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It is well known [1], that if X is a real finite dimensional inner product space and S is a closed subset of X, then S is convex only in case every point in X has a unique nearest point in S.

The following definition was motivated by the search for the answer to the question: Is there a real inner product space X containing a closed nonconvex subset S, such that each point in X has a unique nearest point in S?

In all that follows it is to be understood that X is a real inner product space.

Definition. By a G-system for X we mean an ordered pair (M,S) where M is pairwise disjoint, closed and convex set collection filling up X and S is a subset of X such that each set in M contains exactly one point of S, and if m_1 and m_2 are two sets in M with points s_1 and s_2 from S respectively, then each point p in m_1 is closer to s_1 than p is to s_2.

Suppose S is a closed and convex subset of the finite dimensional space X and s is a point in S. Let K be the set to which x belongs only in case x is in X and s is the nearest point in S to x. Then K is a closed and convex set, and the collection of all such K is a decomposition of X into pairwise disjoint, closed and convex sets, each containing exactly one point of S.

Let X be the euclidean plane, M the set of all vertical lines and S the x-axis. Then (M,S) is a G-system for X. If S′ is any horizontal line, then (M,S′) is a G-system for X. However, if L is a non-horizontal line, then (M,L) is not a G-system for X.

Question 1. If each of (M,S) and (M,S′) is a G-system for X, then how are S and S′ related?

Theorem 1. *If* (M,S) *is a* G-*system for* X, *then* S *is closed.*

Proof. Suppose p is a point in X. Then there is a unique set m in M which contains p. Let s be that point in S which is in m. If $p \neq s$, then every point in S different from s is farther from p, than p is from s, and hence p is not a limit point of S. Hence S is closed.

Theorem 2. *If* (M,S) *is a* G-*system for* X *and* S *is non-degenerate, then* S *is uncountable.*

Proof. Suppose S is nondegenerate and u and v are two points in S. Let [u,v] denote the line interval with end-points u and v. For each point p in [u,v] there is a set m_p in M which contains p. Notice that $m_p \cap [u,v]$ is a closed set and if p and q are two points in [u,v], then either $m_p = m_q$ or m_p does not intersect m_q. We also have that $\bigcup_{p \in [u,v]} [m_p \cap [u,v]] = [u,v]$.

Therefore $\{m_p : p \in [u,v]\}$ is uncountable, because no interval is the union of countably many pairwise disjoint closed sets. Hence S is uncountable.

If X is separable, then except for at most a countable subset of S, each point of S is a limit point of S, and hence is a boundary point of that set in M to which it belongs. This then leads to the following theorem.

Theorem 2. *If* (M,S) *is a* G-*system for* X *and* S *is nondegenerate, then each point in* S *is a boundary point of that set in* M *to which it belongs, and hence is a limit point of* S.

Proof. Suppose S is nondegenerate and that there is a point s in S which is not a boundary point of the set m in M to which it belongs. Then s is an interior point of m and therefore there is a number $d > 0$ such that $S_d(s) = \{x: ||x - s|| \leq d\}$ is a subset of m and if $E > 0$, then there is a point q not in m such that $||s - q|| \leq d + E$.

Let b be a boundary point of m and C be the union of all line intervals having one endpoint b and the other endpoint in $S_d(s)$. Since m is convex, C is a subset of m.

Let $C^+ = \bigcup_{\substack{c \in C \\ c \neq b}} S_{||c-s||}(c)$ and p be a point on the boundary of $S_{||b-s||}(b)$, different from $2b - s$ and not in m. Let w be the point common to the boundary of $S_{||b-s||}(b)$ and boundary of $S_d(s)$ lying in the plane determined by s, b and p, and closest to p.

The line interval [w,b] is a subset of C and hence for each t, $0 \leq t \leq 1$, $(1 - t)w + tb = r$ is in C. There is a number t in [0,1] such that $S_{||r-s||}(r)$ contains p, and hence p is in C^+. Notice then that $S_{||b-s||}(b) - \{2b - s\}$ is a subset of C^+, $2b - s$ is not in C^+, and no point of S different from s is in C^+.

Let $R = \{(1 - t)s + tb: t \geq 0\}$ and let $\{q_i\}_{i=1}^{\infty}$ be a sequence such that

1. for each $i = 1,2\cdots$ q_i is in $R - m$
2. $\lim_{i\to\infty} q_i = b$, and
3. if for each i, m_i is that set in M containing q_i, then $m_i \neq m_j$ if $i \neq j$.

For each i, let s_i be that point in S that is in m_i.

Then

$$\lim_{i\to\infty}||s_i - q_i|| = ||s - b||,$$

$$\lim_{i\to\infty}||q_i - b|| = 0,$$

and if $E > 0$, then there is $N > 0$ such that if $n > N$, then $||s_n - s|| \leq 2||s - b|| + E$.

Hence the point sequence $\{s_i\}_{i=1}^{\infty}$ must tend to the boundary of $S_{||s-b||}(b)$. Recall that no point of S different from s is in C^+. Therefore, $\lim_{i\to\infty} s_i = 2b - s$ and since S is closed, $2b - s$ is in S. We then have a contradiction since $||s - b|| = ||(2b - s) - b||$ and $2b - s \neq s$.

It is easy to show that if (M,S) is a G-system for X and S is nondegenerate and connected, then each point in S is a boundary point of that set in M to which it belongs. This then raises the following question.

Question 2. If (M,S) is a G-system for X, then is S connected?

The proof for theorem 2 also raises the following question.

Question 3. If (M,S) is a G-system for X, m is a set in M which contains two points, s is that point in S which is in m and p is a point in m different from s, then is $\{(1 - t)s + tp: t \geq 0\}$ a subset of m?

Theorem 4. *Suppose* (M,S) *is a G-system for* X, $\{s_i\}_{i=1}^{\infty}$ *is a convergent sequence from* S, *and* $\{w_i\}_{i=1}^{\infty}$ *is a convergent sequence such that for each* i, w_i *and* s_i *belong to the same set in* M. *If* $\lim_{i\to\infty} s_i = s$ *and* $\lim_{i\to\infty} w_i = w$, *then* w *and* s *belong to the same set in* M.

Proof. Let m be that set in M which contains s. And $\overline{m}$ that set in M which contains w. Suppose $\overline{m} \neq m$, and let $\overline{s}$ be that point in S which is in $\overline{m}$. Then
$d = ||s - w|| - ||s - \overline{w}|| > 0.$
Now

$$\begin{aligned} ||w_i - \overline{s}|| &\leq ||w_i - w|| + ||w - \overline{s}|| = \\ &||w_i - w|| + ||s - w|| - d < ||w_i - w|| \\ &+ ||s_i - s|| + ||s_i - w|| - d. \end{aligned}$$

Choose N > 0 such that if n > N, then $||w_n - w|| < d/4$ and $||s_n - s|| < d/4$.
Hence

$$\begin{aligned} ||w_n - \overline{s}|| &< d/4 + d/4 + ||s_n - w|| - d \\ &< ||s_n - w_n|| + ||w_n - w|| - d/2 \\ &< ||s_n - w_n|| - d/4 \end{aligned}$$

Therefore $||w_n - \overline{s}|| < ||s_n - w_n||$, which is a contradiction.

Question 4. If (M,S) is a G-system for X and $\{w_i\}_{i=1}^{\infty}$ is a convergent sequence in X must $\{s_i\}_{i=1}^{\infty}$ be a convergent sequence where for each i, s_i and w_i belong to the same set in M?

Theorem 5. *Suppose* (M,S) *is a* G-*system for* X *such that if* m_1 *and* m_2 *are two sets in* M *with points* s_1 *and* s_2 *from* S *respectively, then the unique nearest point for* s_2 *in* m_1 *is* s_1 *and the unique nearest point for* s_1 *in* m_2 *is* s_2. *Then* S *is convex.*

Proof. Let s_1 and s_2 be two points in S and q a point between s_1 and s_2 on the line interval $[s_1,s_2]$. If s_1 is in m_1 and s_2 is in m_2, then q is not in m_1 or m_2 since q is closer to s_1 than s_2 is to s_1 and q is closer to s_2 than s_1 is to s_2. Let m be that set in M which contains q and s that point in S which is in m. Then $||s_1 - s|| + ||s_1 - s_2|| \geq ||s_1 - q|| + ||q - s_2||$ and equality holds only in case s is in $[s_1,s_2]$. Hence $s = q_1$ and therefore S is convex.

Notice that in order to construct a closed nonconvex set have the unique nearest point property, the hypothesis of theorem 5 must not hold.

References

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