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by

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In [1], Gale gave a condition which could be used to replace equicontinuity in a less restrictive version of Ascoli's Theorem, namely where the range space is regular, rather than a metric or uniform space. Gale's Theorem 1 is stated below for the sake of completeness. Throughout this note, Y^X will denote the collection of all functions from X to Y with the product topology and if $F \subset Y^X$, then $\overline{F}$ will denote the closure of F in this topology, i.e., the pointwise closure of F.

Theorem 1 (Gale). *If* X *is a* k-*space and* Y *is regular, then a collection of continuous functions* F *from* X *to* Y *is compact in the compact-open topology if and only if*

(1) F *is closed.*

(2) F(x) *is compact for each* x *in* X.

(3) *If* G *is closed in* F *and* U *is open in* Y *then* $\cap\{g^{-1}(U) | g \in G\}$ *is open in* X.

In the proof of this theorem, Gale showed that if a collection F is continuous and satisfies condition (3) then the compact-open and pointwise topologies agree on F. This condition was abstracted by Yang in [5] and renamed property (G).

Definition. $F \subset Y^X$ is said to have property (G) if for each U open in Y, and each pointwise closed subset G of F, $\cap\{g^{-1}(U) | g \in G\}$ is open in X.

Note that in the above definition, the topology being considered is the pointwise, rather than the compact-open, and F is not required to be a closed collection, as was the case in Gale's Theorem 1. Since F is not necessarily closed, the phrase "pointwise closed subset G of F" admits two distinct interpretations. Either

(1) the closure of G in Y^X lies in F, or

(2) G is closed in the relative topology on F induced by Y^X.

The purpose of this note is to examine this ambiguity.

Kelley, to whom Yang refers for all definitions not specified in [5], defines pointwise closed [4, p. 218] to mean closed in Y^X, so that interpretation (1) of property (G) seems to be intended. Yet the proof of Theorem 1 of [5] employes interpretation (2), and in fact is false using interpretation (1), as our Example B will show.

In order to sort out these difficulties, we will introduce two versions of the definition of Property (G).

Using interpretation (1) we will say $F \subset Y^X$ has property (G_1) if for each open U in Y, and for each $G \subset F$ such that $\overline{G} = G$, $\cap\{g^{-1}(U) \mid g \in G\}$ is open in X.

Similarly we will say $F \subset Y^X$ has property (G_2) if for each open U of Y and for each $G \subset F$ such that $G = \overline{G} \cap F$, $\cap\{g^{-1}(U) \mid g \in G\}$ is open in X.

It is clear that if F satisfies property (G_2) then F must also satisfy property (G_1), but the converse fails as the following example shows.

Example A. For each $n \in N$, define f_n: $[0,1] \to [0,1]$ by

$$f_n(x) = \begin{cases} 1/2n & , \quad x \in [0,1/2n] \\ x & , \quad x \in [1/2n,1] \end{cases}$$

and let $F = \{f_n | n \in N\}$. Then F has property (G_1) trivially because the only subsets G of F for which $\overline{G} \subset F$ are the finite ones, so the intersection condition is always satisfied. But letting $U_0 = (0,1) - \{1/(2n+1) | n \in N\}$ and noting that $F = \overline{F} \cap F$, we have that

$$\cap\{f_n^{-1}(U_0) | n \in N\} = [0,1) - \{1/(2n+1) | n \in N\}$$

which is not open in X, so that F does not satisfy property (G_2). Also note that F is equicontinuous and pointwise bounded, and therefore regular by the corollary to Theorem 3 of [5].

The proof of Theorem 1 of [5] establishes that a collection satisfying property (G_2) is necessarily regular, but the next example shows that this result fails for property (G_1).

Example B. For each $n \in N$, define f_n: $[0,1] \to [0,1]$ by

$$f_n(x) = \begin{cases} 4nx & , \quad x \in [0,1/4n] \\ 2-4nx & , \quad x \in [1/4n,1/2n] \\ 0 & , \quad x \in [1/2n,1] \end{cases}$$

and let $F = \{f_n | n \in N\}$. Then F has property (G_1) trivially, but is not equicontinuous at $x = 0$, and hence by Theorem 5 of [2] is not regular there.

Examples A and B also show that the corollary following Theorem 6 of [5] fails under either interpretation of property (G). However it is the case that whenever X is a k-space and Y is regular, if F is evenly continuous (or regular, by Theorem A of [3]) and $\overline{F(x)}$ is compact for each x in X, then F satisfies property (G_1). This holds because if F is evenly continuous, then so is $\overline{F}$ by [4, Theorem 19, p. 235], and hence the product topology and the compact-open topology coincide on $\overline{F}$. Thus $\overline{F}$ is compact by [6, Theorem B] and therefore satisfies property (G_1) by Gale's Theorem 1. It follows from the definition that F must also satisfy property (G_1).

References

1. D. Gale, *Compact sets of functions and function rings*, Proc. Amer. Math. Soc. 1 (1950), 303-308.
2. M. Henry, D. Reynolds and G. Trapp, *Equicontinuous and regular collections of functions*, Top. Proc. 7 (1982), 71-81.
3. S. K. Kaul, *Compact subsets in function spaces*, Canad. Math. Bull. 12 (1969), 461-466.
4. J. L. Kelley, *General Topology*, Van Nostrand, Princeton, N.J., 1955.
5. J. S. Yang, *Property* (G), *regularity, and semi-equicontinuity*, Canad. Math. Bull. 16 (1973), 587-594.
6. J. D. Weston, *A generalization of Ascoli's theorem*, Mathematika 6 (1959), 19-24.

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