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A NOTE ON NORMALITY AND COLLECTIONWISE NORMALITY

Franklin D. Tall[1]

In this note we make several unrelated observations concerning circumstances under which normality implies collectionwise normality. All spaces are assumed Hausdorff.

I. Screenable Spaces

In [T_1] I proved that screenable normal spaces are collectionwise Hausdorff. I can now improve this.

Theorem 1. *Screenable normal spaces are collectionwise normal with respect to countably metacompact closed sets.*

Proof. Let $\mathcal{M}$ be a discrete collection of countably metacompact closed sets. Let $\mathcal{S} = \cup_{n<\omega}\mathcal{S}_n$ be a σ-disjoint refinement of the canonical cover. The closed subspace $\cup\mathcal{M}$ is normal and countably metacompact. Since point-finite open covers of normal spaces can be shrunk, there are open $\{T_n\}_{n<\omega}$, $\overline{T}_n \subseteq \cup\mathcal{S}_n \cap \cup\mathcal{M}$. $\cup\mathcal{M}$ is closed so the closure sign is unambiguous. $\{M \cap \overline{T}_n: M \in \mathcal{M}\}$ for each n is a discrete collection separated by the open sets $S_{n,M} = \cup\{S \in \mathcal{S}_n: S \cap M \neq \emptyset\}$. By normality the $S_{n,M}$'s may be shrunk to a discrete separation $\{S'_{n,M}: M \in \mathcal{M}\}$. Then $\cup\{S'_{n,M}: M \in \mathcal{M}\}$ is σ-discrete and so yields a separation by standard arguments.

Theorem 1 cannot be improved in ZFC to get collectionwise normality; assuming $\diamondsuit^{++}$, M. E. Rudin [R_1] constructed

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a screenable normal space which is not countably metacompact. Assuming there is such a space, in [R_2] she then produces a screenable normal non-collectionwise normal space.

II. Getting By With Less

In this section we give several examples of how results that are obvious if Nyikos' "Product Measure Extension Axiom" [N_1] is assumed, can be obtained from weaker set-theoretic assumptions with a bit more topology. The crucial ingredient is the following theorem, which is due to the author and W. Weiss [TW]. Also see W. Fleissner [F_6]. Somewhat weaker results were earlier obtained by Carlson [C]. In the first version of this note I used his work and the measure-extension techniques of Nyikos to obtain collectionwise normality results. It has since become clear that these methods are unnecessary and that stronger results may be obtained directly. Some of the measure extension proofs still appear in [T_4].

Theorem 2. *Adjoin* ρ *Cohen (random) reals to a model of set theory. If* X *is normal and* $\mathcal{Y}$ *is a discrete collection such that* $|\cup\mathcal{Y}|$ < *cofinality of* ρ *and each point of* $\cup\mathcal{Y}$ *has character less than* ρ, *then* X *is collectionwise normal with respect to* $\mathcal{Y}$. *If* ρ *is weakly compact, it suffices to have* $|\cup\mathcal{Y}| \leq \rho$.

Using this theorem, we obtain the following result:

Theorem 3. *Adjoin* ρ *many Cohen (random) reals to a model of* V = L, ρ *regular. Then normal spaces of character less than* ρ *are collectionwise normal with respect to* *discrete collections of sets of cardinality less than* ρ.

Proof. By theorem 2 we can take care of collections with unions of size less than ρ, so in particular, X is $<\rho$-collectionwise Hausdorff. If we collapse a set of cardinality less than ρ in a space of character less than ρ to a point, that resulting point in the quotient space has character $2^{<\rho}$, which in this model is just ρ. In L[A], $A \subseteq \rho$, GCH holds at ρ and above, while $\diamondsuit$ for stationary systems holds for regular cardinals $\geq\rho$. By [F_1], normal spaces of character $\leq\rho$ which are $<\rho$-collectionwise Hausdorff are then collectionwise Hausdorff. The Theorem follows.

In [F_2] Fleissner proved that in the model obtained by Lévy-collapsing an inaccessible cardinal over a model of V = L, the character of copies of ω_1 in first countable spaces was $\aleph_1$, and (hence) that in normal first countable spaces, discrete collections of copies of ω_1 could be separated. In [T_2] I observed that the latter result could be obtained without an inaccessible by adjoining $\aleph_3$ Cohen subsets of ω_1 with countable conditions, collapsing $\aleph_3$ to $\aleph_2$ by conditions of size $\leq\aleph_1$, and then adding κ^+ Cohen subsets of κ for all regular $\kappa \geq \aleph_2$ by reverse Easton forcing. It follows from the previous Theorem that adjoining $\aleph_2$ Cohen (random) reals to a model of V = L will also do the trick.

The following topological lemma will enable us to extract more results from Theorems 2 and 3. This lemma is essentially proved in [B], but we give the proof for the reader's convenience.

Lemma 4. *If* X *is hereditarily collectionwise Hausdorff and each point of* X *has a neighbourhood of weight* $<\kappa$, *then* X *is the topological sum of spaces of weight* $\leq\kappa$.

Proof. Let $\mathcal{U}_0$ be a maximal disjoint collection of open sets of weight $<\kappa$. Suppose $\mathcal{U}_\beta$, $\beta < \alpha$, $\alpha < \kappa$ have been defined to be unions of $\leq\kappa$ many disjoint collections of open sets of weight $<\kappa$. Let $F_\alpha = X - \cup_{\beta<\alpha}\cup\mathcal{U}_\beta$. Let $\mathcal{V}_\alpha$ be a collection of open subsets of X of weight $<\kappa$ such that $\{F_\alpha \cap V: V \in \mathcal{V}_\alpha\}$ is a maximal disjoint collection of relatively open subsets of F_α. Fix a dense set D_V of power $<\kappa$ in each $F_\alpha \cap V$. Any selection of points, one from each D_V, yields a set which is closed and discrete in $\cup\mathcal{V}$, which may then be separated by disjoint open sets of weight $<\kappa$. We may therefore cover $\cup\{D_V: V \in \mathcal{V}_\alpha\}$ by a collection $\mathcal{U}_\alpha$ of open sets of weight $<\kappa$, such that $\mathcal{U}_\alpha$ is the union of $\leq\kappa$ collections of disjoint open sets. Claim $X = \cup_{\alpha<\kappa}\cup\mathcal{U}_\alpha$. Suppose $x \notin \cup_{\alpha<\kappa}\cup\mathcal{U}_\alpha$. Let W be a neighbourhood of x of weight $<\kappa$. Then there is an $\alpha < \kappa$ such that $F_\alpha \cap W = F_{\alpha+1} \cap W$, since the F_α's are descending and W has hereditary Lindelöf number $<\kappa$. Then $\cup\mathcal{U}_{\alpha+1} \cap W$ is empty so $F_{\alpha+1} \cap W$ is empty, because $\cup\mathcal{U}_{\alpha+1} \cap F_{\alpha+1}$ is dense in $F_{\alpha+1}$. But that's a contradiction since $x \in \cap_{\alpha<\kappa}F_\alpha$.

$\cup_{\alpha<\kappa}\mathcal{U}_\alpha$ is the union of $\leq\kappa$ many collections of disjoint open sets. Each member of $\cup_{\alpha<\kappa}\mathcal{U}_\alpha$ intersects fewer than κ many elements of each such collection. It follows by standard arguments that X is the sum of subspaces, each of which is the union of $\leq\kappa$ open sets of weight $<\kappa$. Each such subspace then has weight $\leq\kappa$.

The fruit of these results is

Theorem 5. *Adjoin* κ^{++} *Cohen (random) reals to a model of* V = L. *Then hereditarily normal spaces of local weight* $\leq\kappa$ *are hereditarily collectionwise normal.*

Proof. Since character is $\leq$ local weight, by Theorem 3 such a space X is hereditarily collectionwise Hausdorff. By Theorem 4 it decomposes into subspaces of weight $\leq\kappa^+$ and then X is collectionwise normal by Theorem 3 or 2.

Theorem 6. *Adjoin weakly compact many Cohen (random) reals to a model of* V = L. *Then hereditarily normal spaces of local weight* $<2^{\aleph_0}$ *are hereditarily collectionwise normal.*

Of course by making the significantly stronger assumption of the consistency of the existence of a strongly compact cardinal, there is a model where "hereditarily" may be omitted in both places and local weight replaced by character. That is the original Kunen-Nyikos result for random reals [K] (or see [F_5]), [N_1]. L does not enter the picture in that case. For the non-logicians, we note that the consistency of the existence of a weakly compact cardinal implies that such a cardinal consistently exists in L (see e.g. [D]), and hence that that many reals may be adjoined to L.

To prove the Theorem, as before the space decomposes into the sum of clopen pieces, each of which is the union of $\leq 2^{\aleph_0}$ sets, each of which has weight $<2^{\aleph_0}$ and hence cardinality $\leq 2^{\aleph_0}$, since in this model $2^{<2^{\aleph_0}} = 2^{\aleph_0}$. Hence each piece has cardinality $\leq 2^{\aleph_0}$, so by Theorem 2 we are done.

Theorem 7. *Adjoin weakly compact many Cohen (random reals) to a model of set theory. Then normal locally connected spaces of character* $<2^{\aleph_0}$ *are collectionwise normal if they locally have cardinality* $\leq 2^{\aleph_0}$.

Note that local cellularity or local Lindelöf number $<2^{\aleph_0}$ yields local cardinality $\leq 2^{\aleph_0}$ since $2^{<2^{\aleph_0}} = 2^{\aleph_0}$ here. (For the case of manifolds, this result is due to Nyikos [N_2]. To prove the Theorem it suffices to look at each component. But by an argument of Reed and Zenor [RZ], each component has cardinality $2^{\aleph_0}$.

I conjecture that normal manifolds are consistently collectionwise normal without large cardinals. I can get this result for normal manifolds which e.g. have cellularity $\leq \aleph_1$ or in which each closed set is the intersection of $\leq \aleph_1$ open sets, by a closer analysis of the proof of Lemma 4.

In [T_3] I proved that $\aleph_2$ random reals adjoined to L ensure that every locally compact perfectly normal space is collectionwise normal.

Restricting in a different direction, we have

Theorem 8. *Adjoin* $\aleph_2$ *Cohen (random) reals to a model of* V = L. *Then normal manifolds which are* $\delta\theta$*-refinable are collectionwise normal.*

Recall $\delta\theta$-refinability is a simultaneous generalization of θ-refinability and metalindelöfness, namely every open cover has an open refinement which is the union of countably many covers, such that for each point there is an n such that it's in only countably many members of the n'th cover. θ-refinable normal manifolds are collectionwise normal by standard arguments: since they're locally developable, they're developable [WW] and hence perfectly normal; since they're locally compact, locally connected, and perfectly normal, they're collectionwise normal with respect to compact sets [AZ]; since they're θ-refinable, locally compact, and collectionwise normal with respect to compact sets, they're paracompact. In fact, by a more difficult argument, θ-refinable normal, locally compact, locally connected spaces are paracompact [G]. Metalindelöf manifolds are collectionwise normal by an even easier argument: they're locally separable and metalindelöf, so paracompact.

We prove the Theorem by a blend of [RZ] and [AP]. It suffices to show that $\delta\theta$-refinable normal locally second countable connected spaces have weight $\leq \aleph_1$. We can do this if we can construct second countable open sets $\{U_\alpha\}_{\alpha<\omega_1}$ such that $\overline{U}_\alpha \subset U_{\alpha+1}$. For then $\cup_{\alpha<\omega_1} U_\alpha = \cup_{\alpha<\omega_1} \overline{U}_\alpha$, which by first countability is closed, so $\cup_{\alpha<\omega_1} U_\alpha = X$. But then the union of the bases for the U_α's is a basis for the whole space. It suffices to prove that the closure of a second countable open set is Lindelöf for then, given U_α, we cover $\overline{U}_\alpha$ by second countable open sets $\{V_{\alpha n}\}_{n<\omega}$, and let $U_{\alpha+1} = \cup_{n<\omega} V_{\alpha n}$. Suppose then that U is open and second countable, but $\overline{U}$ is not Lindelöf. $\overline{U}$ is $\delta\theta$-refinable, so it has an uncountable closed discrete subspace [Au]. But $\overline{U}$ is separable, normal, and by Theorem 2, $\aleph_1$-collectionwise Hausdorff, so it cannot have such a subspace.

III. Observations on a Theorem of Shelah

Fleissner [F_3], [F_4] and Shelah [S] have investigated the question of under what circumstances does $\aleph_1$-collectionwise Hausdorff imply collectionwise Hausdorff. For example, Shelah proves that in the model obtained by Lévy-collapsing a supercompact cardinal to ω_2, an $\aleph_1$-collectionwise Hausdorff space is collectionwise Hausdorff if it is locally countable (i.e. each point has a neighbourhood of cardinality $\leq \aleph_0$). Similar questions can be asked for collectionwise normality. We can prove

Theorem 9. V = L *implies that if* X *is hereditarily normal, (hereditarily) collectionwise normal with respect to discrete collections of* $\leq \aleph_1$ *sets which are each of cardinality* $\leq \aleph_1$*, locally hereditarily Lindelöf, and locally hereditarily separable, then it is (hereditarily) collectionwise normal.*

Proof. Since separable regular spaces have weight $\leq 2^{\aleph_0}$ (see e.g. [J]), X has character $\leq \aleph_1$, so it is hereditarily collectionwise Hausdorff. By the proof of Lemma 4, without loss of generality X may be assumed to be the union of $\leq \aleph_1$ hereditarily Lindelöf subspaces. Hereditarily Lindelöf regular spaces have cardinality $\leq 2^{\aleph_0}$ (see e.g. [Ju]), so by CH and hypothesis, X is collectionwise normal. All properties are hereditary, so we also have the hereditary version.

More in the spirit of Shelah's results we have

Theorem 10. *In the model obtained by Lévy-collapsing a supercompact to* ω_2, *if* X *is a locally countable space which hereditarily is collectionwise normal with respect to discrete collections of* $\leq \aleph_1$ *sets, each of cardinality* $\leq \aleph_1$, *then* X *is hereditarily collectionwise normal.*

Proof. By hypothesis X is hereditarily $\aleph_1$-collectionwise Hausdorff. By Shelah X is hereditarily collectionwise Hausdorff. By the proof of Lemma 4, X decomposes into the sum of subspaces of cardinality $\leq\aleph_1$, so by hypothesis it is hereditarily collectionwise normal.

Shelah's method works for spaces satisfying somewhat less stringent conditions than local countability; thus Theorem 10 can be improved. However, since the details of his argument do not appear in [S], it would take us too far afield to develop them here.

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