TOPOLOGY PROCEEDINGS
Volume 7, 1982
Pages 381–390
http://topology.auburn.edu/tp/

PROBLEM SECTION

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

PROBLEM SECTION

CONTRIBUTED PROBLEMS

The contributor's name is in parentheses immediately following the statement of the problem. In most cases, there is an article by the contributor in this volume containing material realted to the problem.

A. Cardinal Invariants

15. (*van Douwen*) Is there for each $\kappa \geq \omega$ a (preferably homogeneous, or even groupable) hereditarily paracompact (or hereditarily normal) space X with $w(X) = \kappa$ and $|X| = 2^{\kappa}$? [Yes to all questions if $2^{\kappa} = \kappa^{+}$. Also, $w(X) \leq \kappa < |X|$ is always possible.]

16. (*van Douwen*) Is there for each $\kappa \geq \omega$ a homogeneous compact Hausdorff space X with $\chi(X) = \kappa$ and $w(X) = 2^{\kappa}$? Or is ω the only value of κ for which this is true?

17. (*van Douwen*) Is there a regular space without a Noetherian base? (Noetherian: no infinite ascending chains) [Yes if there is a strongly inaccessible cardinal: the first such cardinal is the first ordinal without a Noetherian base.]

See also problems under the next three headings, and H8, H10, P19, R6, and S12.

B. Generalized Metric Spaces and Metrization

22. (*Nyikos*) Is every weakly θ-refinable space with a BCO quasi-developable (equivalently, by an old theorem of Bennett and Berney, hereditarily weakly θ-refinable)?

23. (*Nyikos*) Is every collectionwise normal, countably paracompact space with a σ-locally countable base metrizable (equivalently, by an old theorem of Fedorchuk, paracompact)?

See also A15, D30, D31, and K6.

C. Compactness and Generalizations

40. (*Dow*) Is there a first countable, H-closed space of cardinality $\aleph_1$? Equivalently: is there a compact Hausdorff space that can be partitioned into $\aleph_1$ nonempty zero-sets? [Yes if CH].

41. (*van Douwen*) Is there a regular (noncompact, countably compact) space which is homeomorphic to each of its closed noncompact subsets, and is not orderable? (The orderable such spaces are regular cardinals.)

See also A16, B24, D33, H10, and R4.

D. Paracompactness and Generalizations

30. (*Nyikos*) Is there a first countable space (or even a space of countable pseudocharacter) that is weakly θ-refinable and countably metacompact, but not subparacompact?

31. (*Nyikos*) Is there a quasi-developable, countably metacompact space which is not subparacompact?

32. (*Nyikos*) Is every collectionwise normal, countably paracompact, quasi-developable space paracompact?

33. (*Nyikos*) Does MA imply every locally compact Hausdorff space of weight $<c$ is either subparacompact or contains a countably compact noncompact subspace? [If one substitutes "cardinality" for "weight" the answer is affirmative (Balogh).]

34. (*van Douwen*) Is there a non-paracompact, collectionwise normal space that is "not trivially so"? Such a space would be realcompact and countably paracompact, and each closed subspace F would be irreducible (i.e. every open cover has an open refinement with no proper subcover) or at least satisfy $\hat{L}(F) = \hat{e}(F)$ where

$\hat{L}(F) = \min\{\kappa$: each open cover of F has a subcover of cardinality $\kappa\}$

$\hat{e}(F) = \min\{\kappa$: no closed discrete subspace of F has cardinality $\kappa\}$

It would be even better if the space is a D-space, i.e. for every neighbornet there is a closed discrete subspace D such that the restriction of the neighbornet to D covers the space.

See also A15, B22, B23.

E. Separation and Disconnectedness

10. (*van Douwen*) Characterize internally the class $T_3 \vdash T_4$ of regular spaces X such that every regular continuous image of X is normal. (Note that ω_1 and $\omega \times (\omega_1+1)$ are examples but their direct sum is not.)

See also H10, P16, P18, and P19.

F. Continua Theory

7. (*Rhee*) Does admissibility of a metric continuum imply property c?

See also J2, J3, J4, P16, and the article by Brechner in this Problem Section.

G. Mappings of Continua and Euclidean Spaces

13. (*Rhee*) For each fiber map $\alpha: X \to C(X)$, does there exist a continuous fiber map $\beta: X \to C(X)$ such that $\beta(x) \subset \alpha(x)$ for each $x \in X$?

14. (*Gibson*) Give necessary and sufficient conditions for the extension $I^2 \to I$ of a connectivity function $I \to I$ to be a connectivity function. In particular, is it necessary for the function $I \to I$ to have the CIVP?

15. (*Keesling*) Can the maps in Theorem 5.2 of the paper by Keesling and Wilson (this volume) be made monotone or cell-like?

See also M5 and P16.

H. Mappings of More General Spaces

8. (*Ball and Yokura*) Let X be the one-point compactification of a discrete space of cardinality κ. If $\kappa < \aleph_\omega$, there is a subset F of C(X) with $|F| \leq \kappa$ such that every element of C(X) is the composition of an element of F followed by a map of $\mathbb{R}$ into $\mathbb{R}$. Can the restriction $\kappa < \aleph_\omega$ be dropped?

9. (*van Douwen*) Characterize the spaces X such that the projection map $\pi_1: X^2 \to X$ preserves Borel sets.

10. (*van Douwen*) One can show that a compact zero-dimensional space X is the continuous image of a compact orderable space if X has a clopen family which is T_0-point-separating (i.e. if $x \neq y$ then there is $S \in \mathcal{S}$ such that $|S \cap \{x,y\}| = 1$) and of rank 1 (i.e. two members are either disjoint or comparable). Is the converse false?

See also P17 and R5.

J. Group Actions

2. (*Lewis*) Does every zero-dimensional compact group act effectively on the pseudo-arc?

3. (*Lewis*) If a compact group acts effectively on a chainable (tree-like) continuum, must it act effectively on the pseudo-arc?

4. (*Lewis*) Under what conditions does a space X with a continuous decomposition into pseudo-arcs admit an effective p-adic Cantor group action which is an extension of an action on individual pseudo-arcs of the decomposition?

See also P16 and the article by Brechner in this Problem Section.

K. Connectedness

6. (*Nyikos, attributed to M. E. Rudin*) Does MA + $\neg$CH imply every compact, perfectly normal, locally connected space is metrizable?

L. Topological Algebra

6. (*van Douwen*) If $(G_i)_{i\in I}$ is a collection of topological groups, the coproduct-topology on the weak product $\sum_{i\in I} G_i = \{x \in \Pi_{i\in I} G_i: x_i = e_i$ for all but finitely many $i \in I\}$ is defined to be the finest group topology such that the relative topology on each finite subproduct is the product topology. Is it possible to have families $(G_i)_{i\in I}$ and $(H_i)_{i\in I}$ of (preferably Abelian) topological groups such that G_i and H_i have the same underlying space and the same identity for all $i \in I$, yet the coproduct topologies on the respective weak products are unequal? non-homeomorphic?

M. Manifold Theory

5. (*Haver*) Do there exist versions of the standard engulfing theorems in which the engulfing isotopy depends continuously on the given open sets and embeddings?

P. Products, Hyperspaces, Remainders, and Similar Constructions

16. (*Lewis*) Is the space of homeomorphisms of the pseudo-arc totally disconnected?

17. (*van Douwen*) Let Y be a Hausdorff continuous image of the compact Hausdorff space X.

(a) If $\square^{\omega}X$ is paracompact [resp. normal], is $\square^{\omega}Y$ paracompact [resp. normal]?

(b) If the G_{δ}-modification of X is paracompact (or normal), is the same true for that of Y?

18. (*van Douwen*) Can normality of a $\sum$-product depend on the choice of the base point?

19. (*Nyikos*) Is there a chain of clopen subsets of ω^* of uncountable cofinality whose union is regular open? [Yes if $p > \omega_1$ or there is a scale or in any model obtained by adding uncountably many mutually Cohen reals.]

20. (*van Douwen*) Can one find in ZFC a point p of ω^* such that $\beta(\omega^* - \{p\}) \neq \omega^*$ or, better yet, $\zeta(\omega^* - \{p\}) \neq \omega^*$? It is known that p is as required if it has a local base of cardinality ω_1, and that it is consistent for there to be $p \in \omega^*$ with $\beta(\omega^* - \{p\}) = \omega^*$.

See also G13 and the article by Brechner in this Problem Section

Q. Generalizations of Topological Spaces

2. (*Hermann*) Characterize those topological spaces (X,T) such that sh = T (resp. u = T).

3. (*Hermann*) Characterize those topological spaces (X,T) such that $rc \times rc = rc(T \times T_z)$ (the r.c. structure generated by $T \times T_z$) and those such that $u \times u = u(T \times T_z)$, $sh \times sh = sh(T \times T_z)$.

4. (*Hermann*) Characterize those topological spaces for which sh [resp. u] is pseudotopological, pretopological, or topological.

R. Dimension Theory

4. (*Rubin*) Does there exist a separable metric space, compact or not, which has infinite cohomological dimension?

5. (*Keesling*) If $f(X) = Y$ is a mapping between compact metric spaces such that $m \leq \dim f^{-1}(y) \leq n$ for all $y \in Y$, then is there a closed set K in X such that $\dim K \leq n-m$ and $\dim f(K) = \dim Y$? [*Note*: an affirmative solution has been announced by Eiji Kurihara.]

6. (*van Douwen*) For which sequences $(k_n)_{n=1}^{\infty}$ of integers is there a separable metrizable space X such that $\dim X^n = k_n$ for all n? For example, is $\lim_n k_n/n = \sqrt{2}$ possible? What if X is also compact?

S. Problems Closely Related to Set Theory

12. (*Nyikos*) For each cardinal κ let u_κ be the least cardinality of a base for a uniform ultrafilter on a set of cardinality κ. Is it consistent to have $\lambda < \kappa$, yet $u_\lambda > u_\kappa$? How about in the case $\lambda = \omega$, $\kappa = \omega_1$?

See also P19 and P20.

T. Algebraic and Geometric Topology

8. (*Perko*) Is every minimal-crossing projection of an alternating knot alternating?

9. (*Perko*) Is the minimal crossing number additive for composition of primes?

10. (*Perko*) Does the bridge number equal the minimal number of Wirtinger generators? (This has been resolved for two-bridged knots by M. Boileau.)

See also M5 and R4.

INFORMATION ON EARLIER PROBLEMS

A5, Volume 3 (*Arhangel'skii*) Let c(X) denote the cellularity of X. Does there exist a space X such that $c(X^2) > c(X)$? *Solution.* There are compact Hausdorff examples X with $c(X) = 2^{\aleph_0}$ and $c(X) = \text{cof } 2^{\aleph_0}$ in every model of ZFC. (Todorčevic, AMS Abstracts 4 (April 1983) 291.)

P15 (*Nyikos*) Is it consistent that $\beta\omega - \omega$ is the union of a chain of nowhere dense sets? *Solution.* Yes, in fact this is equivalent to being able to cover $\beta\omega - \omega$ by $\leq c$ nowhere dense sets. On the one hand, any chain of nowhere dense sets covering $\beta\omega - \omega$ must have cofinality $\leq c$ because $\beta\omega - \omega$ has a dense subspace of cardinality c; on the other hand, there is Theorem 3.5(iv) of the paper *The space of ultrafilters on* N *covered by nowhere dense sets*, by B. Balcar, J. Pelant, and P. Simon [Fund. Math. 110 (1980), 11-24]. This paper also gives a good indication of how extensive this class of models is: if it is impossible to cover $\beta\omega - \omega$ with $\leq c$ nowhere dense sets, then there are more than c selective P_c-points [ibid., Theorem 3.7]. So any model without such points (in particular, any model in which there is no scale of cofinality $\mathfrak{c}$) gives an affirmative solution to P15. Examples are the usual "Cohen real" and "random real" models.

C39 (*van Douwen*) Let μ be the least cardinality of a compact space that is not sequentially compact. It is known that $2^{\mathfrak{t}} \leq \mu \leq 2^{\mathfrak{s}}$. [Here $\mathfrak{t}$ denotes the least cardinality of a chain of subsets of ω (with respect to almost-containment: $A \subset^* B$ iff $A \backslash B$ is finite) such that no infinite subset of ω is almost contained in every one, while $\mathfrak{s}$ is the least cardinality of a splitting family $\mathcal{S}$ of subsets of ω: a family such that for each infinite $A \subset \omega$, there exists $S \in \mathcal{S}$ such that $A \cap S$ and $A \backslash S$ are both infinite.] What else can be said about μ? *Partial solution.* (*Nyikos & Simon*) Let $\mathfrak{t}\pi$ denote the least height of a tree π-base for $\beta\omega - \omega$. Then $\mathfrak{t}\pi \leq \mathfrak{s}$, and there is a family of $\mathfrak{t}\pi$ compact sequential spaces of cardinality $\leq \mathfrak{c}$ whose product is not sequentially compact. Thus $\mu \leq 2^{\mathfrak{t}\pi}$. Also, $\mathfrak{t}\pi$ is equal to the least cardinality of a family of sequentially compact spaces whose product is not sequentially compact, as well as the least cardinality of a family of nowhere dense subsets of $\beta\omega - \omega$ whose union is dense. For additional information on $\mathfrak{t}\pi$, see the paper of Balcar, Pelant, and Simon referenced above, where it is denoted $\kappa(N^*)$, and Peter Dordal's thesis, where it is denoted d, and where it is shown that $\mathfrak{t} < \mathfrak{t}\pi$ is consistent. It seems to be an open problem whether $\mathfrak{t}\pi < \mathfrak{s}$ is consistent.

A further improvement is that $\mu \leq \mathfrak{b}$, where $\mathfrak{b} = \min\{|B|: B$ is the set of branches in some tree π-base for $\beta\omega - \omega\}$. It is easy to see that $\beta \leq 2^{t\pi}$. Moreover (Nyikos and Shelah) it is consistent to have $\beta < 2^{t\pi}$.