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## THE CO-LESS-THAN- $\lambda$ TOPOLOGY AND PIXLEY-ROY SPACES

by

PEG DANIELS

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**Web:** <http://topology.auburn.edu/tp/>

**Mail:** Topology Proceedings  
Department of Mathematics & Statistics  
Auburn University, Alabama 36849, USA

**E-mail:** [topolog@auburn.edu](mailto:topolog@auburn.edu)

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## THE CO-LESS-THAN- $\lambda$ TOPOLOGY AND PIXLEY-ROY SPACES

Peg Daniels

### I. Introduction

Given a space  $X$ , let  $\mathcal{P}(X)$  be the collection of all nonempty subsets of  $X$ . For each  $A \in \mathcal{P}(X)$  and each open set  $U$  of  $X$ , let  $[A, U] = \{B \in \mathcal{P}(X) : A \subseteq B \subseteq U\}$ . Then  $\{[A, U] : A \in \mathcal{P}(X) \text{ and } U \text{ is an open set in } X\}$  forms a basis for a topology on  $\mathcal{P}(X)$ , called the Pixley-Roy topology on  $\mathcal{P}(X)$ . Let  $PR(X)$  denote  $\mathcal{P}(X)$  with this topology. It is well-known that if  $X$  is a  $T_1$ -space, then each element of this basis is clopen, and so  $PR(X)$  is completely regular and zero-dimensional.

Most researchers have considered the Pixley-Roy topology restricted to the finite subsets of a space. These spaces are always hereditarily metacompact. Much work has been done investigating when these spaces are normal, collectionwise-Hausdorff, paracompact, ccc, or metrizable (see e.g. [B], [HJ], [L], [PR]). E. van Douwen has shown, however, that these are not the only interesting Pixley-Roy spaces [vD].

For an infinite cardinal  $\kappa$ , let  $\kappa(\omega)$  denote the co-finite topology on  $\kappa$ , and let  $PR_3(\kappa(\omega))$  denote the subspace of  $PR(\kappa(\omega))$  consisting of all the nonempty subsets of  $\kappa$  of cardinality less than or equal to two. In [D] we proved that if  $\kappa > \omega$ , then  $PR_3(\kappa(\omega))$  is not collectionwise-Hausdorff, and hence, not paracompact. We also mentioned

that if  $\kappa \leq \omega_1$ , then  $PR_\omega(\kappa(\omega))$ , the subspace of  $PR(\kappa(\omega))$  consisting of all the nonempty, finite subsets of  $\kappa$ , is subparacompact, but if  $\kappa > \omega_1$  then  $PR_3(\kappa(\omega))$  is not subparacompact.

In this paper we generalize those results by (a) considering the "co-less-than- $\lambda$ " topology on an infinite cardinal  $\kappa$  for various infinite cardinals  $\lambda$ , and (b) including infinite subsets of  $\kappa$  in our Pixley-Roy spaces.

Let  $\mathcal{P}^\lambda(\kappa) = \{x \subseteq \kappa : |\kappa - x| < \lambda\}$ ;  $\mathcal{P}^\lambda(\kappa)$  forms a topology on  $\kappa$ . We consider Pixley-Roy spaces over  $\kappa$  with this topology. For each  $A \in \mathcal{P}(\kappa)$  and each  $x \in \mathcal{P}^\lambda(\kappa)$ , let  $\mathcal{U}(A, \kappa - x) = [A, x]$ .  $\{\mathcal{U}(A, \kappa - x) : A \in \mathcal{P}(\kappa), x \in \mathcal{P}^\lambda(\kappa)\}$  forms a basis for a Pixley-Roy topology on  $\mathcal{P}(\kappa)$ , which we denote by  $PR(\kappa(\lambda))$ . Given a cardinal  $\sigma$ , we let  $PR_\sigma(\kappa(\lambda))$  denote the subspace of  $PR(\kappa(\lambda))$  consisting of all subsets of  $\kappa$  of cardinality less than  $\sigma$ .

$PR(\kappa(\lambda))$  is completely regular and zero-dimensional. In this paper we study the properties of paracompactness, subparacompactness, and collectionwise-Hausdorffness in the spaces  $PR_\sigma(\kappa(\lambda))$  for various cardinals  $\kappa$ ,  $\lambda$ , and  $\sigma$ . As is common with investigations of Pixley-Roy spaces, the results are intimately connected with properties of the underlying spaces; in this case the combinatorial properties of sets play a large role.

The results may be summarized as follows:

For any infinite cardinal  $\lambda$ ,  $PR_\sigma(\lambda(\lambda))$  is paracompact for each cardinal  $\sigma \leq \text{cf } \lambda$ ; in fact,  $\{x \in PR(\lambda(\lambda)) \mid \sup x < \lambda\}$  is hereditarily paracompact and monotonically normal; it

is  $\lambda$ -metrizable if  $\lambda$  is regular or  $\lambda = \omega$ . If  $\lambda$  is regular, or singular with  $\text{cf } \lambda > \omega$ , then  $\text{PR}_\sigma(\lambda(\lambda))$  is neither collectionwise-Hausdorff (and hence, not paracompact) nor subparacompact for each cardinal  $\sigma \geq (\text{cf } \lambda)^+$ . If  $\text{cf } \lambda = \omega$  then we have no results on the subparacompactness of  $\text{PR}_\sigma(\lambda(\lambda))$  for any cardinal  $\sigma \geq (\text{cf } \lambda)^+ = \omega_1$ , but these spaces are not collectionwise-Hausdorff, and hence, not paracompact.

For cardinals  $\kappa$  and  $\lambda$  with  $\kappa > \lambda \geq \omega$  and  $\text{cf } \lambda > \omega$ ,  $\text{PR}_\sigma(\kappa(\lambda))$  is neither subparacompact nor collectionwise-Hausdorff for any cardinal  $\sigma \geq 3$ . For the case  $\text{cf } \lambda = \omega$ , we have the following partial results:  $\text{PR}_\sigma(\kappa(\lambda))$  is not collectionwise-Hausdorff, and hence, not paracompact, for any cardinal  $\sigma \geq 3$ ; if  $\kappa \geq (\lambda^+)^+$ , then  $\text{PR}_\sigma(\kappa(\lambda))$  is not subparacompact for any cardinal  $\sigma \geq 3$ ; if  $\kappa = \lambda^+$ , then  $\text{PR}_\sigma(\kappa(\lambda))$  is subparacompact for any cardinal  $\sigma \leq \omega$ , and  $\text{PR}(\kappa(\lambda))$  is not subparacompact; we are thus left with the question of the subparacompactness of  $\text{PR}_\sigma(\kappa(\lambda))$  for  $\omega_1 \leq \sigma \leq \kappa$ .

Note that we need not consider the case where  $\kappa$  and  $\lambda$  are infinite cardinals and  $\kappa < \lambda$ , since the space  $\text{PR}(\kappa(\lambda))$  is then a discrete space.

Also note that if each of  $\kappa$ ,  $\lambda$ ,  $\theta$ ,  $\sigma$ , and  $\tau$  is a cardinal, with  $\omega \leq \lambda < \kappa < \theta$ , and  $\sigma < \tau$ , then (a)  $\text{PR}_\sigma(\kappa(\lambda))$  is a closed subspace of  $\text{PR}_\tau(\kappa(\lambda))$ , and hence the paracompactness (subparacompactness, collectionwise-Hausdorffness) of the latter implies that of the former, and (b)  $\text{PR}_\sigma(\kappa(\lambda))$  is a closed subspace of  $\text{PR}_\sigma(\theta(\lambda))$  and similar remarks apply.

In [vD], E. van Douwen also looked at the space  $\text{PR}(\omega(\omega))$ , which he referred to as  $A[\omega^\#]$ , and a related space, which he

called  $\Omega$ .  $\Omega$  is homeomorphic to the space  $PR(\omega(\omega))$  with the point  $\emptyset$  included and the topology as previously defined. We could include the point  $\emptyset$  in any space  $PR(\kappa(\lambda))$  in a similar fashion, and we may study such spaces in a future paper. For now we note that including  $\emptyset$  may drastically change the properties of these spaces: E. van Douwen has pointed out in a private communication that, for example, if we include  $\emptyset$  in the space  $PR_{\omega}(\kappa(\omega_1))$ , then this space is Lindelöf for each  $\kappa \geq \omega_1$ . But without the point  $\emptyset$ ,  $PR_3(\omega_2(\omega_1))$  is neither subparacompact nor collectionwise-Hausdorff, as mentioned previously.

## II. $PR(\lambda(\lambda))$

The usual set-theoretic notation is followed, e.g.  $|F|$  denotes the cardinality of  $F$ ; cardinals are initial ordinals; if  $x$  is a set of ordinals, then  $\sup x$  is the least ordinal greater than or equal to each element of  $x$ ; for each cardinal,  $cf \lambda$  is the cofinality of  $\lambda$ , and  $\lambda^+$  is the successor of  $\lambda$ ; and so on.

*Theorem 1.* *If  $\lambda$  is an infinite cardinal, then  $PR_{\sigma}(\lambda(\lambda))$  is hereditarily paracompact and monotonically normal for any cardinal  $\sigma \leq cf \lambda$ . In fact,  $\{x \in PR(\lambda(\lambda)) : \sup x < \lambda\}$  is hereditarily paracompact and monotonically normal; moreover it is  $\lambda$ -metrizable if  $\lambda$  is regular or  $\lambda = \omega$ .*

*Proof.* Suppose  $\lambda$  is an infinite cardinal. Since  $PR_{cf \lambda}(\lambda(\lambda))$  is a closed subspace of  $\{x \in PR(\lambda(\lambda)) : \sup x < \lambda\}$ , we show only that the latter space, which we denote by  $X$ , is hereditarily paracompact and monotonically normal.

Suppose that  $Z \subseteq X$  and  $\mathcal{U} = \{\mathcal{U}(x, F_x) : x \in Z\}$  is an open cover of  $Z$ . We show that  $\mathcal{U}$  has a disjoint clopen refinement. Let  $\{\lambda_\beta : \beta < \text{cf } \lambda\}$  be an increasing sequence of ordinals converging to  $\lambda$ . Let  $A_0 = \{\mathcal{U}(x, F_x \cup (\lambda_0 - x)) : x \in Z \text{ and } x \subseteq \lambda_0\}$ .  $A_0$  is a disjoint clopen collection. For each  $\beta < \text{cf } \lambda$ , let  $A_\beta = \{\mathcal{U}(x, F_x \cup (\lambda_\beta - x)) : x \in Z, \beta \text{ is the least ordinal such that } x \subseteq \lambda_\beta, \text{ and } x \not\subseteq \bigcup_{\alpha < \beta} (A_\alpha)\}$ . It is easy to check that  $\mathcal{U}\{A_\beta : \beta < \text{cf } \lambda\}$  is a disjoint clopen refinement of  $\mathcal{U}$ . Since every open cover of  $Z$  has such a refinement,  $Z$  is paracompact, and hence  $X$  is hereditarily paracompact. (If  $\lambda$  is regular or  $\lambda = \omega$ , let  $\lambda_\beta = \beta$  for each  $\beta < \lambda$ ; it is easy to check that  $\bigcup_{\alpha < \lambda} \{\mathcal{U}(x, \alpha - x) : x \in X \text{ and } x \subseteq \alpha\}$  is a  $\lambda$ -discrete base for  $X$ ; since  $\lambda(\lambda)$  is a space in which the intersection of less than  $\lambda$  open sets is open,  $\text{PR}(\lambda(\lambda))$  also has this property, and hence  $X$  is  $\lambda$ -metrizable, i.e., it is a regular space that has a  $\lambda$ -locally finite base, and is such that the intersection of less than  $\lambda$  open subsets is open. All  $\lambda$ -metrizable spaces are hereditarily paracompact and monotonically normal (see [H], [S]).

For each  $x \in X$  and  $F \subseteq \lambda - x$  of cardinality less than  $\lambda$ , assign  $\mathcal{U}(x, F)$  to  $\mathcal{U}(x, F \cup (\lambda_{\alpha_x} - x))$ , where  $\alpha_x$  is the least  $\alpha$  such that  $x \subseteq \lambda_\alpha$ . Such an assignment satisfies the property that if  $\mathcal{U}(x, F \cup (\lambda_{\alpha_x} - x)) \cap \mathcal{U}(y, G \cup (\lambda_{\alpha_y} - y)) \neq \emptyset$ , then either  $x \in \mathcal{U}(y, G)$  or  $y \in \mathcal{U}(x, F)$ , and hence  $X$  is monotonically normal.

Originally we proved that the space  $X$  above is paracompact and monotonically normal. T. Przymusiński pointed out that the proof could be simplified and strengthened by

showing that for regular  $\lambda$ ,  $X$  is  $\lambda$ -metrizable, and that suitable modifications could be made for the singular case.

At the next level, however, paracompactness is lost. In order to prove the next theorem, we need two preliminary lemmas:

*Lemma.* Every subparacompact P-space is paracompact ( $X$  is a P-space if  $\bigcap_{n \in \omega} U_n$  is open in  $X$  whenever each  $U_n$  is open in  $X$ ).

*Sketch of Proof.* Suppose  $X$  is a subparacompact P-space and  $\mathcal{U}$  is an open cover of  $X$ . Let  $\langle V_n \rangle_{n \in \omega}$  be a sequence of open refinements of  $\mathcal{U}$  such that for each  $x \in X$  there is an  $n \in \omega$  such that  $\text{st}(x, V_n)$  is contained in some element of  $\mathcal{U}$ . Let  $\mathcal{V} = \{\bigcap_{i \in \omega} V_i : \text{for each } i \in \omega, V_i \in \mathcal{V}_i\}$ . Then  $\mathcal{V}$  is a point-star refinement of  $\mathcal{U}$ , and hence  $X$  is paracompact.

Note that if  $X$  is a P-space, then  $\text{PR}(X)$  is a P-space.

The second lemma is a result from set theory called the Free Set Lemma, and can be found in [J] (If  $F: X \rightarrow \mathcal{P}(X)$ , then a set  $S \subseteq X$  is free for  $F$  provided that for any two elements  $x$  and  $y$  of  $S$ , we have that  $x \notin F(y)$  and  $y \notin F(x)$ ):

*Free Set Lemma.* If  $F: X \rightarrow \{x \subseteq X: |x| < \eta\}$  where  $\eta < |X|$ , then there is a free set  $S \subseteq X$  for  $F$  with  $|S| = |X|$ .

*Theorem 2.* If  $\lambda$  is an infinite cardinal, then  $\text{PR}_\sigma(\lambda(\lambda))$  is not collectionwise-Hausdorff, and hence, not paracompact, for any cardinal  $\sigma \geq (\text{cf } \lambda)^+$ ; furthermore, if  $\text{cf } \lambda > \omega$ , then these spaces are also not subparacompact.

*Proof.* Suppose  $\lambda$  is an infinite cardinal. Note that if  $\text{cf } \lambda > \omega$ , then  $\lambda(\lambda)$ , and hence  $\text{PR}(\lambda(\lambda))$  is a P-space,

and so it suffices to show that  $PR_{(cf \lambda)^+(\lambda(\lambda))}$  is not collectionwise-Hausdorff to conclude that, by the previous lemma, it is also not subparacompact.

First let us assume that  $\lambda$  is regular, or that  $\lambda = \omega$ . Let  $X = PR((\lambda \times 2)(\lambda))$  denote the Pixley-Roy space over  $\lambda \times 2$  with the co-less-than- $\lambda$  topology. Since  $|\lambda \times 2| = \lambda$ , clearly  $X$  is homeomorphic to  $PR(\lambda(\lambda))$ : we show that  $X$  is not collectionwise-Hausdorff. Let  $C = \{f \subseteq \lambda \times 2: f \text{ is a function with } \text{dom}(f) = \lambda\}$ .

To see that  $C$  is closed in  $X$ , consider a point  $f \in X - C$ . Then either there is an  $\alpha < \lambda$  such that  $f \cap (\{\alpha\} \times 2) = \emptyset$ , in which case  $\mathcal{U}(f, (\{\alpha\} \times 2))$  misses  $C$ , or there is an  $\alpha < \lambda$  such that  $f \cap (\{\alpha\} \times 2) = \{\alpha\} \times 2$ , in which case  $\mathcal{U}(f, \emptyset)$  misses  $C$ .  $C$  is clearly discrete, since if  $f$  and  $g$  are two elements of  $C$ , then  $f \not\subseteq g$ .

We claim that the points of  $C$  cannot be separated by disjoint open sets. Suppose that, on the contrary,  $\{\mathcal{U}(f, F_f): f \in C\}$  is pairwise disjoint. For each  $f \in C$ ,  $|F_f| < \lambda$ , and so there is an  $\alpha_f < \lambda$  such that  $F_f \subseteq (\alpha_f + 1) \times 2$ ; let  $H_f = [(\alpha_f + 1) \times 2] - f$ .  $\{\mathcal{U}(f, H_f): f \in C\}$  is pairwise disjoint. It is easy to check that if  $f$  and  $g$  are two elements of  $C$  with  $f|_{\alpha_g+1} = g|_{\alpha_g+1}$ , then since  $f \cap H_g$  must be empty,  $g \cap H_f$  is nonempty, and so  $\alpha_f > \alpha_g$ . We construct a sequence  $\langle f_\alpha: \alpha \leq \lambda \rangle$  of elements of  $C$  as follows. Let  $f_0 \in C$ . Suppose  $\alpha < \lambda$  and for each  $\beta < \alpha$ ,  $f_\beta \in C$  has been chosen such that if  $\gamma < \beta$ , then  $f_\beta \neq f_\gamma$ , but  $f_\beta|_{\alpha_{f_\gamma}+1} = f_\gamma|_{\alpha_{f_\gamma}+1}$ . Let  $f'_\alpha \in C$  be such that for each  $\beta < \alpha$ ,



$f'_\alpha|_{\alpha_{f_\beta}+1} = f'_\beta|_{\alpha_{f_\beta}+1}$ . If  $\alpha$  is a limit, let  $\gamma = \sup_{\beta < \alpha} \alpha_{f_\beta}$ , and if  $\alpha$  is a successor, say  $\alpha = \alpha' + 1$ , let  $\gamma = \alpha_{f_{\alpha'}} + 1$ ; in either case,  $f'_\alpha|_\gamma$  has  $2^\lambda$  extensions in  $C$ , and so we may choose one, call it  $f_\alpha$  such that  $f_\alpha \neq f_\beta$  for any  $\beta < \alpha$ . Note that  $\alpha_{f_\alpha} > \alpha_{f_\beta}$  for each  $\beta < \alpha$ . In this way we define  $\langle f_\alpha : \alpha < \lambda \rangle$ ; now  $f_\lambda \in C$  is completely determined, i.e.,  $f_\lambda$  is the element of  $C$  such that  $f_\lambda|_{\alpha_{f_\beta}+1} = f_\beta|_{\alpha_{f_\beta}+1}$  for each  $\beta < \lambda$ . But then  $\alpha_{f_\lambda} \geq \sup_{\beta < \lambda} \alpha_{f_\beta} = \lambda$ , which is a contradiction. We conclude that the points of  $C$  cannot be separated, so  $X$  is not collectionwise-Hausdorff, and hence  $PR(\lambda(\lambda))$  is not collectionwise-Hausdorff.

Now let us assume that  $\lambda$  is singular. The main ideas are very similar to those presented above, but this case is not as simple. Let  $\langle \lambda_\alpha : \alpha < cf \lambda \rangle$  be an increasing sequence of regular cardinals, each bigger than  $cf \lambda$ , converging to  $\lambda$ . Since  $|U_{\alpha < cf \lambda} (\{\alpha\} \times \lambda_\alpha)| = \lambda$ ,  $PR((U_{\alpha < cf \lambda} (\{\alpha\} \times \lambda_\alpha))(\lambda))$  is homeomorphic to  $PR(\lambda(\lambda))$ . Let  $X = U_{\alpha < cf \lambda} (\{\alpha\} \times \lambda_\alpha)$ . We claim that  $C = \{f \subseteq X : f \text{ is a function with domain } cf \lambda\}$  is a closed discrete set in  $PR_{(cf \lambda)+}(X(\lambda))$  that cannot be separated by disjoint open sets. The proof that  $C$  is closed and discrete is similar to the one presented above.

Suppose that  $\{U(f, F_f) : f \in C\}$  is pairwise disjoint. For each  $f \in C$ ,  $|F_f| < \lambda$ , so let  $\alpha_f < cf \lambda$  be such that  $|F_f| < \lambda_{\alpha_f}$ ; let  $G_f = F_f - U_{\alpha \leq \alpha_f} (\{\alpha\} \times \lambda_\alpha)$  and let  $H_f = [U_{\alpha \leq \alpha_f} (\{\alpha\} \times \lambda_\alpha) - f] \cup G_f$ .  $\{U(f, H_f) : f \in C\}$  is pairwise disjoint. Note that for each  $f \in C$  and  $\beta > \alpha_f$ ,

$|\{\gamma \in \lambda_\beta : (\beta, \gamma) \in G_f\}| < \lambda_{\alpha_f} < \lambda_\beta$ , so we may let  $k_f \subseteq X$  be a function with domain  $\text{cf } \lambda$  that "bounds"  $G_f$ , i.e., for each  $\beta > \alpha_f$ ,  $k_f(\beta) > \sup\{\gamma \in \lambda_\beta : (\beta, \gamma) \in G_f\}$ . As in the previous argument, we want to construct a sequence of  $f$ 's with increasing  $\alpha_f$ 's. In order to do this, we need the following lemma:

*Lemma.* For each  $f \in \mathcal{C}$  there is a  $\phi_f \in \mathcal{C}$  such that for each  $\beta > \alpha_f$ ,  $\phi_f(\beta) > k_f(\beta)$ , and with the property that if  $g$  is an element of  $\mathcal{C}$  with  $g|_{\alpha_{f+1}} = f|_{\alpha_{f+1}}$  such that for each  $\beta > \alpha_f$ ,  $g(\beta) > \phi_f(\beta)$ , then  $\alpha_g > \alpha_f$ .

*Proof of Lemma.* Suppose  $f \in \mathcal{C}$  and there is no such function  $\phi_f$ . Let  $\phi_0 \in \mathcal{C}$  be such that for each  $\beta > \alpha_f$ ,  $\phi_0(\beta) > k_f(\beta)$ . Let  $g_0 \in \mathcal{C}$  be such that  $g_0|_{\alpha_{f+1}} = f|_{\alpha_{f+1}}$  and for each  $\beta > \alpha_f$ ,  $g_0(\beta) > \phi_0(\beta)$ , and  $\alpha_{g_0} \leq \alpha_f$ . Suppose  $\gamma < \lambda_{\alpha_{f+1}}$  and for each  $\delta < \gamma$ ,  $\phi_\delta$  and  $g_\delta$  have been defined. Now for each  $\beta > \alpha_f$ ,  $|\{g_\delta(\beta) : \delta < \gamma\}| \leq \lambda_{\alpha_f} < \lambda_\beta$ , so we may let  $\phi_\gamma$  be an element of  $\mathcal{C}$  such that for each  $\beta > \alpha_f$ ,  $\phi_\gamma(\beta) > \sup_{\delta < \gamma} g_\delta(\beta)$ . Then there is a  $g_\gamma \in \mathcal{C}$  with  $g_\gamma|_{\alpha_{f+1}} = f|_{\alpha_{f+1}}$  such that for each  $\beta > \alpha_f$ ,  $g_\gamma(\beta) > \phi_\gamma(\beta)$ , and  $\alpha_{g_\gamma} \leq \alpha_f$ . In this way we define  $g_\gamma$  for each  $\gamma < \lambda_{\alpha_{f+1}}$ .

For each  $\gamma < \lambda_{\alpha_{f+1}}$ , let  $K_\gamma = \{\delta < \lambda_{\alpha_{f+1}} : g_\delta \cap G_\gamma \neq \emptyset\}$ ; we claim that  $|K_\gamma| \leq |G_\gamma|$ . Suppose this is not so; then there are two elements  $\delta$  and  $\rho$ ,  $\delta < \rho$ , of  $\lambda_{\alpha_{f+1}}$  and an element  $(\sigma, \tau)$  of  $X$  such that  $(\sigma, \tau) \in G_\gamma \cap g_\delta \cap g_\rho$ . Since  $g_\delta|_{\alpha_{f+1}} = f|_{\alpha_{f+1}} = g_\rho|_{\alpha_{f+1}} = g_\gamma|_{\alpha_{f+1}}$ , we must have that

$\sigma > \alpha_f$ . But then  $\tau = g_\rho(\sigma) > \phi_\rho(\sigma) > \sup_{\nu < \rho} g_\nu(\sigma) \geq g_\delta(\sigma) = \tau$ , a contradiction. So for each  $\gamma < \lambda_{\alpha_f+1}$ ,  $|K_\gamma| \leq |G_\gamma| < \lambda_{\alpha_{g_\gamma}}$   
 $\leq \lambda_{\alpha_f} < \lambda_{\alpha_f+1}$ . By the Free Set Lemma of Section II, there are two elements  $\delta$  and  $\rho$  of  $\lambda_{\alpha_f+1}$  such that  $\delta \notin K_\rho$  and  $\rho \notin K_\delta$ . Thus  $g_\delta \cap G_\rho = \emptyset$ , and  $g_\rho \cap G_\delta = \emptyset$ . But then we also have that  $g_\delta \cap H_\rho = \emptyset$  and  $g_\rho \cap H_\delta = \emptyset$ , and so  $g_\delta \cup g_\rho \in \mathcal{U}(g_\delta, H_\delta) \cap \mathcal{U}(g_\rho, H_\rho)$ , which gives us a contradiction. We conclude that the lemma is true.

*Continuation of Proof of Theorem 2.* Let  $f_0 \in \mathcal{C}$ , and let  $\phi_{f_0}$  be as in the lemma. Suppose that  $\gamma < \text{cf } \lambda$  and for each  $\delta < \gamma$  we have defined  $f_\delta$  (and its corresponding  $\phi_{f_\delta}$ ) such that for each  $\sigma < \delta$  we have that  $f_\sigma|_{\alpha_{f_\sigma}+1} = f_\delta|_{\alpha_{f_\sigma}+1}$  and that for  $\beta > \alpha_{f_\sigma}$ ,  $f_\delta(\beta) > \phi_{f_\sigma}(\beta)$ , and so  $\alpha_{f_\delta} > \alpha_{f_\sigma}$ . Let  $f'_\gamma \in \mathcal{C}$  be such that for each  $\delta < \gamma$ ,  $f'_\gamma|_{\alpha_{f_\delta}+1} = f_\delta|_{\alpha_{f_\delta}+1}$ . If  $\gamma$  is a limit, let  $\psi = \sup_{\delta < \gamma} \alpha_{f_\delta}$ , and if  $\gamma$  is a successor, say  $\gamma = \gamma' + 1$ , let  $\psi = \alpha_{f_{\gamma'}} + 1$ . Suppose  $\sigma < \gamma$  and  $\beta$  is such that  $\alpha_{f_\sigma} < \beta < \psi$ . Let  $\delta < \gamma$  such that  $\beta \leq \alpha_{f_\delta}$ . Note that  $\sigma < \delta$ , and so  $f_\delta(\beta) > \phi_{f_\sigma}(\beta)$ ; thus  $f'_\gamma(\beta) > \phi_{f_\sigma}(\beta)$ . For each  $\beta \geq \psi$ ,  $|\{\phi_{f_\delta}(\beta) : \delta < \gamma\}| < \text{cf } \lambda < \lambda_\beta$ , so we may let  $f_\gamma$  be an element of  $\mathcal{C}$  such that  $f_\gamma|_\psi = f'_\gamma|_\psi$  and such that for each  $\beta \geq \psi$ ,  $f_\gamma(\beta) > \sup\{\phi_{f_\delta}(\beta) : \delta < \gamma\}$ . Thus we have that for each  $\delta < \gamma$ ,  $f_\gamma|_{\alpha_{f_\delta}+1} = f_\delta|_{\alpha_{f_\delta}+1}$ , and for each  $\beta > \alpha_{f_\delta}$ ,  $f_\gamma(\beta) > \phi_{f_\delta}(\beta)$ , and so  $\alpha_{f_\gamma} > \alpha_{f_\delta}$ . In this way we define the

sequence  $\langle f_\gamma : \gamma < cf \lambda \rangle$ . Again we let  $f_{cf \lambda}$  be the function determined by this sequence, i.e.,  $f_{cf \lambda}$  is the element of  $C$  such that for each  $\gamma < cf \lambda$ ,  $f_{cf \lambda} \upharpoonright_{\alpha_{f_\gamma} + 1} = f_\gamma \upharpoonright_{\alpha_{f_\gamma} + 1}$ . But then  $\alpha_{f_{cf \lambda}} \geq \sup_{\gamma < cf \lambda} \alpha_{f_\gamma} = cf \lambda$ , which is a contradiction. We must conclude that  $C$  cannot be separated, so  $PR_{(cf \lambda)^+}(X(\lambda))$  is not collectionwise-Hausdorff, and hence  $PR_{(cf \lambda)^+(\lambda(\lambda))}$  is not collectionwise-Hausdorff.

We originally proved Theorem 2 with the assumption of GCH, and also noted that for limit cardinals, it was enough to assume these cardinals were strong limits; in fact under these assumptions we can prove that the space  $PR_{(cf \lambda)^+(\lambda(\lambda))}$  is not even weakly  $\lambda^+$ -collectionwise-Hausdorff (a space  $X$  is  $\kappa$ -collectionwise-Hausdorff provided that every closed discrete subset of cardinality  $\kappa$  can be separated by disjoint open sets; it is *weakly*  $\kappa$ -collectionwise-Hausdorff provided that for every closed discrete subset  $A$  of cardinality  $\kappa$  there is a subset  $B \subseteq A$  of cardinality  $\kappa$  that can be separated by disjoint open sets). K. Kunen pointed out that it is enough to assume that there is a  $\lambda$ -Canadian-tree (a tree of height and size  $\lambda$ ), for a regular cardinal  $\lambda$ , to show that  $PR(\lambda(\lambda))$  is not weakly  $\lambda^+$ -collectionwise-Hausdorff. Thus, under the above assumptions,  $PR_{(cf \lambda)^+(\lambda(\lambda))}$  is  $\lambda$ -collectionwise-Hausdorff, but not  $\lambda^+$ -weakly-collectionwise-Hausdorff.

K. Kunen was also able to show without any extra set-theoretic assumption that  $PR(\omega_1(\omega_1))$  and  $PR_{(cf \lambda)^+(\lambda(\lambda))}$ , where  $cf \lambda = \omega$ , are not collectionwise-Hausdorff. His

proofs could be easily generalized to any successor cardinal and any singular cardinal. We then came up with the proof above that works for all regular cardinals.

We are left with the following question:

*Question 3.* Assume that  $\lambda$  is an infinite cardinal with  $\text{cf } \lambda = \omega$ . Is  $\text{PR}_\sigma(\lambda(\lambda))$  subparacompact for  $(\text{cf } \lambda)^+ \leq \sigma \leq \lambda^+$ ?

The simplest form of this question is whether  $\text{PR}(\omega(\omega))$ , the Pixley-Roy space over the collection of all nonempty subsets of  $\omega$  with the co-finite topology, is subparacompact. E. van Douwen has also looked at this space and a related space, which in [vD] he called  $A[\omega^\#]$  and  $\Omega$ , respectively. He and B. Scott independently raised the question whether  $\Omega$  is countably metacompact. The countable metacompactness of  $\Omega$  can be shown to be equivalent to the countable metacompactness of  $\text{PR}(\omega(\omega))$ , so if  $\text{PR}(\omega(\omega))$  is subparacompact we would also have a positive answer to their question.

It should also be pointed out that E. van Douwen essentially showed in [vD] that  $\text{PR}(\omega(\omega))$  is not normal by showing that it is separable but contains a closed discrete set of cardinality  $c = 2^\omega$  by identifying  $\omega$  with  ${}^{<\omega}2$  and using the branches of this tree as the closed discrete set.

### III. $\text{PR}(\kappa(\lambda)), \kappa > \lambda$

We now consider spaces of the form  $\text{PR}(\kappa(\lambda))$ , with  $\kappa > \lambda$ .

*Theorem 4.* If  $\kappa$  and  $\lambda$  are infinite cardinals,  $\kappa > \lambda$ , and  $\text{cf } \lambda > \omega$ , then  $\text{PR}_\sigma(\kappa(\lambda))$  is neither subparacompact nor

*collectionwise-Hausdorff for any cardinal  $\alpha \geq 3$ .*

*Proof.* Suppose  $\kappa$  and  $\lambda$  are infinite cardinals,  $\kappa > \lambda$ , and  $\text{cf } \lambda > \omega$ . Since  $\text{PR}(\kappa(\lambda))$  is a P-space, it again suffices to show that  $\text{PR}_3(\kappa(\lambda))$  is not collectionwise-Hausdorff

The collection  $\{\{\alpha\}: \alpha < \lambda^+\}$  is a discrete closed subset of  $\text{PR}_3(\kappa(\lambda))$ . Suppose that  $\{\mathcal{U}(\{\alpha\}, F_\alpha): \alpha < \lambda^+\}$  is a collection of pairwise disjoint open sets. For each pair of points  $\alpha, \beta$ , either  $\alpha \in F_\beta$  or  $\beta \in F_\alpha$ . Since  $|\bigcup_{\alpha < \lambda} F_\alpha| \leq \lambda$ , let  $\beta \in \lambda^+ - (\bigcup_{\alpha < \lambda} F_\alpha \cup \lambda)$ . Then for each  $\alpha < \lambda$ ,  $\alpha \in F_\beta$ , contradicting the fact that  $|F_\beta| < \lambda$ . Since  $\{\{\alpha\}: \alpha < \lambda^+\}$  cannot be separated,  $\text{PR}_3(\kappa(\lambda))$  is not collectionwise-Hausdorff.

*Remark.* A similar argument shows that  $\kappa(\lambda)$  is not weakly separated in the sense of Tkacenko [T]:  $X$  is *weakly separated* if for every point  $x \in X$  one can choose a neighborhood  $V_x$  so that if  $y \in V_x$  and  $x \in V_y$ , then  $x = y$ . Przymusiński has noted that  $\text{PR}_3(X)$  is (hereditarily) collectionwise-Hausdorff if, and only if,  $X$  is weakly separated (see also [P]).

We again note that these spaces are in fact not weakly  $\lambda^+$ -collectionwise-Hausdorff, and that they are  $\lambda$ -collectionwise-Hausdorff.

Theorem 4 leaves unanswered various questions about the case of  $\text{cf } \lambda = \omega$ . Partial results of this case are given in the next theorem.

*Theorem 5.* Suppose  $\lambda$  is an infinite cardinal, cf  $\lambda = \omega$ , and  $\kappa > \lambda$ . Then  $PR_{\sigma}(\kappa(\lambda))$  is not collectionwise-Hausdorff for any cardinal  $\sigma \geq 3$  and the following hold:

(a) if  $\kappa > \lambda^+$ , then  $PR_{\sigma}(\kappa(\lambda))$  is not subparacompact for any cardinal  $\sigma \geq 3$ .

(b)  $PR_{\sigma}(\lambda^+(\lambda))$  is not paracompact for any cardinal  $\sigma \geq 3$ .

(c)  $PR_{\sigma}(\lambda^+(\lambda))$  is subparacompact for any cardinal  $\sigma \leq \omega$ .

(d)  $PR(\lambda^+(\lambda))$  is not subparacompact.

*Proof.* Suppose  $\lambda$  is an infinite cardinal, cf  $\lambda = \omega$ , and  $\kappa > \lambda$ . The proof that  $PR_3(\kappa(\lambda))$  is not collectionwise-Hausdorff is the same as that in Theorem 4.

(a) Suppose  $\kappa > \lambda^+$ . It suffices to show that  $PR_3(\kappa(\lambda))$  is not subparacompact. Suppose that it is.  $\mathcal{U} = \{\mathcal{U}(\{\alpha\}, \emptyset) : \alpha < \kappa\}$  is an open cover of  $PR_3(\kappa(\lambda))$ . Suppose that  $\langle \mathcal{V}_n \rangle_{n \in \omega}$  is a sequence of open refinements of  $\mathcal{U}$  such that for each  $x \in PR_3(\kappa(\lambda))$  there is an  $n \in \omega$  such that  $st(x, \mathcal{V}_n)$  is contained in some element of  $\mathcal{U}$ .

For each  $\alpha < \beta < \kappa$ , let  $n_{\alpha\beta} \in \omega$  be such that  $st(\{\alpha, \beta\}, n_{\alpha\beta})$  is contained in some element of  $\mathcal{U}$ . For each  $\alpha < \kappa$  and  $m \in \omega$ , let  $F_{\alpha m} \subseteq \kappa - \{\alpha\}$  be a set of cardinality less than  $\lambda$  such that  $\mathcal{U}(\{\alpha\}, F_{\alpha m})$  is contained in some element of  $\mathcal{V}_m$ .

$|\bigcup_{\alpha < \lambda^+} (\{\alpha\} \cup \bigcup_{m \in \omega} F_{\alpha m})| \leq \lambda^+$ , so let  $\beta \in \kappa - \bigcup_{\alpha < \lambda^+} (\{\alpha\} \cup \bigcup_{m \in \omega} F_{\alpha m})$ . Let  $m \in \omega$  be such that  $\{\alpha < \lambda^+ : n_{\alpha\beta} = m\}$  has cardinality  $\lambda^+$ . If  $\alpha$  is in this set, then either  $st(\{\alpha, \beta\}, \mathcal{V}_m) \subseteq \mathcal{U}(\{\alpha\}, \emptyset)$  or  $st(\{\alpha, \beta\}, \mathcal{V}_m) \subseteq \mathcal{U}(\{\beta\}, \emptyset)$ .

Therefore, either  $\{\alpha, \beta\} \notin \mathcal{U}(\{\alpha\}, F_{\alpha m})$  or  $\{\alpha, \beta\} \notin \mathcal{U}(\{\beta\}, F_{\beta m})$ ,

i.e. either  $\beta \in F_{\alpha m}$  or  $\alpha \in F_{\beta m}$ . Since  $\beta \notin F_{\alpha m}$ , we must have  $\alpha \in F_{\beta m}$ . But since  $|F_{\beta m}| < \lambda$ , we have a contradiction. Thus  $PR_3(\kappa(\lambda))$  is not subparacompact.

(b) Since  $PR_3(\lambda^+(\lambda))$  is not collectionwise-Hausdorff, it is not paracompact. However, neither the methods of proof of Theorem 4, nor those of part (a) enable us to establish that this space is not subparacompact.

(c) It suffices to show that  $PR_\omega(\lambda^+(\lambda))$  is subparacompact. Let  $X = PR_\omega(\lambda^+(\lambda))$  and suppose  $\mathcal{U}$  is an open cover of  $X$ . For each  $\alpha < \lambda^+$ , since  $|\alpha| = \lambda$ , we may let  $\alpha = \bigcup_{n \in \omega} C_{\alpha n}$ , where  $C_{\alpha 0} \subseteq C_{\alpha 1} \subseteq \dots$ , and  $|C_{\alpha n}| < \lambda$  for each  $n \in \omega$ . For each  $\alpha \in \lambda^+$ , let  $F_{\{\alpha\}}$  be a subset of  $\lambda^+ - \{\alpha\}$  of cardinality less than  $\lambda$  such that  $\mathcal{U}(\{\alpha\}, F_{\{\alpha\}})$  is contained in some set in  $\mathcal{U}$ . Suppose  $F_x$  has been defined for each element of  $X$  of cardinality less than or equal to  $n$ . Suppose  $x \in X$ ,  $|x| = n + 1$ . Let  $\{\alpha_0, \dots, \alpha_{n-1}, \alpha_n\}$  be an increasing enumeration of  $x$ . Let  $F_x$  be a subset of  $\lambda^+ - x$  of cardinality less than  $\lambda$  such that  $\mathcal{U}(x, F_x)$  is contained in some set in  $\mathcal{U}$  and such that  $F_{\{\alpha_0, \dots, \alpha_j\}} - x \subseteq F_x$  for each integer  $j \leq n - 1$ . In this way we assign a set  $F_x$  for every  $x \in X$ ; these  $F_x$ 's have the property that if  $x = \{\alpha_0, \dots, \alpha_n\}$  is an increasing finite sequence of ordinals in  $\lambda^+$  and  $y = \{\alpha_0, \dots, \alpha_j\}$  for some  $j \leq n$  is such that  $x \cap F_y = \emptyset$ , then  $\mathcal{U}(x, F_x) \subseteq \mathcal{U}(y, F_y)$ .

For each  $n \in \omega$  and  $x \in X$ , let  $F_{x_n} = F_x \cup (\bigcup \{C_{\alpha n} : \alpha \in x\} - x)$ , and let  $\mathcal{V}_n = \{\mathcal{U}(x, F_{x_n}) : x \in X\}$ . Each  $\mathcal{V}_n$  is an open refinement of  $\mathcal{U}$ . Suppose  $x \in X$ . We show that there is an  $n \in \omega$  such that  $st(x, \mathcal{V}_n)$  is contained in some element



of  $\mathcal{U}$ . Let  $\{\alpha_j: j \leq m\}$  be an increasing enumeration of  $x$ . For each pair of integers  $i, j$  with  $i < j$ , let  $n_{ij} \in \omega$  be such that  $\alpha_i \in C_{\alpha_j, n_{ij}}$ . Since there are only finitely many such pairs, let  $n = \max\{n_{ij}: i < j \leq m\}$ . If  $x \in \mathcal{U}(y, \cup\{C_{\alpha_n}: \alpha \in y\} - y)$ , then  $y$  must be an initial segment of  $x$ . Let  $k$  be the least integer less than or equal to  $m$  such that  $x \cap F_{\{\alpha_j: j \leq k\}n} = \emptyset$ . It is easy to show that  $\text{st}(x, \mathcal{V}_n) \subseteq \mathcal{U}(\{\alpha_j: j \leq k\}, F_{\{\alpha_j: j \leq k\}n})$ . The  $C_{\alpha_n}$ 's ensure that if  $x \in \mathcal{U}(y, F_{y_n})$ , then  $y$  is an initial segment of  $x$ , and the  $F_y$ 's ensure that the open sets containing these initial segments (and  $x$ ) are nested.

(d) Suppose  $\text{PR}(\lambda^+(\lambda))$  is subparacompact. As in Theorem 2 we may construct a discrete closed subset  $D = \{d_\alpha: \alpha < (\lambda^+)^+\}$  of  $\text{PR}(\lambda^+(\lambda^+))$  that cannot be separated by disjoint open sets, and such that for each  $x \in \text{PR}(\lambda^+(\lambda^+)) - D$  there is a subset  $F_x$  of  $\lambda^+ - x$  of cardinality less than or equal to two such that  $\mathcal{U}(x, F_x)$  meets no point of  $D$ , and such that for each pair  $\alpha, \beta$  of  $(\lambda^+)^+$ ,  $d_\alpha \notin d_\beta$ . Since  $\mathcal{U} = \{\mathcal{U}(x, F_x): x \in \text{PR}(\lambda^+(\lambda)) - D\} \cup \{\mathcal{U}(d_\alpha, \emptyset): \alpha < (\lambda^+)^+\}$  is an open cover of  $\text{PR}(\lambda^+(\lambda))$ , let  $\langle \mathcal{V}_n \rangle_{n \in \omega}$  be a sequence of open refinements of  $\mathcal{U}$  such that for each  $x \in \text{PR}(\lambda^+(\lambda))$ , there is an  $n \in \omega$  such that  $\text{st}(x, \mathcal{V}_n)$  is contained in some set in  $\mathcal{U}$ . For each  $\alpha < (\lambda^+)^+$  and each  $n \in \omega$ , let  $F_{\alpha n} \subseteq \lambda^+ - d_\alpha$  be a set of cardinality less than  $\lambda$  such that  $\mathcal{U}(d_\alpha, F_{\alpha n})$  is contained in some set in  $\mathcal{V}_n$ .

For each  $\alpha < (\lambda^+)^+$ ,  $|\cup_{n \in \omega} F_{\alpha n}| \leq \lambda$ , and so  $\mathcal{U}(d_\alpha, \cup_{n \in \omega} F_{\alpha n})$  is an open set in  $\text{PR}(\lambda^+(\lambda^+))$ . Let  $\alpha$  and  $\beta$  be two elements

of  $(\lambda^+)^+$ , and let  $n \in \omega$  be such that  $\text{st}(d_\alpha \cup d_\beta, \mathcal{V}_n)$  is contained in some element of  $\mathcal{U}$ . Since  $d_\alpha$  and  $d_\beta$  are not both elements of any element in  $\mathcal{U}$ , either  $d_\alpha \cup d_\beta \notin \mathcal{U}(d_\alpha, F_{\alpha n})$  or  $d_\alpha \cup d_\beta \notin \mathcal{U}(d_\beta, F_{\beta n})$ . So either  $d_\beta \cap F_{\alpha n} \neq \emptyset$  or  $d_\alpha \cap F_{\beta n} \neq \emptyset$ . Thus, for each pair  $\alpha$  and  $\beta$ , either  $d_\alpha \cap (\bigcup_{n \in \omega} F_{\beta n}) \neq \emptyset$  or  $d_\beta \cap (\bigcup_{n \in \omega} F_{\alpha n}) \neq \emptyset$ . But then  $\{\mathcal{U}(d_\alpha, \bigcup_{n \in \omega} F_{\alpha n}) : \alpha < (\lambda^+)^+\}$  is a collection of pairwise disjoint open subsets of  $\text{PR}(\lambda^+(\lambda^+))$  separating  $\{d_\alpha : \alpha < (\lambda^+)^+\}$ , a contradiction. So  $\text{PR}(\lambda^+(\lambda))$  is not subparacompact.

These results still leave us with the following question:

*Question 6.* If  $\lambda$  is an infinite cardinal, cf  $\lambda = \omega$ , and  $\omega_1 \leq \sigma \leq \lambda^+$ , is the space  $\text{PR}_\sigma(\lambda^+(\lambda))$  subparacompact?

The simplest form of this question is whether the Pixley-Roy space over the collection of all nonempty countable subsets of  $\omega_1$  with the co-finite topology is subparacompact.

Note that an affirmative answer to Question 6 would give an affirmative answer to Question 3.

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Auburn University

Auburn, Alabama 36849