since  $\text{cl } A_\beta \subset F_\beta$ , clearly  $A_\beta \cap U_J = \emptyset$ . Since  $\mathcal{A}$  is a  $K$ -locally finite refinement of  $\mathcal{G}$  it follows that  $X$  has the Katětov property. This completes the proof.

In [7] Krajewski defined a topological space  $X$  to be *expandable* if for every locally finite family  $(F_\alpha)_{\alpha \in I}$  of subsets of  $X$  there is a locally finite family  $(G_\alpha)_{\alpha \in I}$  of open subsets of  $X$  with the property that  $F_\alpha \subset G_\alpha$  for all  $\alpha \in I$ . We will show that (3) *implies* (2) in Proposition 10 if the space  $X$  is expandable. This will follow from a characterization of expandable spaces. But first we need the following result of Katětov (see [6] or [1], Theorem 12.2).

*Theorem 12. Let  $\mathcal{F} = (F_\alpha)_{\alpha \in I}$  be a family of subsets of a topological space  $X$ . The following are equivalent:*

- (1) *The family  $\mathcal{F}$  is  $K$ -locally finite.*
- (2) *There is a locally finite family of open subsets*

$\mathcal{G} = (G_\alpha)_{\alpha \in I}$  of  $X$  such that  $\text{cl } F_\alpha \subset G_\alpha$  for all  $\alpha \in I$ .

As a result of Theorem 12 and the definition of expandable we have the following.

*Theorem 13. Let  $X$  be a topological space.  $X$  is expandable if and only if every locally finite collection is  $K$ -locally finite.*

If  $X$  is normal and if  $\mathcal{G} = (G_\alpha)_{\alpha \in I}$  is a locally finite open cover of  $X$  then there exists a locally finite open cover  $\mathcal{H} = (H_\alpha)_{\alpha \in I}$  such that  $\text{cl } H_\alpha \subset G_\alpha$  for each  $\alpha \in I$ . This observation together with Theorem 12 yields the following interesting result from [9].

*Theorem 14 ([9, 3.13]). If  $X$  is normal and if  $\mathcal{G} = (G_\alpha)_{\alpha \in I}$  is a locally finite open cover of  $X$  then  $\mathcal{G}$  has a  $K$ -locally finite refinement.*

A result similar to Theorem 14 for weakly  $K$ -even covers is the following.

*Theorem 15. If  $\mathcal{G} = (G_\alpha)_{\alpha \in I}$  is a point finite 2-even cover of a topological space  $X$ , then there exists a weakly  $K$ -even (and hence locally finite) open cover  $\mathcal{H} = (H_\alpha)_{\alpha \in I}$  of  $X$  such that  $\text{cl } H_\alpha \subset G_\alpha$  for all  $\alpha \in I$ .*

*Proof.* As in the proof of [10, Theorem 3.5] we let

$$H_\alpha = \cup \{W_2(x) : W_1(x) \subset G_\alpha\}$$

for all  $\alpha \in I$  where  $W_2$  and  $W_1$  are open symmetric neighborhoods of the diagonal of  $X$  such that  $W_2^2 \subset W_1$  and  $(W_1(x))_{x \in X}$  refines  $\mathcal{G}$ . We need only observe that the proof of Theorem

3.5 in [10] shows that  $\#$  is finite with respect to  $(W_2(x))_{x \in X}$ .

We can also use Theorem 12 to prove that a weakly  $K$ -2-even collection is  $K$ -locally finite but first we need to define the following concepts.

*Definition.* If  $n \in \underline{\mathbb{N}}$  and if  $\mathcal{J}$  is a collection of subsets of a topological space, we say that  $\mathcal{J}$  is weakly  $K$ - $n$ -even if there exist neighborhoods  $W_1, \dots, W_n$  of the diagonal of  $X$  such that  $W_i^2 \subset W_{i-1}$  for  $i = 2, \dots, n$  and  $\mathcal{J}$  is finite with respect to  $(W_1(x))_{x \in X}$ . We say that  $\mathcal{J}$  is  $K$ - $n$ -even if there exists a locally finite  $n$ -even open cover  $\mathcal{G}$  of  $X$  such that  $\mathcal{J}$  is finite with respect to  $\mathcal{G}$ .

*Theorem 16.* If  $X$  is a topological space and if  $\mathcal{F} = (F_\alpha)_{\alpha \in I}$  is a weakly  $K$ -2-even family then there exists a neighborhood  $U$  of the diagonal of  $X$  such that  $(U(F_\alpha))_{\alpha \in I}$  is locally finite.

*Proof.* By hypothesis there exist open symmetric neighborhoods  $W_1$  and  $W_2$  of the diagonal of  $X$  such that  $W_2^2 \subset W_1$  and  $\mathcal{F}$  is finite with respect to  $(W_1(x))_{x \in X}$ . We show that  $(W_2(F_\alpha))_{\alpha \in I}$  is locally finite. Let  $x \in X$ . Since  $\mathcal{F}$  is finite with respect to  $(W_1(x))_{x \in X}$  there exists a finite subset  $K_x$  of  $I$  such that  $W_1(x) \cap F_\alpha = \emptyset$  if  $\alpha \notin K_x$ . We assert that  $W_2(x) \cap W_2(F_\alpha) = \emptyset$  if  $\alpha \notin K_x$ . Suppose that  $y \in W_2(x) \cap W_2(F_\alpha)$ . Then  $(x, y) \in W_2$  and there exists  $z \in F_\alpha$  such that  $(z, y) \in W_2$ . Therefore  $(x, z) \in W_2^2 \subset W_1$  so  $z \in W_1(x) \cap F_\alpha$  whence  $W_1(x) \cap F_\alpha \neq \emptyset$ . Therefore  $\alpha \in K_x$  and the proof is complete.

From Theorem 16 and the definition of expandable we have the following corollary.

*Corollary.* If a collection of subsets of a topological space  $X$  is weakly  $K$ -2-even, then it is expandable.

We know that a weakly  $K$ -even collection of a topological space need not be  $K$ -locally finite, however, we can prove the following result.

*Theorem.* If  $\mathcal{F} = (F_\alpha)_{\alpha \in I}$  is a weakly  $K$ -2-even family of a topological space  $X$  then  $\mathcal{F}$  is  $K$ -locally finite.

*Proof.* By Theorem 16 there exists an open symmetric neighborhood  $U$  of the diagonal of  $X$  such that  $(U(F_\alpha))_{\alpha \in I}$  is locally finite. We need only show that  $\text{cl } F_\alpha \subset U(F_\alpha)$  for all  $\alpha \in I$ , whence Theorem 12 will yield the desired result. If  $x \in \text{cl } F_\alpha$  then  $U(x)$  is a neighborhood of  $x$  and thus  $U(x) \cap F_\alpha \neq \emptyset$ . If  $y \in U(x) \cap F_\alpha$  then  $x \in U(y) \subset U(F_\alpha)$ .

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