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s-CONNECTED SPACES AND THE FIXED POINT PROPERTY

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We wish to establish a general procedure for showing that certain spaces have the fixed point property. In particular, we will be interested in spaces which resemble products.

Consider the topological disk $D = [0,1] \times [0,1]$. It is a well known property of D that if H is a closed subset of D then either some component of $D - H$ intersects both the top and bottom of D or some component of H intersects both the right and left sides of D . One can use this property to show that D has the fixed point property. The argument goes like this. Let $f: D \rightarrow D$ be a continuous function. Let $H = \{x \in D \mid \pi_2(x) = \pi_2 f(x)\}$. The set H is non-empty since $\pi_2: D \rightarrow [0,1]$ is universal. No component C of $D - H$ can intersect both top and bottom, for otherwise we could write C as a union of mutually separated sets $\{x \in C \mid \pi_2(x) < \pi_2 f(x)\}$ and $\{x \in C \mid \pi_2(x) > \pi_2 f(x)\}$. Thus, some component K of H intersects both the right and left sides of D . So, $\pi_1|_K$ maps K onto $[0,1]$ and is therefore universal. Hence, there is a point $x \in K$ such that $\pi_1(x) = \pi_1 f(x)$. Since $x \in K \subset H$, $\pi_2(x) = \pi_2 f(x)$. Thus, x is a fixed point for f .

Haskell Cohen [2] used an argument of a similar nature to show that the product of ordered spaces has the fixed point property.

We generalize this property possessed by D to a property which holds for a large class of spaces. In a manner similar to the one outlined above, we establish several fixed point results.

A *continuum* is a nondegenerate compact connected metric space. A continuous function will be referred to as a *map* or *mapping*. A continuum X has the *fixed point property* provided that whenever f is a mapping of X into X , there is a point x in X such that $f(x) = x$. A mapping $f: X \rightarrow Y$ is said to be *universal* provided that whenever $g: X \rightarrow Y$ is a mapping, there is a point $x \in X$ such that $f(x) = g(x)$. In [1], R. E. Basye defined the terms "weakly disconnect," "simple connected," and "simply connected in the weak sense." A definition similar to "weakly disconnect" has been used by F. B. Jones with different terminology. We adopt Jones' terminology and introduce some new definitions of a similar nature.

Let A and B be closed disjoint subsets of the connected topological space X . The closed set H *cuts* A *from* B in X provided that no component of $X - H$ intersects both A and B . The closed set H *cuts weakly between* A *and* B in X provided that whenever C is a closed connected set in X that intersects each of A and B , then C intersects H . Notice that, in our definition of *cuts weakly*, H may intersect $A \cup B$. We say that X *is s-connected between* A *and* B provided that whenever H is a closed set in X that *cuts weakly between* A *and* B , then some component K of H *cuts weakly between* A *and* B . A connected space X is said to be *s-connected* provided that whenever A and B are disjoint closed connected subsets of X , then X is *s-connected between* A *and* B .

We notice that, in a locally connected metric space, the properties of "cutting" and "cutting weakly" coincide.

As an immediate consequence of Theorem 4 in [1], we have

Lemma 1. If a metric space is s-connected, it is unicoherent.

The following theorem generalizes, in the compact case, results of Haskell Cohen [2, Lemma 2] and R. E. Basye [1, Theorem 6].

Theorem 1. A locally connected continuum is unicoherent if and only if it is s-connected.

Proof. Let X be a locally connected continuum. If X is s-connected, by Lemma 1 it is unicoherent.

Suppose that X is unicoherent. We will show that X is s-connected. Let A and B be disjoint continua in X and let H be a closed set that cuts weakly between A and B . Suppose no component of H cuts weakly between A and B . Let $\{W_i\}_{i=1}^{\infty}$ be the components of $X - H$. Since X is locally connected, each W_i is a connected open set. Also, no W_i intersects both A and B , for otherwise, we could construct an arc in W_i which intersects both A and B , contradicting the fact that H cuts weakly between A and B .

Let $C_0 = X$ and $U_1 = W_1$. Since U_1 does not intersect both A and B , one of A or B must be contained in a component C_1 of $C_0 - U_1$. Now, C_1 cuts A from B . Proceeding by induction, assume that continua $\{C_i\}_{i=0}^k$ and connected open sets $\{U_i\}_{i=1}^k$ have been defined so that, for $1 \leq i \leq k$,

- i) $U_i = W_j$ for some $j \geq 1$,
- ii) $U_i \subset C_{i-1}$,
- iii) C_i is a component of $C_{i-1} - U_i$, and
- iv) C_i cuts A from B.

Notice that each C_i is a component of $X - \bigcup_{r=1}^i U_r$.

Suppose that, for some $j \geq 1$, $W_j \cap C_k \neq \emptyset$. Then $C_k \cup W_j$ is a subset of $X - \bigcup_{r=1}^k U_r$. Since C_k is a component of $X - \bigcup_{r=1}^k U_r$, it follows that $W_j \subset C_k$.

Now, by assumption, $C_k \not\subset H$; so, let U_{k+1} be the first member of $\{W_i\}_{i=1}^{\infty}$ such that $U_{k+1} \cap C_k \neq \emptyset$. Then we have that $U_{k+1} \subset C_k$. Now, U_{k+1} cannot intersect both A and B. Assume that $A \cap U_{k+1} = \emptyset$. If $A \cap (C_k - U_{k+1}) \neq \emptyset$, let C_{k+1} be any component of $C_k - U_{k+1}$ that intersects A. Otherwise, let I_A be an irreducible continuum from A to C_k and let C_{k+1} be any component of $C_k - U_{k+1}$ that intersects I_A . We need to show that C_{k+1} cuts A from B in order to complete the inductive step. Suppose that C_{k+1} does not cut A from B in X. Let Q be a continuum in $X - C_{k+1}$ that intersects both A and B. Since C_k cuts A from B, Q must intersect C_k . Let I_Q be an irreducible continuum in Q from A to C_k (I_Q could be degenerate). Since $Q \cap C_{k+1} = \emptyset$, $I_Q \cap C_{k+1} = \emptyset$. So, I_Q intersects a different component of $C_k - U_{k+1}$ than C_{k+1} . Let R be the component of $X - U_{k+1}$ that contains A. If C_{k+1} intersects A, then R intersects C_{k+1} . If $C_{k+1} \cap A = \emptyset$, then R intersects $C_{k+1} \cup I_A$. In either case, we have that R intersects two components of $C_k - U_{k+1}$. So, $R \not\subset C_k$. Let S be the union of C_k and all components (if any) of $X - U_{k+1}$ other than R. Now, R and S are continua

whose union is X . Also, $R - S \neq \emptyset$ and $S - R \neq \emptyset$. By the unicoherence of X , $R \cap S$ must be a continuum. But $R \cap S \subset C_k - U_{k+1}$ and $R \cap S$ intersects two components of $C_k - U_{k+1}$, which is a contradiction. Hence, C_{k+1} cuts A from B and the induction step is complete.

Let $C = \bigcap_{i=1}^{\infty} C_i$. Now, since each C_i cuts A from B , it follows that C is a continuum which cuts A from B . We claim that $C \subset X - \bigcup_{i=1}^{\infty} W_i$. Let $x \in C$ and suppose there is an integer j such that $x \in W_j$. Then $x \in W_j \cap C_i$ for each $i \geq 1$. As we have previously seen, W_j must be a subset of C_i for each $i \geq 1$; i.e., $W_j \subset C$. However, there must be an integer $n \geq 1$ such that W_j is the first member of $\{W_i\}_{i=1}^{\infty}$ such that $W_j \subset C_n$. By construction of the C_i 's, C_{n+1} is a component of $C_n - W_j$. But then $W_j \cap C_{n+1} = \emptyset$, a contradiction. Hence, C is a subcontinuum of H that cuts A from B . This contradicts our original assumption. Thus, X is s -connected.

We are indebted to Eldon J. Vought and E. E. Grace who suggested the proof above, which greatly simplified the original proof of the author.

Theorem 2. In a metric space X , let $\{S_i\}_{i=1}^{\infty}$, $\{A_i\}_{i=1}^{\infty}$, and $\{B_i\}_{i=1}^{\infty}$ be monotonic decreasing sequences of continua with respective intersections S , A , and B . Suppose that, for each $i \geq 1$, A_i and B_i are disjoint, $A_i \cup B_i \subset S_i$, and S_i is s -connected between A_i and B_i . Then S is s -connected between A and B .

Proof. Let F be a closed set in S that cuts weakly between A and B in S . Let $\{D_i\}_{i=1}^{\infty}$ be a sequence of open

sets in X whose intersection is F . For each $i \geq 1$, the closed set \bar{D}_i cuts weakly between A_j and B_j in S_j for some $j \geq i$. For suppose otherwise. Then, for each $j \geq i$, there is a continuum C_j in S_j which intersects each of A_j and B_j but does not intersect \bar{D}_i . Some subsequence of $\{C_j\}_{j=i}^{\infty}$ has a sequential limiting set C . The set C is a subcontinuum of S which intersects each of A and B . Since F is a subset of D_i , it follows that C does not intersect F . But this contradicts the fact that F cuts weakly between A and B in S .

For each $i \geq 1$, let S_{n_i} be the first member of $\{S_j\}_{j=1}^{\infty}$ such that $n_i \geq i$ and \bar{D}_i cuts weakly between A_{n_i} and B_{n_i} in S_{n_i} . Since each S_{n_i} is s -connected between A_{n_i} and B_{n_i} , there is a component d_i of $S_{n_i} \cap \bar{D}_i$ that cuts weakly between A_{n_i} and B_{n_i} in S_{n_i} . Some subsequence of $\{d_i\}_{i=1}^{\infty}$ has a sequential limiting set d . Now, $d \subset F$ and d cuts weakly between A and B in S . For suppose that L is a subcontinuum of S which intersects each of A and B but does not intersect d . Then there is an integer r such that d_r and L are disjoint. Since L intersects each of A and B , it follows that L intersects each of A_j and B_j for each $j \geq i$. Also, recall that $L \subset S$. This implies that d_r does not cut weakly between A_{n_r} and B_{n_r} in S_{n_r} , which is a contradiction. Thus, S is s -connected between A and B .

Corollary 2.1. In a metric space, if \mathcal{G} is a monotonic decreasing sequence of continua each of which is s -connected, then $\bigcap \mathcal{G}$ is s -connected.

Corollary 2.2. Every plane continuum which does not separate the plane is s-connected.

Proof. Since each nonseparating planar continuum is a countable intersection of nested topological disks, this corollary follows from Theorem 1 and Corollary 2.1.

Theorem 3. If X is an inverse limit of absolute retracts, then X is s-connected.

Proof. It is a well known fact that if X is an inverse limit of absolute retracts, then X is the intersection of a monotonic decreasing sequence of absolute retracts (see [7, Lemmas 1.152 & 1.153]). Since absolute retracts are unicoherent and locally connected, this result follows from Theorem 1 and Corollary 2.1.

Theorem 4. The monotone image of an s-connected space is s-connected.

Proof. The proof is straightforward and is omitted.

The property of being s-connected is especially useful in obtaining fixed point results in certain spaces. In particular, we will demonstrate a general procedure which works nicely in cones and products. The theorems we establish generalize existing results in this area.

Let X be a continuum. We say that Z is the cone over X if $Z = X \times [0,1] /_{X \times \{1\}}$. The surjective semi-span of X is zero, denoted by $\sigma_0^*(X) = 0$, provided that whenever C is a continuum in $X \times X$ such that $\pi_1(C) = X$, then C intersects the diagonal in $X \times X$. We will say that a continuum X is tree-like (arc-like) provided that X is

an inverse limit of trees (arcs); see [7, 1.162 & 1.163] and [6].

Theorem 5. If $\sigma_0^*(X) = 0$ and Z is the cone over X , then Z has the fixed point property.

Proof. Let $\eta: X \times [0,1] \rightarrow Z$ be the identification mapping and let $v = \eta(X \times \{1\})$. Let $\pi_1: Z - \{v\} \rightarrow X$ and $\pi_2: Z \rightarrow [0,1]$ be the natural projection mappings.

Suppose that $f: Z \rightarrow Z$ is a fixed point free mapping. Let $H = \{z \in Z \mid \pi_2 f(z) = \pi_2(z)\}$. The set H is not empty since $\pi_2: Z \rightarrow [0,1]$ is a universal mapping.

Suppose that $v \in H$. Then $\pi_2 f(v) = \pi_2(v) = 1$. But then $f(v) = v$, which is a contradiction. So, $v \notin H$. Similarly, $v \notin f(H)$.

Suppose there is a continuum C in $Z - H$ that intersects both $\{v\}$ and $X \times \{0\}$. Then $\pi_2(C) = [0,1]$. Since $C \subset Z - H$, we may write C as a union of sets

$$R = \{z \in C \mid \pi_2 f(z) > \pi_2(z)\} \text{ and}$$

$$S = \{z \in C \mid \pi_2 f(z) < \pi_2(z)\}.$$

Now, $v \in S$, $C \cap (X \times \{0\}) \subset R$, and each of R and S is an open set relative to C . This contradicts the fact that C is connected. Hence, H cuts weakly between $\{v\}$ and $X \times \{0\}$ in Z .

Since $\sigma_0^*(X) = 0$, X is tree-like. Now, Z can be realized as an inverse limit of cones over trees. Hence, by Theorem 3, Z is s -connected. So, there is a continuum K in H that cuts weakly between $\{v\}$ and $X \times \{0\}$.

Suppose that $x \in X$ and $\{x\} \times [0,1]$ does not intersect K . Then $\{x\} \times [0,1]$ is a continuum that intersects

both $\{v\}$ and $X \times \{0\}$ but does not intersect K , a contradiction. So, $\pi_1(K) = X$. Since $\sigma_0^*(X) = 0$, $\pi_1: K \rightarrow X$ is universal. Hence, there is a point $z \in K$ such that $\pi_1(z) = \pi_1 f(z)$. Also, since $z \in H$, $\pi_2(z) = \pi_2 f(z)$. We have that $z = f(z)$, which is a contradiction.

Corollary 5.1. If X is either

- (1) *weakly chainable and in Class(W), or*
- (2) *weakly chainable and tree-like,*

and $\sigma_0(Y) = 0$ for each proper subcontinuum Y of X , then the cone over X has the fixed point property.

Proof. Oversteegen and Tymchatyn [8] have shown that $\sigma_0(X) = 0$ in each of the cases listed above. Since $\sigma_0(X) = 0$ implies that $\sigma_0^*(X) = 0$, the result follows immediately from Theorem 6.

Jack Segal [10] and J. T. Rogers, Jr. [9] have shown that the hyperspace of subcontinua of an arc-like continuum has the fixed point property. Rogers' proof works equally well for the cone over an arc-like continuum. As a corollary to Theorem 5, we get Rogers' result.

Corollary 5.2. The cone over an arc-like continuum has the fixed point property.

The proof of the next theorem is similar to the proof of Theorem 5. However, a few modifications are necessary. Also, the method of proof further illustrates the general procedure mentioned in the introduction of this paper.

Theorem 6. If $\sigma_0^*(X) = 0$, Y is arc-like, and $Z = X \times Y$, then Z has the fixed point property.

Proof. Suppose that $f: Z \rightarrow Z$ is a fixed point free mapping. Let ρ be a metric for Z and d a metric for Y . Let ϵ be a positive number such that $\rho(z, f(z)) \geq \epsilon$ for $z \in Z$. Since Y is arc-like there is a mapping $g: Y \rightarrow [0, 1]$ such that, for each $t \in [0, 1]$, $\text{diam}(g^{-1}(t)) < \epsilon$. We refer to g as an ϵ -map.

Let $H = \{z \in Z \mid g\pi_2(z) = g\pi_2 f(z)\}$. The set H is not empty since $g\pi_2: Z \rightarrow [0, 1]$ is a universal mapping. Let $p \in g^{-1}(0)$ and $q \in g^{-1}(1)$. Let $X_p = X \times \{p\}$ and $X_q = X \times \{q\}$.

Suppose there is a continuum C in $Z - H$ that intersects both X_p and X_q . Then $g\pi_2(C) = [0, 1]$ and C is the union of sets

$$R = \{z \in C \mid g\pi_2 f(z) > g\pi_2(z)\} \text{ and}$$

$$S = \{z \in C \mid g\pi_2 f(z) < g\pi_2(z)\}.$$

Now, $C \cap X_p \subset R$, $C \cap X_q \subset S$, and each of R and S is an open set relative to C . This contradicts the fact that C is connected. Hence, H cuts weakly between X_p and X_q in Z .

As in Theorem 5, Z is s -connected. So, there is a continuum K in H that cuts weakly between X_p and X_q .

Suppose that $x \in X$ and $\{x\} \times Y$ does not intersect K . Then $\{x\} \times Y$ is a continuum that intersects both X_p and X_q but does not intersect K , a contradiction. So, $\pi_1(K) = X$. Since $\pi_1: K \rightarrow X$ is universal, there is a point $z \in K$ such that $\pi_1(z) = \pi_1 f(z)$. Also, since $z \in H$, $g\pi_2(z) = g\pi_2 f(z)$. Since g is an ϵ -map, it follows that $d(\pi_2(z), \pi_2 f(z)) < \epsilon$. But then $\rho(z, f(z)) < \epsilon$, which is a contradiction.

Let D be the unit disk in the plane with polar coordinates; i.e., $D = \{(r, \theta) \mid 0 \leq r \leq 1\}$. Let $\pi: D - \{(0,0)\} \rightarrow S^1$ be radial projection and let $\alpha: D \rightarrow [0,1]$ be projection into the first coordinate.

A mapping f from a continuum X onto D is said to be *AH-essential* provided that $f|_{f^{-1}(S^1)}: f^{-1}(S^1) \rightarrow S^1$ cannot be extended to a mapping $F: X \rightarrow S^1$.

Theorem 7. Suppose that $X = \varprojlim \{X_i, g_i^{i+1}\}$, where for each $i \geq 1$, $X_i = D$, $g_i^{i+1}(S^1) = S^1$, and $(g_i^{i+1})^{-1}(0,0) = \{(0,0)\}$. Suppose also that for each $i \geq 1$, $g_i^{i+1}\pi = \pi g_i^{i+1}$ on $D - \{(0,0)\}$. Then X has the fixed point property.

Proof. For each $i \geq 1$, let $g_i: X \rightarrow X_i$ be projection onto the i^{th} coordinate. Let v be the point of X such that $g_i(v) = (0,0)$ for each $i \geq 1$. We let d denote the metric on X . Also, we write S^1 for each set $\{x \in X_i \mid \alpha(x) = 1\}$.

Suppose there is an integer m such that if $n \geq m$, then $g_n|_{g_n^{-1}(S^1)}: g_n^{-1}(S^1) \rightarrow S^1$ is essential. We claim that, for $n \geq m$, $g_n: X \rightarrow D$ is AH-essential. Suppose that g_n is not AH-essential. Let $g: X \rightarrow S^1$ be an extension of $g_n|_{g_n^{-1}(S^1)}$. Since X is disk-like and Čech cohomology with integer coefficients is continuous, it follows that $H^1(X) \approx 0$. By [3, 8.1], g is inessential. Since g is an extension of $g_n|_{g_n^{-1}(S^1)}$, $g_n|_{g_n^{-1}(S^1)}$ is inessential. But this contradicts our assumption. So, for $n \geq m$, $g_n: X \rightarrow D$ is AH-essential and by [5] g_n is universal. It follows from Lemma 1 in [4] that X has the fixed point property.

Suppose for each positive integer m , there is an $n > m$ such that g'_n is inessential. Suppose that $f: X \rightarrow X$ is a fixed point free mapping and ϵ is a positive number such that $d(x, f(x)) \geq \epsilon$ for each $x \in X$. Let n be an integer such that g_n is an ϵ -map and g'_n is inessential.

Let $H = \{x \in X \mid \alpha g_n(x) = \alpha g_n f(x)\}$. The set H is not empty since αg_n is universal. Let $X_0 = \lim_{\leftarrow} \{S^1, g_i^{i+1} \mid S^1\}$. Then X_0 is a subcontinuum of X and $X_0 \subset g_n^{-1}(S^1)$. Now, as in the proof of Theorem 5, H cuts weakly between $\{v\}$ and X_0 . Since X is s -connected, there is a continuum K in H that cuts weakly between $\{v\}$ and X_0 .

Since g'_n is inessential, so is $g'_n|_{X_0}$. Let $\hat{g}_n = g'_n|_{X_0}$ and let $\psi: X_0 \rightarrow E^1$ be a mapping such that $\hat{g}_n(x) = e^{i\psi(x)}$ for each x in X_0 . Let $\eta: X - \{v\} \rightarrow X_0$ be defined by $g_i \eta(x) = \pi g_i(x)$ for each $i \geq 1$. Now, $\psi(X_0)$ is an arc or a point; so, $\psi|_K: K \rightarrow \psi(X_0)$ is universal. Hence, there is a point $x \in K$ such that $\psi \eta(x) = \psi \eta f(x)$. Thus,

$$\hat{g}_n(\eta(x)) = e^{i\psi \eta(x)} = e^{i\psi \eta f(x)} = \hat{g}_n(\eta f(x)).$$

By definition of η , this gives us that $\pi g_n(x) = \pi g_n f(x)$. Since $x \in K$, $\alpha g_n(x) = \alpha g_n f(x)$. These last two equalities give us that $g_n(x) = g_n f(x)$. But then $d(x, f(x)) < \epsilon$, which is a contradiction.

J. T. Rogers, Jr. [9] has shown that the hyperspace of subcontinua of a circle-like continuum has the fixed point property. Again, Rogers' proof works equally well for the cone over a circle-like continuum. We get Rogers' result as a corollary to Theorem 7.

Corollary 7.1. The cone over a circle-like continuum has the fixed point property.

Proof. Suppose that X is the cone over a circle-like continuum; let $X = \text{cone}(X_0)$, where $X_0 = \varprojlim\{S_i, f_i^{i+1}\}$ with each $S_i = S^1$. Now, X is homeomorphic to $\varprojlim\{\text{cone}(S_i), f_i^{i+1} \times \text{id}\}$. For each $i \geq 1$, $\text{cone}(S_i)$ is homeomorphic to D with the vertex of $\text{cone}(S_i)$ mapping to $(0,0)$. The other conditions in the hypothesis of Theorem 7 follow easily. Hence, X has the fixed point property.

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