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In 1976 E. V. Scepin [S_1] defined a capacity on a T_3-space X to be a function ε (called the capacity) from $X \times \mathcal{F}$ (where $\mathcal{F}$ is the class of all closed subsets of X) into the set of non-negative real numbers such that

(c-1) $\varepsilon(x,F) > 0$ if and only if $x \in \text{Int}(F)$ (Interior of F)

(c-2) if $F_1 \subseteq F_2$, then $\varepsilon(x,F_1) \leq \varepsilon(x,F_2)$

(c-3) if F is fixed, then ε is continuous in the first variable, and

(c-4) if $\{F_\alpha\}$ is a family of closed sets linearly ordered by set inclusion, then $\varepsilon(x,\cap F_\alpha) = \inf_\alpha\{\varepsilon(x,F_\alpha)\}$, where $x \in X$ and each $F_\alpha \in \mathcal{F}$.

A space with a capacity is called a capacity space. If (X,d) is a metric space then a capacity may be defined on X by

$$\varepsilon(x,F) = d(x,X-F)$$

Thus a capacity space is a generalization of a metric space.

Recall that a closed subset A of space X is regularly closed if A = Cl(Int(A)) and the complement of a regularly closed set is called a regularly open set (where Cl(A) is the closure of A in X). Also, recall that a space is perfectly κ-normal if any two non-intersecting regularly closed sets have non-intersecting neighborhoods and every regularly closed set is the countable intersection of regularly open sets. Capacity spaces were evidently introduced as a tool to study perfectly κ-normal spaces.

Using (c-1) and (c-3) it can be shown that each capacity space is perfectly κ-normal.

It can be shown that Heath's sticker space, Example 1 of [H], is not perfectly κ-normal (and hence not a capacity space). It can also be shown that the Moore plane [W, exercise 4B] does have a capacity. Thus the property of having a capacity is a differentiating feature for the class of Moore spaces.

In [S_1] there were nine theorems (Theorems 5-13) involving capacity spaces that were given without proof. One of these theorems asserted that a LOTS (=linearly ordered topological space) with a capacity is metrizable. In [BL] this result is obtained as a corollary to the more general result.

Theorem 1. *A* GO-*space (=generalized ordered space) with a capacity has a* G_δ-*diagonal and is perfect (=closed sets are* G_δ-*sets).*

The question of what subspaces of a capacity space have a capacity naturally arises.

Theorem 2. *Let* Y *be a subspace of a capacity space* X. *If* Y *is either a regularly closed, or open subset of* X, *then* Y *is a capacity space.*

Proof. Let ε be a capacity on X. If Y is a regularly closed subspace of X define

$$\eta(x,F) = \varepsilon(x,F \cup Cl(X-F))$$

for $x \in Y$ and F closed in Y.

If Y is an open subspace of X define

$$\eta(x,F) = \varepsilon(x,F \cup (X-F))$$

for $x \in Y$ and f closed in Y. In either case it is not difficult to verify that η is a capacity in Y.

It will be shown later that closed subspaces of capacity spaces need not have a capacity. To solve the dense subspace problem we need the following definition.

Definition. A capacity for a space X is *faithful* if (c-5) $F_1, F_2 \in \mathcal{F}$ such that Int F_1 = Int F_2, then $\varepsilon(x,F_1) = \varepsilon(x,F_2)$.

Theorem 3. *If* Y *is a dense subspace of a faithful capacity space* X, *then* Y *is a faithful capacity space.*

Proof. Let ε be a faithful capacity on X. Define $\eta(x,F) = \varepsilon(x,Cl(F,X))$ where $Cl(F,X)$ is the closure of F in X. It is routine to verify that η is a faithful capacity on Y.

The condition (c-5) leads to the following question.

Question 1. Does there exist a capacity space that *does not* have a faithful capacity?

The following example gives a non-faithful capacity of [0,1] with the usual topology. Since this space is metric it also has a faithful capacity.

Example. For a closed set F in X = [0,1] with the usual topology let $\varepsilon(x,F) = d(x,X-F) \cdot m^*(F)$ where d is the usual metric on [0,1] and $m^*(F)$ is the outer Lebesgue measure of F.

It is straight forward to verify that this is a non-faithful capacity on [0,1].

In 1980 in another paper [S_2] by Scepin, the notion of a κ-metric on a completely regular space X is introduced. Let $\mathcal{C}$ be the class of all regularly closed subsets of X. Then a nonnegative real-valued function ρ with domain $X \times \mathcal{C}$ is a κ-metric on X if

(k-1) $\rho(x,C) = 0$ if and only if $x \in C$,

(k-2) if C_1 and C_2 are in $\mathcal{C}$ and $C_1 \subset C_2$, then $\rho(x,C_1) \geq \rho(x,C_2)$

(k-3) if C is fixed, then ρ is continuous in the first variable, and

(k-4) if $\{C_\alpha\}$ is a transfinite increasing collection of elements of $\mathcal{C}$, then $\rho(x,\mathrm{Cl}(\bigcup_\alpha\{C_\alpha\})) = \inf_\alpha\{\rho(x,C_\alpha)\}$.

In [S_2] proofs for the theorems in [S_1] are indicated except for Theorem 12 of [S_1] which is stated as an open problem in [S_2]. These proofs are given in the κ-metrizable setting rather than the capacity setting. This is due, perhaps, to the unproven statement that spaces with a capacity "are identical with κ-metrizable spaces," ([S_2], p. 411).

The following theorem indicates that this may not be true and if the answer to Question 1 is yes then the theorems in [S_1] do need to be addressed in a capacity space setting.

Theorem 4. *A completely regular space* X *has a faithful capacity if and only if* X *is* κ-*metrizable*.

Proof. Let X have a faithful capacity ε. For each

$C \in \mathcal{C}$ let

$$\rho(x,C) = \varepsilon(x,Cl(X-C))$$

Conditions (k-1), (k-2), and (k-3) are readily obtainable. To see that (k-4) is true, let $\{C_\alpha\}$ be an increasing collection of elements of $\mathcal{C}$. Then $\rho(x,Cl(\cup C_\alpha)) = \varepsilon(x,Cl(X-Cl(\cup C_\alpha)))$. On the other hand

$$\inf_\alpha\{\rho(x,C_\alpha)\} = \inf_\alpha\{\varepsilon(x,Cl(X-C_\alpha)\} = \varepsilon(x,\cap Cl(X-C_\alpha))$$

since $\{Cl(X-C_\alpha)\}$ is a decreasing collection of closed sets. Since ε is faithful capacity we must show

$$Int(Cl(X-Cl(\cup C_\alpha))) = Int(\cap Cl(X-C_\alpha))$$

To this end let $z \in Int(Cl(X-Cl(\cup C_\alpha)))$. Then there is an open set U containing z such that $U \subset Cl(X-Cl(\cup C_\alpha))$. Since each C_α is regularly closed and $\{C_\alpha\}$ is an increasing collection, $U \cap C_\alpha = \emptyset$ for each α. Thus $U \subset X-C_\alpha \subset Cl(X-C_\alpha)$. Hence $U \subset \cap Cl(X-C_\alpha)$ and it follows that $z \in Int(\cap Cl(X-C_\alpha))$.

If $z \in Int(\cap Cl(X-C_\alpha))$ then there is an open set U containing z such that $U \subset \cap Cl(X-C_\alpha)$. Hence for each α, $U \subset Cl(X-C_\alpha)$ and, since C_α is regularly closed, $U \cap C_\alpha = \emptyset$. Thus $U \cap (\cup C_\alpha) = \emptyset$. It follows that $U \cap Cl(\cup C_\alpha) = \emptyset$. Thus $U \subset X-Cl(\cup C_\alpha)$. Hence $z \in Int(Cl(X-Cl(\cup C_\alpha)))$. Hence $Int(Cl(X-Cl(\cup C_\alpha))) = Int(\cap Cl(X-C_\alpha))$ and, since ε is faithful, we have $\varepsilon(x,Cl(X-Cl(\cup C_\alpha))) = \varepsilon(x,\cap Cl(X-C_\alpha))$ from which it follows that $\rho(x,Cl(\cup C_\alpha)) = \inf\{\rho(x,C_\alpha)\}$.

Conversely, let ρ be a κ-metric on X. For each $x \in X$ and closed set F in X let $\varepsilon(x,F) = \rho(x,Cl(X-F))$. Notice that ε is well-defined since if F is closed then $Cl(X-F)$ is regularly closed. Conditions (c-1), (c-2) and (c-3) are readily obtainable.

To see that (c-4) is true let $\{F_\alpha\}$ be a decreasing sequence of closed sets. Then

$$\varepsilon(x,\cap F_\alpha) = \rho(x,\mathrm{Cl}(X-\cap F_\alpha))$$

and

$$\inf_\alpha\{\varepsilon(x,F_\alpha)\} = \inf_\alpha\{\rho(x,\mathrm{Cl}(X-F_\alpha)\} = \rho(x,\mathrm{Cl}(\cup \mathrm{Cl}(X-F_\alpha))).$$

Thus, it must be shown that

$$\mathrm{Cl}(X-\cap F_\alpha) = \mathrm{Cl}(\cup \mathrm{Cl}(X-F_\alpha)).$$

To this end let $z \in \mathrm{Cl}(X-\cap F_\alpha) = \mathrm{Cl}(\cup(X-F_\alpha))$. For each open set U containing z, it follows that $U \cap (\cup(X-F_\alpha)) = \emptyset$. Hence

$$z \in \mathrm{Cl}(\cup(X-F_\alpha)) \subseteq \mathrm{Cl}(\cup \mathrm{Cl}(X-F_\alpha)).$$

Let $z \in \mathrm{Cl}(\cup \mathrm{Cl}(X-F_\alpha))$. Then for each open set U containing z there is a member M(U) of the well-ordered indexing set to which α belongs, such that if M(U) precedes α in the well-ordering, then $U \cap \mathrm{Cl}(X-\cap F_\alpha) \neq 0$ and $z \in \mathrm{Cl}(X-\cap F_\alpha)$. Hence ε is a capacity for X.

If F_1 and F_2 are closed sets such that Int F_1 = Int F_2, then $\mathrm{Cl}(X-F_1) = \mathrm{Cl}(X-F_2)$. Thus, if $x \in X$,

$$\rho(x,\mathrm{Cl}(X-F_1)) = \rho(x,\mathrm{Cl}(X-F_2)).$$

From this it follows that

$$\varepsilon(x,F_1) = \varepsilon(x,F_2)$$

and ε is a faithful capacity on X.

In [S_2] it is shown that the product of κ-metrizable spaces is a κ-metrizable space. This fact is used in the next example.

Example. There is a κ-metrizable space X with a closed subspace Y that does not have a capacity.

Let Z be an uncountable subset of [0,1] whose only compact (with regard to the usual topology) subsets are countable (see [K]). Let Y = Cl(Z,[0,1]. Topologize Y with a finer topology τ than the relative Euclidean topology by letting points of Y-Z be discrete. It follows that (Y,τ) is a quasi-developable, [B_1], GO-space which is not metrizable. If (Y,τ) had a capacity, then, by Theorem 1, (Y,τ) would be perfect and, hence, developable [B_1]. Since (Y,τ) is a GO-space it would be metrizable. From this contradiction it follows that (Y,τ) cannot have a capacity.

It is not difficult to prove that (Y,τ) is a Lindelöf space and, hence, realcompact. Thus (Y,τ) can be closed embedded in a product of real lines. Let $X = \prod_\alpha R_\alpha$ (where each R_α is a copy of the real line) contain a homeomorphic copy of (Y,τ) as a closed subset. Since each R_α is κ-metrizable it follows that X is κ-metrizable [S_2, pg. 408]. Hence closed subspaces of κ-metrizable spaces need not have a capacity.

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