TOPOLOGY PROCEEDINGS

Volume 8, 1983

Pages 85–97

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

s-CONNECTED SPACES AND THE FIXED POINT PROPERTY

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We wish to establish a general procedure for showing that certain spaces have the fixed point property. In particular, we will be interested in spaces which resemble products.

Consider the topological disk $D = [0,1] \times [0,1]$. It is a well known property of D that if H is a closed subset of D then either some component of D - H intersects both the top and bottom of D or some component of H intersects both the right and left sides of D. One can use this property to show that D has the fixed point property. The argument goes like this. Let $f: D \to D$ be a continuous function. Let $H = \{x \in D | \pi_2(x) = \pi_2 f(x)\}$. The set H is non-empty since $\pi_2: D \to [0,1]$ is universal. No component C of D - H can intersect both top and bottom, for otherwise we could write C as a union of mutually separated sets $\{x \in C | \pi_2(x) < \pi_2 f(x)\}$ and $\{x \in C | \pi_2(x) > \pi_2 f(x)\}$. Thus, some component K of H intersects both the right and left sides of D. So, $\pi_1|_K$ maps K onto [0,1] and is therefore universal. Hence, there is a point $x \in K$ such that $\pi_1(x) = \pi_1 f(x)$. Since $x \in K \subset H$, $\pi_2(x) = \pi_2 f(x)$. Thus, x is a fixed point for f.

Haskell Cohen [2] used an argument of a similar nature to show that the product of ordered spaces has the fixed point property.

We generalize this property possessed by D to a property which holds for a large class of spaces. In a manner similar to the one outlined above, we establish several fixed point results.

A *continuum* is a nondegenerate compact connected metric space. A continuous function will be referred to as a *map* or *mapping*. A continuum X has the *fixed point property* provided that whenever f is a mapping of X into X, there is a point x in X such that f(x) = x. A mapping f: X → Y is said to be *universal* provided that whenever g: X → Y is a mapping, there is a point x ∈ X such that f(x) = g(x). In [1], R. E. Basye defined the terms "weakly disconnect," "simple connected," and "simply connected in the weak sense." A definition similar to "weakly disconnect" has been used by F. B. Jones with different terminology. We adopt Jones' terminology and introduce some new definitions of a similar nature.

Let A and B be closed disjoint subsets of the connected topological space X. The closed set H *cuts* A *from* B in X provided that no component of X - H intersects both A and B. The closed set H *cuts weakly between* A *and* B in X provided that whenever C is a closed connected set in X that intersects each of A and B, then C intersects H. Notice that, in our definition of cuts weakly, H may intersect A ∪ B. We say that X *is* s-*connected between* A *and* B provided that whenever H is a closed set in X that cuts weakly between A and B, then some component K of H cuts weakly between A and B. A connected space X is said to be s-*connected* provided that whenever A and B are disjoint closed connected subsets of X, then X is s-connected between A and B.

We notice that, in a locally connected metric space, the properties of "cutting" and "cutting weakly" coincide.

As an immediate consequence of Theorem 4 in [1], we have

Lemma 1. *If a metric space is* s-*connected, it is unicoherent.*

The following theorem generalizes, in the compact case, results of Haskell Cohen [2, Lemma 2] and R. E. Basye [1, Theorem 6].

Theorem 1. *A locally connected continuum is unicoherent if and only if it is* s-*connected.*

Proof. Let X be a locally connected continuum. If X is s-connected, by Lemma 1 it is unicoherent.

Suppose that X is unicoherent. We will show that X is s-connected. Let A and B be disjoint continua in X and let H be a closed set that cuts weakly between A and B. Suppose no component of H cuts weakly between A and B. Let $\{W_i\}_{i=1}^{\infty}$ be the components of X - H. Since X is locally connected, each W_i is a connected open set. Also, no W_i intersects both A and B, for otherwise, we could construct an arc in W_i which intersects both A and B, contradicting the fact that H cuts weakly between A and B.

Let $C_0 = X$ and $U_1 = W_1$. Since U_1 does not intersect both A and B, one of A or B must be contained in a component C_1 of $C_0 - U_1$. Now, C_1 cuts A from B. Proceeding by induction, assume that continua $\{C_i\}_{i=0}^{k}$ and connected open sets $\{U_i\}_{i=1}^{k}$ have been defined so that, for $1 \leq i \leq k$,

i) $U_i = W_j$ for some $j \geq 1$,

ii) $U_i \subset C_{i-1}$,

iii) C_i is a component of $C_{i-1} - U_i$, and

iv) C_i cuts A from B.

Notice that each C_i is a component of $X - \cup_{r=1}^{i} U_r$.

Suppose that, for some $j \geq 1$, $W_j \cap C_k \neq \emptyset$. Then $C_k \cup W_j$ is a subset of $X - \cup_{r=1}^{k} U_r$. Since C_k is a component of $X - \cup_{r=1}^{k} U_r$, it follows that $W_j \subset C_k$.

Now, by assumption, $C_k \not\subset H$; so, let U_{k+1} be the first member of $\{W_i\}_{i=1}^{\infty}$ such that $U_{k+1} \cap C_k \neq \emptyset$. Then we have that $U_{k+1} \subset C_k$. Now, U_{k+1} cannot intersect both A and B. Assume that $A \cap U_{k+1} = \emptyset$. If $A \cap (C_k - U_{k+1}) \neq \emptyset$, let C_{k+1} be any component of $C_k - U_{k+1}$ that intersects A. Otherwise, let I_A be an irreducible continuum from A to C_k and let C_{k+1} be any component of $C_k - U_{k+1}$ that intersects I_A. We need to show that C_{k+1} cuts A from B in order to complete the inductive step. Suppose that C_{k+1} does not cut A from B in X. Let Q be a continuum in $X - C_{k+1}$ that intersects both A and B. Since C_k cuts A from B, Q must intersect C_k. Let I_Q be an irreducible continuum in Q from A to C_k (I_Q could be degenerate). Since $Q \cap C_{k+1} = \emptyset$, $I_Q \cap C_{k+1} = \emptyset$. So, I_Q intersects a different component of $C_k - U_{k+1}$ than C_{k+1}. Let R be the component of $X - U_{k+1}$ that contains A. If C_{k+1} intersects A, then R intersects C_{k+1}. If $C_{k+1} \cap A = \emptyset$, then R intersects $C_{k+1} \cup I_A$. In either case, we have that R intersects two components of $C_k - U_{k+1}$. So, $R \not\subset C_k$. Let S be the uinon of C_k and all components (if any) of $X - U_{k+1}$ other than R. Now, R and S are continua whose union is X. Also, $R - S \neq \emptyset$ and $S - R \neq \emptyset$. By the unicoherence of X, $R \cap S$ must be a continuum. But $R \cap S \subset C_k - U_{k+1}$ and $R \cap S$ intersects two components of $C_k - U_{k+1}$, which is a contradiction. Hence, C_{k+1} cuts A from B and the induction step is complete.

Let $C = \cap_{i=1}^{\infty} C_i$. Now, since each C_i cuts A from B, it follows that C is a continuum which cuts A from B. We claim that $C \subset X - \cup_{i=1}^{\infty} W_i$. Let $x \in C$ and suppose there is an integer j such that $x \in W_j$. Then $x \in W_j \cap C_i$ for each $i \geq 1$. As we have previously seen, W_j must be a subset of C_i for each $i \geq 1$; i.e., $W_j \subset C$. However, there must be an integer $n \geq 1$ such that W_j is the first member of $\{W_i\}_{i=1}^{\infty}$ such that $W_j \subset C_n$. By construction of the C_i's, C_{n+1} is a component of $C_n - W_j$. But then $W_j \cap C_{n+1} = \emptyset$, a contradiction. Hence, C is a subcontinuum of H that cuts A from B. This contradicts our original assumption. Thus, X is s-connected.

We are indebted to Eldon J. Vought and E. E. Grace who suggested the proof above, which greatly simplified the original proof of the author.

Theorem 2. *In a metric space* X, *let* $\{S_i\}_{i=1}^{\infty}$, $\{A_i\}_{i=1}^{\infty}$, *and* $\{B_i\}_{i=1}^{\infty}$ *be monotonic decreasing sequences of continua with respective intersections* S, A, *and* B. *Suppose that, for each* $i \geq 1$, A_i *and* B_i *are disjoint*, $A_i \cup B_i \subset S_i$, *and* S_i *is* s-*connected between* A_i *and* B_i. *Then* S *is* s-*connected between* A *and* B.

Proof. Let F be a closed set in S that cuts weakly between A and B in S. Let $\{D_i\}_{i=1}^{\infty}$ be a sequence of open sets in X whose intersection is F. For each $i \geq 1$, the closed set $\overline{D}_i$ cuts weakly between A_j and B_j in S_j for some $j \geq i$. For suppose otherwise. Then, for each $j \geq i$, there is a continuum C_j in S_j which intersects each of A_j and B_j but does not intersect $\overline{D}_i$. Some subsequence of $\{C_j\}_{j=i}^{\infty}$ has a sequential limiting set C. The set C is a subcontinuum of S which intersects each of A and B. Since F is a subset of D_i, it follows that C does not intersect F. But this contradicts the fact that F cuts weakly between A and B in S.

For each $i \geq 1$, let S_{n_i} be the first member of $\{S_j\}_{j=1}^{\infty}$ such that $n_i \geq i$ and $\overline{D}_i$ cuts weakly between A_{n_i} and B_{n_i} in S_{n_i}. Since each S_{n_i} is s-connected between A_{n_i} and B_{n_i}, there is a component d_i of $S_{n_i} \cap \overline{D}_i$ that cuts weakly between A_{n_i} and B_{n_i} in S_{n_i}. Some subsequence of $\{d_i\}_{i=1}^{\infty}$ has a sequential limiting set d. Now, $d \subset F$ and d cuts weakly between A and B in S. For suppose that L is a subcontinuum of S which intersects each of A and B but does not intersect d. Then there is an integer r such that d_r and L are disjoint. Since L intersects each of A and B, it follows that L intersects each of A_j and B_j for each $j \geq i$. Also, recall that $L \subset S$. This implies that d_r does not cut weakly between A_{n_r} and B_{n_r} in S_{n_r}, which is a contradiction. Thus, S is s-connected between A and B.

Corollary 2.1. *In a metric space, if* $\mathcal{G}$ *is a monotonic decreasing sequence of continua each of which is* s-*connected, then* $\cap\mathcal{G}$ *is* s-*connected.*

Corollary 2.2. *Every plane continuum which does not separate the plane is* s-*connected.*

Proof. Since each nonseparating planar continuum is a countable intersection of nested topological disks, this corollary follows from Theorem 1 and Corollary 2.1.

Theorem 3. *If* X *is an inverse limit of absolute retracts, then* X *is* s-*connected.*

Proof. It is a well known fact that if X is an inverse limit of absolute retracts, then X is the intersection of a monotonic decreasing sequence of absolute retracts (see [7, Lemmas 1.152 & 1.153]). Since absolute retracts are unicoherent and locally connected, this result follows from Theorem 1 and Corollary 2.1.

Theorem 4. *The monotone image of an* s-*connected space is* s-*connected.*

Proof. The proof is straightforward and is omitted.

The property of being s-connected is especially useful in obtaining fixed point results in certain spaces. In particular, we will demonstrate a general procedure which works nicely in cones and products. The theorems we establish generalize existing results in this area.

Let X be a continuum. We say that Z *is the cone over* X if $Z = X \times [0,1]/_{X\times\{1\}}$. The *surjective semi-span of* X *is zero*, denoted by $\sigma_0^*(X) = 0$, provided that whenever C is a continuum in $X \times X$ such that $\pi_1(C) = X$, then C intersects the diagonal in $X \times X$. We will say that a continuum X is *tree-like (arc-like)* provided that X is an inverse limit of trees (arcs); see [7, 1.162 & 1.163] and [6].

Theorem 5. *If* $\sigma_0^*(X) = 0$ *and* Z *is the cone over* X, *then* Z *has the fixed point property.*

Proof. Let $\eta: X \times [0,1] \to Z$ be the identification mapping and let $v = \eta(X \times \{1\})$. Let $\pi_1: Z - \{v\} \to X$ and $\pi_2: Z \to [0,1]$ be the natural projection mappings.

Suppose that $f: Z \to Z$ is a fixed point free mapping. Let $H = \{z \in Z | \pi_2 f(z) = \pi_2(z)\}$. The set H is not empty since $\pi_2: Z \to [0,1]$ is a universal mapping.

Suppose that $v \in H$. Then $\pi_2 f(v) = \pi_2(v) = 1$. But then $f(v) = v$, which is a contradiction. So, $v \notin H$. Similarly, $v \notin f(H)$.

Suppose there is a continuum C in Z - H that intersects both $\{v\}$ and $X \times \{0\}$. Then $\pi_2(C) = [0,1]$. Since $C \subset Z - H$, we may write C as a union of sets

$$R = \{z \in C | \pi_2 f(z) > \pi_2(z)\} \text{ and}$$
$$S = \{z \in C | \pi_2 f(z) < \pi_2(z)\}.$$

Now, $v \in S$, $C \cap (X \times \{0\}) \subset R$, and each of R and S is an open set relative to C. This contradicts the fact that C is connected. Hence, H cuts weakly between $\{v\}$ and $X \times \{0\}$ in Z.

Since $\sigma_0^*(X) = 0$, X is tree-like. Now, Z can be realized as an inverse limit of cones over trees. Hence, by Theorem 3, Z is s-connected. So, there is a continuum K in H that cuts weakly between $\{v\}$ and $X \times \{0\}$.

Suppose that $x \in X$ and $\{x\} \times [0,1]$ does not intersect K. Then $\{x\} \times [0,1]$ is a continuum that intersects both $\{v\}$ and $X \times \{0\}$ but does not intersect K, a contradiction. So, $\pi_1(K) = X$. Since $\sigma_0^*(X) = 0$, $\pi_1: K \to X$ is universal. Hence, there is a point $z \in K$ such that $\pi_1(z) = \pi_1 f(z)$. Also, since $z \in H$, $\pi_2(z) = \pi_2 f(z)$. We have that $z = f(z)$, which is a contradiction.

Corollary 5.1. *If* X *is either*

(1) *weakly chainable and in Class(W), or*

(2) *weakly chainable and tree-like,*

and $\sigma_0(Y) = 0$ *for each proper subcontinuum* Y *of* X, *then the cone over* X *has the fixed point property.*

Proof. Oversteegen and Tymchatyn [8] have shown that $\sigma_0(X) = 0$ in each of the cases listed above. Since $\sigma_0(X) = 0$ implies that $\sigma_0^*(X) = 0$, the result follows immediately from Theorem 6.

Jack Segal [10] and J. T. Rogers, Jr. [9] have shown that the hyperspace of subcontinua of an arc-like continuum has the fixed point property. Rogers' proof works equally well for the cone over an arc-like continuum. As a corollary to Theorem 5, we get Rogers' result.

Corollary 5.2. *The cone over an arc-like continuum has the fixed point property.*

The proof of the next theorem is similar to the proof of Theorem 5. However, a few modifications are necessary. Also, the method of proof further illustrates the general procedure mentioned in the introduction of this paper.

Theorem 6. *If* $\sigma_0^*(X) = 0$, Y *is arc-like, and* $Z = X \times Y$, *then* Z *has the fixed point property.*

Proof. Suppose that $f\colon Z \to Z$ is a fixed point free mapping. Let ρ be a metric for Z and d a metric for Y. Let ε be a positive number such that $\rho(z,f(z)) \geq \varepsilon$ for $z \in Z$. Since Y is arc-like there is a mapping $g\colon Y \to [0,1]$ such that, for each $t \in [0,1]$, $\operatorname{diam}(g^{-1}(t)) < \varepsilon$. We refer to g as an ε-map.

Let $H = \{z \in Z \mid g\pi_2(z) = g\pi_2 f(z)\}$. The set H is not empty since $g\pi_2\colon Z \to [0,1]$ is a universal mapping. Let $p \in g^{-1}(0)$ and $q \in g^{-1}(1)$. Let $X_p = X \times \{p\}$ and $X_q = X \times \{q\}$.

Suppose there is a continuum C in $Z - H$ that intersects both X_p and X_q. Then $g\pi_2(C) = [0,1]$ and C is the union of sets

$$R = \{z \in C \mid g\pi_2 f(z) > g\pi_2(z)\} \text{ and}$$
$$S = \{z \in C \mid g\pi_2 f(z) < g\pi_2(z)\}.$$

Now, $C \cap X_p \subset R$, $C \cap X_q \subset S$, and each of R and S is an open set relative to C. This contradicts the fact that C is connected. Hence, H cuts weakly between X_p and X_q in Z.

As in Theorem 5, Z is s-connected. So, there is a continuum K in H that cuts weakly between X_p and X_q.

Suppose that $x \in X$ and $\{x\} \times Y$ does not intersect K. Then $\{x\} \times Y$ is a continuum that intersects both X_p and X_q but does not intersect K, a contradiction. So, $\pi_1(K) = X$. Since $\pi_1\colon K \to X$ is universal, there is a point $z \in K$ such that $\pi_1(z) = \pi_1 f(z)$. Also, since $z \in H$, $g\pi_2(z) = g\pi_2 f(z)$. Since g is an ε-map, it follows that $d(\pi_2(z),\pi_2 f(z)) < \varepsilon$. But then $\rho(z,f(z)) < \varepsilon$, which is a contradiction.

Let D be the unit disk in the plane with polar coordinates; i.e., $D = \{(r,\theta) | 0 \leq r \leq 1\}$. Let $\pi: D - \{(0,0)\} \to S^1$ be radial projection and let $\alpha: D \to [0,1]$ be projection into the first coordinate.

A mapping f from a continuum X onto D is said to be AH-*essential* provided that $f|_{f^{-1}(S^1)}: f^{-1}(S^1) \to S^1$ cannot be extended to a mapping $F: X \to S^1$.

Theorem 7. *Suppose that* $X = \varprojlim\{X_i, g_i^{i+1}\}$, *where for each* $i \geq 1$, $X_i = D$, $g_i^{i+1}(S^1) = S^1$, *and* $(g_i^{i+1})^{-1}(0,0) = \{(0,0)\}$. *Suppose also that for each* $i \geq 1$, $g_i^{i+1}\pi = \pi g_i^{i+1}$ *on* $D - \{(0,0)\}$. *Then* X *has the fixed point property.*

Proof. For each $i \geq 1$, let $g_i: X \to X_i$ be projection onto the i^{th} coordinate. Let v be the point of X such that $g_i(v) = (0,0)$ for each $i \geq 1$. We let d denote the metric on X. Also, we write S^1 for each set $\{x \in X_i | \alpha(x) = 1\}$.

Suppose there is an integer m such that if $n \geq m$, then $g_n|_{g_n^{-1}(S^1)}: g_n^{-1}(S^1) \to S^1$ is essential. We claim that, for $n \geq m$, $g_n: X \to D$ is AH-essential. Suppose that g_n is not AH-essential. Let $g: X \to S^1$ be an extension of $g_n' = g_n|_{g_n^{-1}(S^1)}$. Since X is disk-like and Cech cohomology with integer coefficients is continuous, it follows that $H^1(X) \approx 0$. By [3, 8.1], g is inessential. Since g is an extension of g_n', g_n' is inessential. But this contradicts our assumption. So, for $n \geq m$, $g_n: X \to D$ is AH-essential and by [5] g_n is universal. It follows from Lemma 1 in [4] that X has the fixed point property.

Suppose for each positive integer m, there is an $n > m$ such that g_n' is inessential. Suppose that $f: X \to X$ is a fixed point free mapping and ε is a positive number such that $d(x,f(x)) \geq \varepsilon$ for each $x \in X$. Let n be an integer such that g_n is an ε-map and g_n' is inessential.

Let $H = \{x \in X | \alpha g_n(x) = \alpha g_n f(x)\}$. The set H is not empty since αg_n is universal. Let $X_0 = \varprojlim\{S^1, g_i^{i+1}|_{S^1}\}$. Then X_0 is a subcontinuum of X and $X_0 \subset g_n^{-1}(S^1)$. Now, as in the proof of Theorem 5, H cuts weakly between $\{v\}$ and X_0. Since X is s-connected, there is a continuum K in H that cuts weakly between $\{v\}$ and X_0.

Since g_n' is inessential, so is $g_n'|_{X_0}$. Let $\hat{g}_n = g_n'|_{X_0}$ and let $\psi: X_0 \to E^1$ be a mapping such that $\hat{g}_n(x) = e^{i\psi(x)}$ for each x in X_0. Let $\eta: X - \{v\} \to X_0$ be defined by $g_i\eta(x) = \pi g_i(x)$ for each $i \geq 1$. Now, $\psi(X_0)$ is an arc or a point; so, $\psi\eta|_K: K \to \psi(X_0)$ is universal. Hence, there is a point $x \in K$ such that $\psi\eta(x) = \psi\eta f(x)$. Thus,

$$\hat{g}_n(\eta(x)) = e^{i\psi\eta(x)} = e^{i\psi\eta f(x)} = \hat{g}_n(\eta f(x)).$$

By definition of η, this gives us that $\pi g_n(x) = \pi g_n f(x)$. Since $x \in K$, $\alpha g_n(x) = \alpha g_n f(x)$. These last two equalities give us that $g_n(x) = g_n f(x)$. But then $d(x,f(x)) < \varepsilon$, which is a contradiction.

J. T. Rogers, Jr. [9] has shown that the hyperspace of subcontinua of a circle-like continuum has the fixed point property. Again, Rogers' proof works equally well for the cone over a circle-like continuum. We get Rogers' result as a corollary to Theorem 7.

Corollary 7.1. *The cone over a circle-like continuum has the fixed point property.*

Proof. Suppose that X is the cone over a circle-like continuum; let $X = \text{cone}(X_0)$, where $X_0 = \varprojlim\{S_i, f_i^{i+1}\}$ with each $S_i = S^1$. Now, X is homeomorphic to $\varprojlim\{\text{cone}(S_i), f_i^{i+1} \times \text{id}\}$. For each $i \geq 1$, $\text{cone}(S_i)$ is homeomorphic to D with the vertex of $\text{cone}(S_i)$ mapping to $(0,0)$. The other conditions in the hypothesis of Theorem 7 follow easily. Hence, X has the fixed point property.

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