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THE APPLICATION OF ANCEL'S VERSION OF EFFROS'S THEOREM TO NON-HOMOGENEOUS SPACES

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In recent years a theorem now known as Effros's theorem has proven to be a very valuable tool to topologists studying homogeneity properties in continua. Still more recently F. D. Ancel proved a slightly different version of this theorem using only topological techniques. Now it must be admitted that Ancel's version of the theorem is almost a corollary to the original, but the statement and proof of the theorem are much simplified, and it is apparent from this version that the theorem can be applied to some extent to any compact metric space which admits a dense in itself, second category in itself orbit under the action of some complete separable metric topological group.

Laurence Fearnley [T] asked the following question: Is there a result analogous to the Effros theorem for almost homogeneous continua? (A continuum M is *almost homogeneous* = *nearly homogeneous* if for each p,g in M and neighborhood U of g, there is a space homeomorphism taking p into U.) Fearnley's question is easily answered by applying Ancel's version of the Effros theorem, as we shall see later. Also, it seems natural to wonder to what extent the usual theorems on homogeneous continua generalize, not just to nearly homogeneous continua, but also to those compact metric spaces which admit dense orbits, or even dense in themselves, G_δ

orbits, under the action of some complete separable metric topological group. That then will be the purpose of this paper. We will also see that this approach gives us some new information about homogeneous spaces themselves.

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I. Background and Notation

It is well known that if X is a compact metric space, then H(X), the group of homeomorphisms from X onto itself, is a complete separable metric topological group which acts on X [AR]. If G is a complete separable metric topological group which acts on the metric space X, the action of G on X is *transitive* if for each x,y in X, there is some g in G such that g(x) = y. The action of G on X is *micro-transitive* if for every x in X and every neighborhood U of the identity 1 in G, Ux (= {y ∈ X| there is some f in U such that f(x) = y}) is a neighborhood of x in X.

Ancel's Version of the Effros Theorem [AFD]. *Suppose that a separable complete metric topological group acts transitively on a metric space* X. *Then* G *acts micro-transitively on* X *if and only if* X *has a complete metric.* (The reader is referred to [E] for the statement of the original theorem.)

In this paper, all spaces will be separable and metric. A *continuum* will mean a compact, connected, metric space.

The metric on $H(X)$ (where X is a compact metric space) that we will use is the familiar "sup" metric, i.e., if d is a metric on X (compatible with its topology), and h and f are in $H(X)$, then $\rho_d(h,f) = \text{lub}\{d(h(x),f(x)) | x \in X\}$. When no confusion arises, ρ_d will just be ρ.

If H is a complete separable metric topological group which acts on the space X, and $x \in X$, let H_x denote the *stabilizer subgroup* of H, i.e., $H_x = \{h \in H | h(x) = x\}$. If $\varepsilon > 0$, $h \in H$, let $N_\varepsilon(h) = \{f \in H | \rho(h,f) < \varepsilon\}$.

If A is a collection of sets, then A^* will denote the union of the members of A. We will use N to denote the positive integers.

Various other theorems, definitions, etc., will be stated and referenced as the need arises. For a more complete discussion of the background for this paper, and a more complete list of references, the reader is referred to [P1].

One last note: Suppose that the separable metric topological group G acts on a space X. If $x \in X$ and we refer to the Effros property, with respect to Gx, what we mean is that if $\varepsilon > 0$ and y is in Gx, then there is $\delta > 0$ such that if $d(z,y) < \delta$ (where d is a metric on X) and $z \in Gx$, then there is some h in G such that $\rho(h,1) < \varepsilon$ and $h(y) = z$. We make *no claim* to being able to get for a given $\varepsilon > 0$ a single $\delta > 0$ such that if y and z are any 2 points in Gx such that $d(y,z) < \delta$, then there is h in G such that $\rho(h,1) < \varepsilon$ and $h(y) = z$. One could only hope for such a thing in the case where Gx turned out to be compact, and this is not generally the case.

II. More on the Effros Theorem and Some Basic Results

Remarks. Suppose G is a complete separable metric topological group which acts on the complete separable metric space X.

(1) If H is a closed subgroup of G and $x \in X$, then Hx is a Borel subset of X, although it may not be a G_δ subset of X. (A proof of the first part of this statement is contained in the proof of Lemma 2.4 in [E]. We include it here for completeness.)

Proof. Now $D = \{hH_x | h \in H\}$ is a partition of H into mutually exclusive closed sets, and there exists a Borel subset B of H which intersects each member of D in exactly one point. (This follows from a theorem of J. Dixmier [D, Lemma 3].) If $T_x: H \to X$ is defined by $T_x(h) = h(x)$, then $Bx = Hx = T_x(H) = T_x(B)$, and since $T_x|B$ is a one-to-one continuous map from a Borel subset of H, $T_x(B)$ must be Borel in X also [KK, Vol. 1, p. 487].

For an example for the second part of the statement, see [P1, the remark which precedes Theorem 7].

(2) If H is a subgroup of G, then H is complete if and only if H is closed in G. [If H is complete, H is G_δ in G, and H is a dense G_δ-subset of $\overline{H} \subseteq G$. Since $g \in \overline{H}$ implies $gH \subseteq \overline{H}$, $ghH = H$: otherwise $gH \cap H = \emptyset$ and gH, H are mutually exclusive dense G_δ-subsets of $\overline{H}$.]

(3) If $x \in X$ and Gx is an orbit of X under G which is second category in itself, then Gx is a G_δ-set in X. [This follows from a slight modification of Ancel's proof, and is often the easiest way to show that a given orbit is G_δ in the space, or complete.]

(4) If an orbit Gx is both F_σ and G_δ in X, then it is locally closed in X; that is, it is the intersection of an open set and a closed set in X. (This follows from a remark in [KK, vol. 1, p. 418] and the fact that if an orbit is locally closed at one of its points, it is locally closed at each of them.)

(5) There are examples of continua which admit only the identity as self-homeomorphisms. (In fact, H. Cook [C] has an example of a continuum M_2 with the property that if f is a continuous mapping of M_2 into itself, then f is either the identity or f is constant.)

We come then to Fearnley's question. Suppose X is a nearly homogeneous continuum. Then H(X) acts on X, and, in fact, if x ∈ X, H(X)x is dense in X. Now if x ∈ X, H(X) acts transitively on H(X)x, considered as space. Ancel's verion of Effros's theorem tells that H(X) will act micro-transitively on H(X)(x) if and only if H(X)x has a complete metric or equivalently, that H(X)x is a G_δ-set in X. Thus, the Effros result may be applied only to an orbit which is G_δ. As an example, consider the Sierpinski curve. This continuum, under the action of its homeomorphism space, admits exactly 2 orbits, each of which is dense in the space [KJ]. This plan continuum is locally connected and its complement in the plane consists of components whose boundaries are mutually exclusive simple closed curves. Using Krasinkiewicz's convention, we will call the union of the boundaries of the components of the complement the rational part of the Sierpinski curve, and the remainder of the curve the irrational part. The irrational part is a dense G_δ-set in the space, and the rational part is therefore *not* G_δ. Effros may be applied to the G_δ-orbit (in the sense that if x,y are "close" in the G_δ-orbit, then there is a "small" space homeomorphism that will take one onto the other). Effros may *not* be applied to the other orbit. (This is easy to see: Suppose x and y are on 2 different simple closed curves in the rational part. Then even if x and y are close to each other, the sizes of the 2 simple closed curves may be very different. Since any homeomorphism which would take x onto y would have to move one simple closed curve onto the other, the homeomorphism in such a case would have to move some point quite a bit.) But what if x and y are close together on the *same* simple closed curve in the rational part? The situation is different then, and suggests the following corollary to the Effros theorem.

Corollary 1. *Suppose* (X,d) *is a compact metric space and* G *is a complete separable metric topological group which acts on* X. *Suppose further that* $x \in X$ *and* $Gx = \cup_{i=1}^{\infty} A_i$ *where* $E = \{A_i | i \in N\}$ *is a collection of mutually exclusive continua. Then if* $\varepsilon > 0$ *and* $z \in H_x$, *there is* $\delta > 0$ *such that if* $\{y,z\} \subseteq A_\tau$ *for some* $\tau \in N$ *and* $\{y,z\} \subseteq D_\delta(z) = \{\omega \in X | d(z,\omega) < \delta\}$, *there is* h *in* G *such that* $h(z) = y$ *and* $\rho(h,1) < \varepsilon$.

Proof. Suppose $\tau \in N$. Let $B = \{h \in G | h(A_\tau) = A_\tau\}$. (If $h \in G$, either $h(A_\tau) = A_\tau$ or $h(A_\tau) \cap A_\tau = \emptyset$.)

B is a closed subgroup of G and $Bx = A_\tau$. Then B is a complete topological group which acts transitively on the

G_δ set A_τ. Ancel's version of the Effros theorem tells us that B then acts micro-transitively on A_τ, and we are done.

One might note that the facts that the set $\{A_\tau | i \in N\}$ is countable or that each member is a continuum were not really crucial in that proof: the fact that each element of G maps each element of E onto itself or onto some other element of E was the necessary part.

One might also wonder about the following:

Question. If X is a nearly homogeneous continuum, must X admit under the action of H(X) a dense G_δ-orbit?

The following is essentially proved in [AR]. We include it here for completeness, especially since many topologists do not seem to be aware of this fact.

Theorem [AR]. *Suppose* X *is a locally compact separable metric space. Then* H(X), *with the compact open topology, is a complete separable metric topological group which acts on* X.

Proof. Consider the one-point compactification $\hat{X}$ (= X ∪ {∞}) of X. Suppose d is a metric on $\hat{X}$ compatible with its topology.

Now $H(\hat{X})$ is a complete separable metric topological group (with the compact open topology, which is equivalent to the topology generated by the sup metric) which acts on $\hat{X}$.

Then $H(\hat{X})_\infty$ is a closed subgroup of $H(\hat{X})$ and $H(\hat{X})_\infty$ is topologically and algebraically equivalent to H(X). Thus H(X) is a complete, separable, metric topological group which acts on X.

Theorem 2. *If* X *is a locally compact, separable homogeneous metric space which contains a nondegenerate continuum, then* H(X) *is not locally compact.*

Proof. Suppose that H(X) is locally compact. Then by a theorem of Keesling [K], dim H(X) = 0. Since H(X) acts transitively on X, we may use Effros's theorem, and if $x \in X$, u is open in H(X), then u(x) is a neighborhood of x in X.

Now H(X) has a basis of open, compact sets. Suppose that u is open in H(X) such that (1) $1 \in u$; (2) u is compact; (3) $u(x) \neq X$; and (4) there is a continuum C such that $C \cap u(x) \neq \emptyset$ and $C \cap (X-u(x)) \neq \emptyset$.

There is some point q in $\overline{u(x)} - u(x)$. Then there is some sequence $q_1, q_2, \cdots$ in u(x) such that $q_1, q_2, \cdots$ converges to q. For each i, there is h_i in u such that $h_i(x) = q_i$. Some subsequence $h_{p_1}, h_{p_2}, \cdots$ of $h_1, h_2, \cdots$ converges to h in u. Then $h_{p_1}(x), \cdots$ converges to h(x) in u(x). But h(x) = q, and $q \in u(x)$. This is a contradiction.

Theorem 3. *Suppose* X *is a locally compact separable metric uncountable homogeneous space. If* $x \in X$, $H(X)_x$ *is uncountable and dense in itself.*

Proof. Suppose d is a metric on the one-point compactification $\hat{X}$ of X, and we take ρ, the sup metric on H(X), with respect to d. Suppose $A = H(X)_x$ is discrete. Then there is some $\varepsilon > 0$ such that if $h \in A-1$, $\rho(h,1) > \varepsilon$.

Consider $N_\alpha(1)$ where $\alpha > 0$, $\alpha < \varepsilon/4$, and $N_\alpha(1)(x) \neq X$, and $k \in N_\alpha(1)$ implies $k^{-1} \in N_{\varepsilon/4}(1)$. Now $N_\alpha(1)(x)$ is open in X. (This follows very easily from the micro-transitivity of the action of H on X.) Define $T: N_\alpha(1) \to N_\alpha(1)(x)$ by $T(f) = f(x)$ for $f \in N_\alpha(1)$. T is well-defined, continuous and open. T is also one-to-one: Suppose k and ℓ are in $N_\alpha(1)$ such that $k(x) = \ell(x)$. Then $\ell^{-1}k(x) = x$ and $\ell^{-1}k \in N_\varepsilon(1)$, so $\ell^{-1}k = 1$.

It follows that T is a homeomorphism from $N_\alpha(1)$ onto $N_\alpha(1)(x)$, and that H(X) is locally compact. Thus H(X) is zero-dimensional, and X does not contain any nondegenerate continua, so X is also zero-dimensional. X has a basis of open, compact sets. Suppose $\mathcal{O}$ is an open, compact subset of X, $\mathcal{O} \neq X$, $x \in \mathcal{O}$. There is a homeomorphism k in H(X) such that $\rho(k,1) < \alpha$; $k(z) \neq z$ for some z in $X - \mathcal{O}$; $k(t) = t$ for each t in $\mathcal{O}$. (Note that each point of X is a limit point of X.) This contradicts the discreteness of A. Then 1 is a limit point of A-1, and each point of A is a limit point of A. Since A is closed, it must be uncountable.

Corollary 4. *Suppose* X *is a locally compact, separable, homogeneous, uncountable metric space. Then* X *is not uniquely homogeneous.*

Theorem 3 and Corollary 4 extend results in [BR] and [P1]. Alos, recently, Jan van Mill [M] has shown there is an example of a separable metric connected and locally connected topological group which is uniquely homogeneous. This example is not complete, although it is second category in itself. He asks in that paper whether or not there exists a uniquely homogeneous topologically complete space.

Since the proofs to the following theorems are almost the same as those of Theorem 2 and 3, they are omitted.

Theorem 5. *Suppose* X *is a compact metric space and* D *is a* G_δ*-subset of* X *such that* (1) D = H(X)(x) *for some* x *in* X*, and* (2) D *contains a nondegenerate continuum. Then* H(X) *is not locally compact.*

Theorem 6. *Suppose* X *is a compact metric space and* D *is a subset of* X *such that*

(1) D *is locally compact;*

(2) D = H(X)(x) *for some* x *in* X*;*

(3) D *contains a nondegenerate continuum.*

Then $H(X)_x$ *is an uncountable closed subgroup of* H(X)*, and* $H(X)_x$ *is complete and dense in itself.*

Corollary 7. *If* X *is a locally compact, separable, homogeneous, uncountable, metric space, then* H(X) *is not abelian.*

Proof. Now $H(X)_x$ is uncountable and dense in itself. Choose f in H(X) such that f(x) = x, f $\neq$ 1. There is some y in D such that f(y) $\neq$ y. Let f(y) = z. There is some g in H(X) such that g(x) = y. Then gf(x) = g(x) = y and fg(x) = f(x) = z. Thus gf $\neq$ fg and H(X) is not abelian.

Corollary 8. *Suppose* X *and* D *are as in Theorem* 6. *Then* H(X) *is not abelian.*

III. Relations to Connectivity

First we need to recall briefly what the relationships of various homogeneity properties are to various connectivity properties. Suppose that n $\in$ N. The space X is n-*homogeneous* (*strongly* n-*homogeneous*) means that if $\{a_1,\cdots a_n\}$ = $\{b_1,\cdots b_n\}$ are n-point sets in X, then there is some homeomorphisms h in H(X) such that $h(\{a_1,\cdots,a_n\}) = \{b_1,\cdots,b_n\}$ $(h(a_i) = b_i$ for each $i \leq n)$. Suppose X is a separable space, and A and B are two countable dense subsets of X. We will say that A and B are equivalent in X $(A \simeq B)$ if there is some homeomorphism h in H(X) such that h(A) = B. If X admits only a countable collection of equivalence classes of countable dense subsets, then we will say that it is *weakly countable dense homogeneous*. If the separable space X admits exactly one equivalence class of countable dense subsets, then it is *countable dense homogeneous*.

Ralph Bennett [BRB] first studied countable dense homogeneity. Ben Fitzpatrick [FB], shortly after that, proved the following: A connected, locally compact, countable dense homogeneous metric space is locally connected. As a result of Professor Fitzpatrick having asked me about the situation in weakly countable dense homogeneous spaces, I obtained the following generalization [P2]: Suppose that X is a locally compact, connected metric space and X is weakly countable dense homogeneous. Then X is locally connected.

The notion of n-homogeneity seems to have been first studied in the 1950's. Gerald Ungar [U1,U2] with the help of Effros's theorem, was able to settle several old questions associated with this notion. We list those of his results important to us here:

(1) If a continuum X is 2-homogeneous, then it is locally connected.

(2) If a continuum X is not the circle and $n \in N$, then

it is n-homogeneous if and only if it is strongly n-homogeneous.

(3) If X is a continuum, then X is countable dense homogeneous if and only if it is n-homogeneous for all n.

What then can we do in the more general setting we have here?

Theorem 9. *Suppose* X *is a continuum,* O *is an open subset of* X*, and* D *is a dense* G_δ*-subset of* O *such that* (1) *if* {a,b} *and* {c,d} *are 2-point sets of* D*, then there is* h *in* H(X) *such that* h({a,b}) = {c,d}*, and* (2) D *contains a nondegenerate continuum. Then* X *is connected im kleinen at each point of* D.

Proof. Let H = H(X). If x and y are in D, there is some h in H such that h(x) = y: Suppose $z \in D$ and $z \neq x$, $z \neq y$. There is some h_1 in H such that $h_1(\{x,z\}) = \{y,z\}$. Either (1) $h_1(x) = y$ and $h_1(z) = z$, or (2) $h_1(x) = z$ and $h_1(z) = y$. If (1) occurs, let $h_1 = h$, and if (2) occurs, let $h_1 \circ h_1 = h$.

Now H acts on $F^2(X) = \{(x,y) \in X^2 | x \neq y\}$. (Define $\phi: H \times F^2(X) \to F^2(X)$ as follows: $\phi(h,(x,y)) = (h(x),h(y))$.) Since D is a dense G_δ-subset of O, $F^2(D)$ is a dense G_δ-subset of $F^2(O)$. Further, if $(x,y) \in F^2(D)$, $F^2(D) \subseteq H(x,y) \cup H(y,x)$. Note that the coordinate switching homeomorphism of $F^2(X)$ interchanges H(x,y) and H(y,x). Thus, H(x,y) is second category in itself, and we may apply the Effros theorem to conclude that H(x,y) is a G_δ-subset of $F^2(X)$, and H acts micro-transitively on H(x,y).

Either $H(x,y) = H(y,x)$ or $H(x,y) \cap H(y,x) = \emptyset$. If $H(x,y) \cap \overline{H(y,x)} \neq \emptyset$, then $H(x,y) \subseteq \overline{H(y,x)}$. So $H(y,x) \cap F^2(O)$ is a dense G_δ-subset of $F^2(O)$. Using the coordinate switching homeomorphism, it then follows that $H(x,y) \cap F^2(O)$ is also dense G_δ in $F^2(O)$. There cannot exist two mutually exclusive dense G_δ-sets in $F^2(O)$, so $H(x,y) \cap H(y,x) \neq \emptyset$. Thus, $H(x,y) = H(y,x)$, and it is the case that either $H(x,y) = H(y,x)$ or $H(x,y) \cap \overline{H(y,x)} = \emptyset$. It follows that $H(x,y) \cap F^2(D)$ is an open subset of $F^2(D)$.

Assume X is not connected im kleinen at some point of D. Then X is not connected im kleinen at any point of D. Now for some $x \in D$, there is an open subset O'_x of O such tha $x \in O'_x$ and if c is a component of $\overline{O}'_x$, c has no interior in X: otherwise $x \in D$ implies that if u is open in O such that $x \in u$, then some component of $\overline{u}$ has interior in X. Then one can obtain a sequence $N_1 \supset N_2 \supset \cdots$ of open subsets of X such that (1) N_{i+1} is contained in a component of $\overline{N}_i$ for $i \geq 1$; (2) diam $N_i \to 0$ as $i \to \infty$; and (3) $\cap_{i=1}^{\infty} N_i = \{z\}$ for some $z \in D$. (To get the third property here we use the topological completeness of D: this means that the diameters of the N_i's are "small" with respect to some complete metric on D.) But then X is connected im kleinen at z, and this can't be.

Hence, for every point x in D, there is an open subset O'_x of O such that $x \in O'_x$ and if c is a component of O'_x, c has no interior in X.

Suppose x is a point of a nondegenerate continuum in D. Then x lies in a nondegenerate continuum $K \subseteq O'_x \cap D$.

Let $y \in K - \{x\}$. Suppose d is a metric on X compatible with its topology and ε is a positive number such that $d(K, X - O_x') > \varepsilon$. Since H acts micro-transitively on H(x,y) and $H(x,y) \cap F^2(D)$ is an open subset of $F^2(D)$, there is a $\delta > 0$ such that if $(x',y') \in F^2(D)$ and $d(x,x') < \delta$ and $d(y,y') < \delta$, then there is an h in H such that $\rho(h,1) < \varepsilon$ and $h(x,y) = (x',y')$. (Here ρ denotes the sup metric on H with respect to the metric d on X.) Let c be the component of $\overline{O}_x'$ containing x. Then $K \subseteq c$ and $y \in c$. Since c has no interior in X, there is a point y' in $(O_x' \cap D) - c$ such that $d(y,y') < \delta$. It follows that there is an h in H such that $h(x,y) = (x,y')$ and $\rho(h,1) < \varepsilon$. Then $\{x,y'\} \subseteq h(K) \subseteq O_x'$. Therefore, $y' \in h(K) \subseteq c$, which is a contradiction.

Corollary 10. *Suppose* X *is a continuum and* D *is an open subset of* X *such that if* {a,b} *and* {c,d} *are* 2*-point sets in* D*, then there is some* h *in* H(X) *such that* h({a,b}) = {c,d}. *Then* X *is locally connected at each point of* D.

Proof. This follows from the fact (see [MRL]) that if a space is connected im kleinen at each point of some open set, then it is locally connected at each point of that open set.

One might wonder about the following question, since if the answer to it is no, a substantial improvement in Theorem 9 would be possible.

Question. Suppose X is a continuum and O is an open subset of X such that if C is a component of $\overline{O}$, C does not have interior. Let $\mathcal{C} = \{C | C$ is a component of $\overline{O}\}$. Can there possibly be a dense G_δ-subset D of O such that for each $C \in \mathcal{C}$, $C \cap D$ is either degenerate or empty?

Three simple examples of continua which admit such a set D to which the previous results might be applied include the following: (1) the Warsaw circle; (2) the Sierpinski curve; (3) a bouquet of n-spheres. (By a bouquet of n-spheres we mean the following: For each $n \in N$, let $A^n = \{x \in E^n | \|x\| = 2^{-n}\}$; choose $x_n \in A^n$, and then identify the x_n's to obtain a continuum which admits a countably infinite number of orbits. If B denotes this continuum and $\hat{x}$ denotes the "identified" point, $\{\hat{x}\}$ is an orbit and $C = \{A^n - \{\hat{x}\} | n \in N\}$ contains the other orbits. Note that each orbit in C is open in B, but not dense.)

To get analogous results for the countable dense homogeneity properties, we will need some rather technical results first.

Theorem 11. *Let* $\hat{D}$ *be a separable metric subspace of a topological space* Z *with the property that if* A *and* B *are countable dense subsets of* $\hat{D}$, *then* $h(A) = B$ *for some* $h \in H(Z)$. *Let* $H = H(Z)$. *Then* $\{Hx \cap \hat{D} | x \in \hat{D}\}$ *is a countable cover of* $\hat{D}$ *by disjoint relatively open subsets of* $\hat{D}$.

Proof. Suppose $x \in \hat{D}$. Then $Hx \cap \hat{D}$ is open in $\hat{D}$: For suppose $Hx \cap \hat{D}$ is not open in $\hat{D}$. Then $E = (Hx \cap \hat{D}) - \text{int}_{\hat{D}}(Hx \cap \hat{D}) \neq \emptyset$. Choose p from E. There is a sequence $q_1, q_2, \cdots$ in $\hat{D} - Hx$ which converges to p, and there is a countable dense subset B of $\hat{D}$ such that $B \cap E = \emptyset$.

Let $A = B \cup \{p\} \cup \{q_1, q_2, \cdots\}$. Then A is a countable dense subset of $\hat{D}$, and $h(A) = B$ for some $h \in H$. Since

$p \in Hx$, $h(p) \in B \cap Hx$. But then $h(p) \in \mathrm{int}_{\hat{D}}(Hx \cap \hat{D})$. Now $h(q_1), h(q_2), \cdots$ converges to $h(p)$, $h(q_i) \in \mathrm{int}_{\hat{D}}(Hx \cap \hat{D})$ for some i, $h(q_i) \in Hx$, and $q_i \in Hx$ for all i. This can't be.

The conclusion now follows easily.

Next we need some notation and lemmas of G. S. Ungar [U1]. Note that in the hypotheses of the lemmas, the space X is assumed to be complete rather than locally compact. Ungar's proofs actually give this stronger result. We state only the last lemma, which is the only one used directly.

Suppose X is a space. If $n \in N$, let $F^n(X) = \{(x_1, x_2, \cdots, x_n) \in X^n | x_i \neq x_j \text{ unless } i = j\}$. If X is a locally compact separable metric space, then H(X) acts on $F_n(X)$. (Define $\phi: H(X) \times F^n(X) \to F^n(X)$ by $\phi(h,x) = (h(x_1), \cdots h(x_n))$ where $x = (x_1, \cdots, x_n)$.) A subset A of a space X is second category in X at the point x in X if there is an open set u in X such that $x \in u$ and $u - A$ is first category in u.

Lemma C. (Ungar). *Let* X *be a complete separable metric space and let* A *be a first category subset of* $F^n(X)$. *Then there exists a dense countable subset* B *of* X *such that* $F^n(B) \cap A = \emptyset$.

Theorem 12. *Suppose* $n \in N$, X *is a compact metric space and* D *is a dense* G_δ*-subset of some open set* $\hat{O}$ *in* X *such that if* A *and* B *are countable dense subsets of* D, *then there is some* h *in* H(X) *such that* h(A) = B. *Then there is a countable collection* $\mathcal{C}$ *of* $\{Hx | x \in F^n(X)\}$ *(where* $Hx = H(X)(x)$) *such that* (1) $F^n(D) \subseteq \cup_{Hx \in \mathcal{C}} Hx$ *and* (2) *each* $Hx \in \mathcal{C}$ *is open in* $F^n(D)$.

Proof. (1) Suppose $x \in F^n(D)$. Then $Hx \cap F^n(D)$ is a second category subset of $F^n(D)$: If not, there is a countable dense subset B of D such that $F^n(B) \cap Hx = \emptyset$. But then x is in $Hx \cap F^n(D)$, and $B \cup \{x_1, \cdots x_n\}$, where $x = (x_1, \cdots x_n)$, is a countable dense subset of D, and there is h_0 in H such that $h_o(B) = B \cup \{x_1, \cdots x_n\}$. This is a contradiction. Thus, Hx is second category in $F^n(D)$.

(2) Now D is dense G_δ in $\hat{O}$, so $F^n(D)$ is dense G_δ in $F^n(\hat{O})$. Since $Hx \cap F^n(D)$ is second category in $F^n(D)$, it is second category in $F^n(X)$, and Effros's theorem implies that Hx is a G_δ-subset of $F^n(X)$.

(3) There is a maximal open subset u_x of $F^n(X)$ such that Hx is a dense G_δ-subset of u_x: Let $\mathcal{U}$ be a countable basis for $F^n(X)$. Let $W = \{u \in \mathcal{U} | Hx \cap u$ is a first category subset of $F^n(X)\}^*$. Then $Hx \cap \overline{W}$ is a first category subset of $F^n(X)$. Since Hx is a second category subset of $F^n(X)$, $V = F^n(X) - \overline{W} \neq \emptyset$, and $Hx \cap V$ is a dense subset of V. Then $Hx \cap V$ is a dense G_δ-subset of V. Let $u_x = V$. Suppose that $z \not\in V$. If there is some open set u' such that $z \in u'$ and $u' \cap Hx$ is dense in u', then $Hx \cap u'$ is dense G_δ in u' and $u' \cap W = \emptyset$. Thus, u_x is maximal. Note that $u_x = \{hu_x | h \in H\}^*$.

(4) If $x,y \in F^n(D)$ and $Hx \cap Hy = \emptyset$, then $u_x \cap u_y = \emptyset$: Otherwise, both $Hx \cap u_x \cap u_y$ and $H_y \cap u_x \cap u_y$ are dense G_δ-subsets of $u_x \cap u_y$. Then $Hx \cap Hy \neq \emptyset$.

(5) Since the sets in the collection $\{Hx \cap F^n(D) | x \in F^n(D)\}$ partition $F^n(D)$ into disjoint sets, it must be the case that $Hx \cap F^n(D) = u_x \cap F^n(D)$ for $x \in F^n(D)$. Thus

$\{Hx \cap F^n(D) \mid x \in F^n(D)\}$ is a cover of $F^n(D)$ by disjoint relatively open sets, and the cover must be countable since $F^n(D)$ is separable.

Corollary 13. *If* X *is a compact metric space, and* D *is a dense* G_δ*-subset of an open subset* $\hat{O}$ *of* X *which has the property that if* A *and* B *are two countable dense subsets of* D*, then there is a homeomorphism* h *in* H(X) *such that* h(A) = B*, then if* $\{x_1, \cdots x_n\}$ *is an* n*-element subset of* D*, there is an open set* O *with respect to* D *such that* $\{x_1, x_2, \cdots x_n\} \subseteq O$ *and if* $\{y_1, \cdots y_n\}$ *is an* n*-element subset of* $O \cap D$*, then there is some* h *in* H(X) *such that* $h(x_1, x_2, \cdots x_n) = (y_1, \cdots y_n)$.

Proof. This result follows immediately from the previous theorem.

Corollary 14. *Assume the hypothesis of Corollary* 13. *In addition, suppose* $F^n(D)$ *is connected. Then if* $\{x_1, \cdots x_n\}$ *and* $\{y_1, \cdots y_n\}$ *are any* n*-element subsets of* D*, there is some* h *in* H(X) *such that* $h(x_1, \cdots x_n) = (y_1, \cdots y_n)$.

Corollary 15. *Suppose* X *is a continuum and* D *is a dense* G_δ*-subset of some open subset* $\hat{O}$ *of* X *such that* (1) D *has the property that if* A *and* B *are two countable dense subsets of* D*, then there is a homeomorphism* h *in* H(X) *such that* h(A) = B*; and* (2) D *contains a nondegenerate continuum. Then* X *is connected im kleinen at each point of* D*. If* D *is open in* X*,* X *is locally connected at each point of* D.

Proof. To see this, first apply Corollary 13 with n = 2, and then apply Theorem 9.

One thing needs to be pointed out here: In this more general setting (X need not be homogeneous) the set D having the countable-dense-homogeneity property is *not* equivalent to D having the n-homogeneity property for all n, as the bouquet of n-spheres example demonstrates. If $F^n(D)$ is connected for all n, then one does get that the countable-dense-homogeneity property for D implies the n-homogeneity property for all n for D. Probably with additional assumptions on D (having $D = \cup_{Hx \cap D \neq \emptyset} Hx$, for example) some sort of converse is true here.

Also, Theorem 12 and Corollary 15 are generalizations of theorems in [P2].

IV. Implications for Homogeneity

In [P2] the following theorem was obtained: If X is a homogeneous continuum and X admits only a countable collection of equivalence classes of countable dense subsets, then X is countable dense homogeneous.

We obtain here some analogous results.

Theorem 16. *Suppose* X *is a compact separable metric space and* D *is a dense* G_δ*-subset of* X *with the property that if* {a,b} *and* {c,d} *are 2-point sets of* D*, then there is some* h *in* H(X) *such that* h({a,b}) = {c,d}. *If, in addition,* X *is homogeneous,* X *is 2-homogeneous.*

Proof. If X is a space ans $x \in X$, define $T_x: H(X) \to X$ by $T_x(h) = h(x)$ for $h \in H(X)$.

Lemma. Let X *be a homogeneous compact metric space. Let* D *be a dense* G_δ *subset of* X. *Then for each* $x \in X$,

$T_x^{-1}(D)$ *is a dense* G_δ*-subset of* $H(X)$.

Proof of Lemma. Let $x \in X$. Then $T_x^{-1}(D)$ is a G_δ-subset of $H(X)$ since T_x is continuous, and $T_x^{-1}(D)$ is dense in $H(X)$ because T_x is an open map by Effros's theorem.

Rest of the proof of the theorem. Suppose $\{a,b\}$ and $\{c,d\}$ are 2-point sets of X. Since $T_x^{-1}(D)$ is a dense G_δ-subset of $H(X)$ for each x in X, and since $H(X)$ has a complete metric, the Baire Category Theorem gives that $\cap\{T_x^{-1} \cap D \mid x \in \{a,b,c,d\}\} \neq \emptyset$. Suppose $g \in \cap\{T_x^{-1} \cap D \mid x \in \{a,b,c,d\}\}$. Then $g(\{a,b\})$ and $g(\{c,d\})$ are 2-point sets of D. Thus $h(g(\{a,b\})) = g(\{c,d\})$ for some h in $H(X)$, and $g^{-1} \circ h \circ g(\{a,b\}) = \{c,d\}$.

Remark. Wayne Lewis has pointed out that a consequence of Theorem 16 is the following: Suppose X is the pseudo-arc. Then it can not be the case that if D is a dense G_δ-subset of X and $\mathcal{C}$ is the collection of all composants of X, then $C \in \mathcal{C}$ implies $C \cap D$ is either degenerate or empty. [This is because if a and b are points on 2 different composants of X, and c and d are points of X on 2 different composants of X, then there is some h in $H(X)$ such that $h(a) = c$ and $h(b) = d$. [Le]. But the pseudo-arc is not 2-homogeneous.]

Theorem 17. *If* X *is a continuum and* D *is a dense* G_δ*-subset of* X *such that if* A *and* B *are any two countable dense subsets of* D, *then there is some* h *in* $H(X)$ *such that* $h(A) = B$, *and* X *is homogeneous, then* X *is countable dense homogeneous.*

Proof. Let A and B be countable dense subsets of X.

Since $T_x^{-1}(D)$ is a dense G_δ-subset of H(X) for each $x \in X$ (lemma in the previous proof), and since H(X) has a complete metric, then the Baire Category Theorem implies that $\cap\{T_x^{-1}(D) \mid x \in A \cup B\} \neq \emptyset$. Then g(A) and g(B) are countable dense subsets of D. Thus $h[g(A)] = g(B)$ for some h in H(X), and $B = g^{-1} \circ h \circ g(A)$.

Theorem 10 of [P2] is a corollary of Theorem 16 here and Theorem 1 of [P2].

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