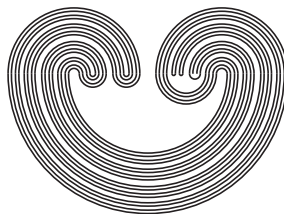

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**THE APOSYNDETTIC DECOMPOSITION
OF HOMOGENEOUS CONTINUA**

F. Burton Jones

The aposyndetic decomposition theorem for homogeneous continua was proved many years ago [3] but the argument given then was tedious and convoluted. Now with the Effros theorem [2] a much simpler and more direct proof is possible. Furthermore, recent use of an extended version has added new interest [4,5].

Theorem. Suppose that M is a decomposable, homogeneous continuum (=compact, connected metric space). Then there exists a nondegenerate continuous collection \mathcal{G} of disjoint subcontinua of M filling up M such that (a) M/\mathcal{G} is a homogeneous, aposyndetic continuum, (b) for each $x \in M$, $L_x \in \mathcal{G}$, and (c) if $G \in \mathcal{G}$ and K is a subcontinuum of M which intersects both G and $M - G$, then $K \supset G$. (Clearly the elements of \mathcal{G} are homogeneous and mutually homeomorphic.)

$[L_x = \{y: M \text{ is not aposyndetic at } y \text{ with respect to } x\}.]$

Proof. If $p \in M$, let G be a proper subcontinuum of M containing p which is maximal with respect to having property (c). To see that G exists, let \mathcal{H} be the set of all proper subcontinua of M containing p and having property (c). If $\text{cl}(U\mathcal{H})$ is an element of \mathcal{H} , then it is obviously maximal with respect to having property (c). Let q be a point of M at which M is aposyndetic with respect to p . Such a point must exist since M is decomposable and

homogeneous. Let N_q be a closed connected neighborhood of q missing p . Clearly N_q intersects no element of \mathcal{H} . So $H = \text{cl}(U\mathcal{H})$ does not contain q and if it does not have property (c), some subcontinuum K of M contains a point of H and a point of $M - H$ but no point of any element of \mathcal{H} . Using the Effros theorem, there exists a homeomorphism of M onto itself which moves K so little that it still misses p , intersects $M - H$ but at the same time intersects some element of \mathcal{H} . So H is maximal and will do for G .

Now let \mathcal{G} be the collection of all such subcontinua G . It is easy to see that the elements of \mathcal{G} are disjoint and $U\mathcal{G} = M$.

Suppose that for some point p of M , L_p does not have property (c). Then there exists a continuum K which contains a point x of L_p , a point y of $M - L_p$ but does not contain L_p . If K does not contain p , then there is a closed connected neighborhood T of y which misses p and the union of Effros images of K intersecting T but covering up a small neighborhood of x would produce a closed connected neighborhood of x missing p . So it must always happen that K contains p and, in fact, the element of \mathcal{G} containing p .

Let U_p denote $M - L_p$. If G_p denotes the element of \mathcal{G} containing p , $U_p \cap G_p = \emptyset$. So $G_p \subset L_p$. By the above, p cuts $L_p - \{p\}$ from U_p . Hence by Bing's "non-domain-cut-point" theorem [1], $L_p - \{p\}$ (and also L_p) has void interior ($\text{rel}M$, of course). So any closed connected neighborhood of a point of L_p intersects U_p . Let p_1 be a point of $L_p - K$ and x be a point of L_{p_1} . If N is a closed connected neighborhood

of x , N must contain p_1 . Since N also intersects U_p , it must contain p . So $L_{p_1} \subset L_p$. None of the points of $K \cap L_p$ belong to L_{p_1} because then K can be fattened up to make a closed connected neighborhood missing p_1 . So $L_{p_1} \subsetneq L_p$. In particular, $p \notin L_{p_1}$.

Let h be a homeomorphism of M onto M such that $h(p) = p_1$. Then $h(L_p) = L_{p_1}$, $h(p_1) = p_2$, $L_{p_2} \subsetneq L_{p_1}$ with $p_1 \notin L_{p_2}, \dots, h(p_n) = p_{n+1} \dots$ continues ω times. Let $L = \bigcap_{i=1}^{\infty} L_{p_i}$ and let p_ω be a point of L . Let x be any point of L_{p_ω} and N be a closed connected neighborhood of x .

Since $p_\omega \in L_{p_i}$ and N contains a point of U_{p_i} (in fact, a point of U_p), N must contain p_i . Hence $L_{p_\omega} \subsetneq L_{p_i}$. Let h^ω be a homeomorphism of M onto M that takes p to p_ω . This starts the process all over again. So it goes on without ever stopping and uncountably many times is too much. So $L_p = G_p$ and has property (c).

It is easy to see that \mathcal{G} is upper-semicontinuous and that each element of \mathcal{G} is homogeneous. Since \mathcal{G} is continuous at some one of its elements, the homogeneity forces \mathcal{G} to be a continuous decomposition. Furthermore, if G_1 and G_2 are different elements of \mathcal{G} , any closed connected neighborhood of a point of G_1 missing a point of G_2 is a closed connected neighborhood of G_1 missing G_2 . Hence M/\mathcal{G} is aposyndetic. Clearly it is homogeneous.

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