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FACTORWISE RIGIDITY OF THE PRODUCT OF TWO PSEUDO-ARCS

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Introduction

I denotes the interval $[0,1]$. A *continuum* is a compact connected metric space. If X and Y are continua, an ε -map $f: X \rightarrow Y$ is a continuous surjective mapping such that for each $p \in Y$, $f^{-1}(p)$ has diameter less than ε . A continuum X is *arclike* or *chainable* if and only if, for each $\varepsilon > 0$, there exists an ε -map $f: X \rightarrow I$. A continuum X is *indecomposable* if and only if it is not the union of two of its proper subcontinua. This is equivalent to each of its proper subcontinua having empty interior. X is *hereditarily indecomposable* if and only if each of its subcontinua is indecomposable. A chainable hereditarily indecomposable continuum is called a *pseudo-arc*. Some background information on pseudo-arcs can be found in [1], [2], [3], [4], [6], and [9]. It is clear that every nondegenerate subcontinuum of a pseudo-arc is a pseudo-arc, and R. H. Bing has shown that all pseudo-arcs are homeomorphic, although we shall not need this fact.

If X and Y are continua then $\pi_1: X \times Y \rightarrow X$ and $\pi_2: X \times Y \rightarrow Y$ will always denote the projection maps. If d_1 and d_2 are the given metrics on X and Y , the metric d on

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$X \times Y$ will always be taken as the maximum metric;

$d((x,y), (u,v)) = \max\{d_1(x,u), d_2(y,v)\}$. Thus, if $f: X \rightarrow X_0$ and $g: Y \rightarrow Y_0$ are ϵ -maps, then $f \times g: X \times Y \rightarrow X_0 \times Y_0$ is an ϵ -map also.

A product of mutually homeomorphic continua is *factorwise rigid* if and only if each homeomorphism of the product is a product of homeomorphisms of the separate factors, followed by a permutation of the factors. Thus, a product $X \times X$ is factorwise rigid if every homeomorphism $H: X \times X \rightarrow X \times X$ is of the form $H(x,y) = (h(x), k(y))$ or $H(x,y) = (h(y), k(x))$ where h and k are self-homeomorphisms of X . Wayne Lewis [7, problem 60] has asked whether every product of pseudo-arcs is factorwise rigid. This article gives an affirmative answer for a product of two pseudo-arcs. It is known that every product of Menger or Sierpinski Universal curves is factorwise rigid [5] and [11].

If X is a continuum and $A \subseteq X$, $T(A)$ is defined to be the complement of the set of points of X which have a continuum neighborhood missing A . A substantial body of literature exists on this set function and related topics. Rather than attempt to give a comprehensive list of references, we refer the interested reader to the articles in Sections II and V of [8], and the bibliographies thereof.

Lemma 1. Suppose W and M are subcontinua of $I \times I$ and that $\pi_1(M) = I$ while $\pi_2(W) = I$. Then $W \cap M \neq \emptyset$.

Proof. This is an immediate consequence of Theorem 28 of [10, p. 156].

Lemma 2. Suppose X and Y are chainable continua and $M, W \subseteq X \times Y$ are continua such that $\pi_1(M) = X$ and $\pi_2(W) = Y$. Then $M \cap W \neq \emptyset$.

Proof. Suppose not. Choose $\epsilon > 0$ such that $\epsilon < d(M, W)$, and let $f: X \rightarrow I$ and $g: Y \rightarrow I$ be ϵ -maps. Then $f \times g: X \times Y \rightarrow I \times I$ is an ϵ -map also, so that $(f \times g)(M) \cap (f \times g)(W) = \emptyset$, since a point in this intersection would have inverse image under $f \times g$ with diameter greater than ϵ .

However, $\pi_1((f \times g)(M)) = f(\pi_1(M)) = I$ and $\pi_2((f \times g)(W)) = g(\pi_2(W)) = I$, so that by Lemma 1, $(f \times g)(M) \cap (f \times g)(W) \neq \emptyset$, a contradiction.

Corollary 3. Let X and Y be chainable continua and suppose W and M are subcontinua of $X \times Y$ such that $\pi_1(W) \subseteq \pi_1(M)$ while $\pi_2(M) \subseteq \pi_2(W)$. Then $W \cap M \neq \emptyset$.

Proof. Subcontinua of chainable continua are chainable (or degenerate) so Lemma 2 applies to $\pi_1(M) \times \pi_2(W)$.

Lemma 4. If X and Y are continua and A is a closed subset of $X \times Y$, and $T(A) = X \times Y$, then either $\pi_1(A) = X$ or $\pi_2(A) = Y$.

Proof. This is a classical argument due to F. B. Jones. Suppose A is closed in $X \times Y$, $\pi_1(A) \neq X$, and $\pi_2(A) \neq Y$. Then there exists a nonempty open subset U of X with $\bar{U} \cap \pi_1(A) = \emptyset$, and there is a point $y \in Y - \pi_2(A)$. Then $(\bar{U} \times Y) \cup (X \times \{y\})$ is a continuum in $X \times Y$ with nonvoid interior missing A so that $T(A) \neq X \times Y$.

Lemma 5. If X and Y are continua and X is indecomposable, then for any $a \in X$, in $X \times Y$, $T(\{a\} \times Y) = X \times Y$.

Proof. If $W \subseteq X \times Y$ is a continuum with nonvoid interior, $\pi_1(W) = X$, since $\pi_1(W)$ must be a subcontinuum of X with interior. Thus, $a \in \pi_1(W)$ so that $W \cap (\{a\} \times Y) \neq \emptyset$.

Lemma 6. Suppose X and Y are indecomposable continua and $a \in X$ and $h: X \times Y \rightarrow X \times Y$ is a homeomorphism. Then either $\pi_1(h(\{a\} \times Y)) = X$ or $\pi_2(h(\{a\} \times Y)) = Y$.

Proof. For any set A and any homeomorphism h , $T(h(A)) = h(T(A))$. Apply Lemmas 4 and 5.

In what follows, P denotes a pseudo-arc.

Lemma 7. Let $h: P \times P \rightarrow P \times P$ be any homeomorphism. Then either $\pi_1(h(\{p\} \times P)) = P$ for every $p \in P$, or $\pi_2(h(\{p\} \times P)) = P$ for every $p \in P$.

Proof. For $i = 1, 2$, let $M_i = \{p \in P \mid \pi_i(h(\{p\} \times P)) = P\}$; by Lemma 6, $M_1 \cup M_2 = P$. Suppose $M_1 \neq P$ and $M_2 \neq P$. Then M_1 and M_2 are closed sets, and so there exist two distinct points $a, b \in M_1 \cap M_2$, since P has no separating point. Then, $\pi_1(h(\{a\} \times P)) = \pi_2(h(\{a\} \times P)) = \pi_1(h(\{b\} \times P)) = \pi_2(h(\{b\} \times P)) = P$. Thus, $h(\{a\} \times P)$ and $h(\{b\} \times P)$ satisfy the hypotheses of Lemma 2, so that $h(\{a\} \times P) \cap h(\{b\} \times P) \neq \emptyset$. But then $(\{a\} \times P) \cap (\{b\} \times P) \neq \emptyset$, so that $a = b$, a contradiction. Hence, either $M_1 = P$ or $M_2 = P$, as claimed.

Let $\theta: P \times P \rightarrow P \times P$ be the homeomorphism which interchanges the factors.

Theorem. Let $H: P \times P \rightarrow P \times P$ be a homeomorphism. Then there exist homeomorphisms $h, k: P \rightarrow P$ such that either $H = (h \times k)$ or $H = \theta \circ (h \times k)$; in other words $P \times P$ is factorwise rigid.

Proof. By Lemma 7, either $\pi_2(H(\{p\} \times P)) = P$ for every $p \in P$ or $\pi_1(H(\{p\} \times P)) = P$ for every $p \in P$. Assume the latter; the other case is similar. Then $\pi_2(\theta \circ H(\{p\} \times P)) = P$ for every $p \in P$. Suppose for some $p \in P$, $\pi_1(\theta \circ H(\{p\} \times P))$ is nondegenerate. Let $\langle W_n \rangle_{n=1}^\infty$ be a decreasing sequence of nondegenerate subcontinua of P whose intersection is $\{p\}$, and let $a \in P$. Then $\langle W_n \times \{a\} \rangle_{n=1}^\infty$ is a sequence of continua in $P \times P$ whose intersection is $\{(p, a)\}$, and so $\langle (\theta \circ H(W_n \times \{a\})) \rangle_{n=1}^\infty$ has intersection equal to $\{\theta \circ H(p, a)\}$, and $\langle \pi_1(\theta \circ H(W_n \times \{a\})) \rangle_{n=1}^\infty$ is a decreasing sequence of continua in P whose intersection is $\{\pi_1(\theta \circ H(p, a))\}$. In particular, $\pi_1(\theta \circ H(p, a)) \in \pi_1(\theta \circ H(W_n \times \{a\}))$ for every n , so that for every n , $\pi_1(\theta \circ H(W_n \times \{a\})) \cap \pi_1(\theta \circ H(\{p\} \times P))$ is nonempty, and so for some n , $\pi_1(\theta \circ H(W_n \times \{a\})) \subseteq \pi_1(\theta \circ H(\{p\} \times P))$, since P is hereditarily indecomposable and $\pi_1(\theta \circ H(\{p\} \times P))$ is nondegenerate and so cannot be a subset of $\pi_1(\theta \circ H(W_m \times \{a\}))$ for every m . Let $q \in W_n$ with $q \neq p$. Then $\pi_1(\theta \circ H(\{q\} \times P)) \cap \pi_1(\theta \circ H(\{p\} \times P)) \neq \emptyset$, so that one of these continua is a subset of the other. Without loss of generality, assume

$$\pi_1(\theta \circ H(\{q\} \times P)) \subseteq \pi_1(\theta \circ H(\{p\} \times P)).$$

But $\pi_2(\theta \circ H(\{p\} \times P)) \subseteq \pi_2(\theta \circ H(\{q\} \times P))$ (since both of these are equal to P). Thus, by Corollary 3,

$$\theta \circ H(\{q\} \times P) \cap \theta \circ H(\{p\} \times P) \neq \emptyset,$$

and consequently $(\{q\} \times P) \cap (\{p\} \times P) \neq \emptyset$, since $\theta \circ H$ is 1 to 1. This is a contradiction since $q \neq p$. Therefore, for every $p \in P$, $\pi_1(\theta \circ H(\{p\} \times P)) = \{x\}$ for some $x \in P$. Define $h: P \rightarrow P$ by $h(p) = \pi_1 \circ \theta \circ H \circ \pi_1^{-1}(p)$, which is well-defined by the above argument. A parallel argument will prove that

$k: P \rightarrow P$ defined by $k(q) = \pi_2 \circ \theta \circ H \circ \pi_2^{-1}(q)$ well-defined. Then, for any $(p, q) \in P \times P$, $\theta \circ H(p, q) = (h(p), k(q))$, or $\theta \circ H = (h \times k)$, so that $H = \theta \circ (h \times k)$. (The other case, that $\pi_2(H(\{p\} \times P)) = P$ for every $p \in P$, yields $H = h \times k$ for some h and k .)

The idea for this paper grew in part out of a conversation between Howard Cook and the first author in the Spring of 1980. At that time, it was proven that there is no homeomorphism of the product $P \times P$ which carries the diagonal to a fiber $\{a\} \times P$. This fact now follows as a corollary to the Theorem in this paper. This is a curious fact, since if X is either a topological group or an n -cell for $n \geq 1$, there is a homeomorphism of $X \times X$ carrying the diagonal to any $\{a\} \times X$, but though the pseudo-arc is homogeneous, and of trivial shape, no such homeomorphism exists.

References

1. D. P. Bellamy, *Mapping hereditarily indecomposable continua onto a pseudo-arc*, Lecture Notes in Math 375 (Topology Conference, Virginia Polytechnic Institute and State University, 1973) Springer Verlag, Berlin, 6-14.
2. R. H. Bing, *Concerning hereditarily indecomposable continua*, Pac. J. Math. 1 (1951), 43-51.
3. _____, *A homogeneous indecomposable plane continuum*, Duke Math. J. 15 (1948), 729-742.
4. _____, *Snake-like continua*, Duke Math. J. 8 (1951), 653-663.
5. K. Kuperberg, W. Kuperberg, and W. R. R. Transue, *On the two-homogeneity of Cartesian products*, Fund. Math. 110 (1980), 131-134.
6. B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fund. Math. 3 (1922), 247-286.

7. I. W. Lewis, editor, *Continua theory problems* (Newsletter) (1983).
8. L. F. McAuley and M. M. Rao, editors, *General topology and modern analysis*, Academic Press, New York (1981).
9. E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Trans. Amer. Math. Soc. 63 (1948), 581-594.
10. R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications, Vol. XIII, Revised Edition. American Mathematical Society, Providence (1962).
11. J. Kennedy Phelps, *Homeomorphisms of products of universal curves*, Houston J. Math. 6 (1980), 127-134.

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