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by

John C. Mayer

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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1. Introduction

In [B-M] Brechner and Mayer show that equivalent embeddings of a nonseparating plane continuum have the same prime end structure (Theorem 2.11). Though not explicitly stated, this fact has been used previously in the literature. For instance, M. Smith [Sm] and W. Lewis [L] have independently shown that there are uncountably many inequivalent embeddings of the pseudo arc into the plane. This result was achieved by exploiting different prime end structures (directly in Lewis' case, indirectly, in terms of differing accessibility of composants by Smith) to distinguish different embeddings.

In this paper we show that the converse of Brechner and Mayer's theorem is false: there are inequivalent embeddings of a nonseparating continuum into the plane that have the same prime end structure, and indeed, that have the same set of accessible points. The following theorems stand in partial contrast to the methods of Smith and Lewis:

1.1 *Theorem. There exist uncountably many inequivalent embeddings of the* sin 1/x *continuum into the plane with the same prime end structure and the same set of accessible points.*

1.2 *Theorem. There exist uncountably many inequivalent embeddings of the Knaster* U*-continuum) (bucket handle)* *into the plane with the same prime end structure. Moreover, the set of accessible points in each of these embeddings is exactly the composant of the* U-*continuum that contains the endpoint of the* U-*continuum.*

In Section 2, we show how to construct uncountably many embeddings of the continuum formed by the sin 1/x curve plus its limit segment. Thereafter, we show these embeddings have the same prime end structure and the same set of accessible points, but that any two are inequivalent, proving Theorem 1.1.

In Section 3, we proceed similarly to prove Theorem 1.2, showing the Knaster U-continuum also has uncountably many inequivalent embeddings with the same prime end structure and the same set of accessible points.

In Section 4, we indicate how Theorem 1.2 can be extended to each of the uncountable class of U-type Knaster continua identified by W. T. Watkins [W].

This paper, together with [M-1] and [M-2], comprise the major portion of the author's dissertation. Section 3, in particular, has benefited from numerous discussions with the author's advisor, Beverly Brechner.

2. The Sin 1/X Continuum

The *standard* embedding of the sin 1/x continuum (Figure 1) consists of a ray R, the graph of (0,1] under the function y = sin 1/x in the xy-plane, plus the limit segment [p,q], the interval [-1,1] on the y-axis. The ray R consists of a number of *loops*, where a loop is a segment of R with exactly one peak and one trough in the standard embedding. For simplicity, we will fix the set of loops as the segments of R between alternate successive points of zero amplitude in order from 1 to 0 on the x-axis. The endpoints of loops limit on (0,0) in the standard embedding, so on the "midpoint" (or, more precisely, some interior point) of [p,q] in any other embedding. Endpoint p of limit segment [p,q] is the limit point of points on R selected from successive troughs, and endpoint q is the limit point of points selected from successive peaks. We suppose R to be coordinatized by the function $g: [0,\infty) \to R$ so that the endpoint of R is g(0) and for $x \in (0,1]$, odd positive integers correspond to $\{(x,1): \sin 1/x = 1\}$, even positive integers correspond to $\{(x,-1): \sin 1/x = -1\}$, and fractions with denominator 2 and odd numerator greater than 1 correspond to $\{(x,0): x = 1/(\pi n),$ for positive integers $n\}$. This embedding, denoted K, and its coordinatization by g are illustrated in Figure 1.

An embedding $e: X \to E^2$ of a continuum X is a homeomorphism into the plane; however, we shall somewhat loosely suppress reference to an embedding function and refer to the image in the plane of a continuum X as the embedding of X.

An embedding of the sin 1/x continuum can be described in terms of how the limit segment and ray is embedded in the plane. We will continue to designate the image of R, [p,q], or a point x in the sin 1/x continuum as R, [p,q], or x, suppressing reference to any particular embedding function.

2.1 *Embeddings of the* sin 1/x *continuum.* Let [p,q] be the vertical line segment between (0,1) and (0,-1) on the y-axis in the xy-plane, identifying (0,1) with q and (0,-1) with p. For each integer $i \geq 1$, let L_i (R_i) be the vertical segment between (-1/i,-1) and (-1/i,1) (between (1/i,-1) and (1/i,1)). The *upper endpoint* of L_i (R_i) is (-1/i,1) ((1/i,1)), and the *lower endpoint* of L_i (R_i) is (-1/i,-1) ((1/i,-1)). For each i, connect the upper endpoints of L_i and R_i by a semicircle, called an *upper semicircle*, centered at q and lying, except for endpoints, above the line y = 1. We will connect the lower endpoints of the L_i's and R_i's in some specified order to produce a ray converging to [p,q].

A *subschema* is a set of directions for connecting a finite number of lower endpoints of L_i's and R_i's by semicircles lying, except for endpoints, below the line y = -1, called *lower semicircles*. In a subschema, *connect* L_k *to* L_{k+r} means that we connect the lower endpoints of L_k and L_{k+r} by a semicircle centered midway between these endpoints on the line y = -1, and lying, except for these endpoints, below the line y = -1. We similarly define *connect* R_k *to* R_{k+r}. By *start at* L_k we mean that the subschema begins with a semicircle from L_k to some L_{k+r}. If a subschema begins with a semicircle from L_k to L_{k+r}, then the last connection instruction will be "connect R_{k+1} to R_{k+r+1}." We say *end at* L_{k+r+1} to indicate that in executing the subschema, the connecting lower semicircles, together with the L_i's and R_i's connected, together with the upper semicircles previously constructed connecting L_{k+1} to $R_{k+1},\cdots,L_{k+r+1}$ to R_{k+r+1}, comprise an arc from the lower endpoint of L_k to the lower endpoint of L_{k+r+1}. The following is a countably infinite list of subschemata:

S_0: Start at L_k;
Connect L_k to L_{k+1}; connect R_{k+1} to R_{k+2};
End at L_{k+2}.

S_1: Start at L_k;
Connect: L_k to L_{k+3}; R_{k+3} to R_{k+2}; L_{k+2} to L_{k+1}; R_{k+1} to R_{k+4};
End at L_{k+4}.

S_2: Start at L_k;
Connect: L_kL_{k+5}; $R_{k+5}R_{k+4}$; $L_{k+4}L_{k+3}$; $R_{k+3}R_{k+2}$; $L_{k+2}L_{k+1}$; $R_{k+1}R_{k+6}$;
End at L_{k+6}.

⋮

S_n: Start at L_k;
Connect: L_kL_{k+2n+1}; $R_{k+2n+1}R_{k+2n}$; $\cdots$; $L_{k+2n-i}L_{k+2n-i-1}$; $R_{k+2n-i-1}R_{k+2n-i-2}$; $\cdots$; $L_{k+2}L_{k+1}$; $R_{k+1}R_{k+2n+2}$;
End at L_{k+2n+2}.

⋮

By executing S_n, the lower endpoints of L_k through L_{k+2n+1} inclusive are connected in pairs by non-intersecting semicircles: one connects L_k to L_{k+2n+1} and the others connect adjacent pairs of L_i's. Similarly, R_{k+1} is connected to R_{k+2n+2} by one semicircle and the intervening R_i's are connected in adjacent pairs by the remaining semicircles. The result is that we have constructed an arc from the lower endpoint of L_k to the lower endpoint of L_{k+2n+2} comprised of vertical segments, semicircles above y = 1, and semicircles below y = -1. By *subschema* S_n we will, somewhat ambiguously, refer to the directions for making connections and the resulting arc from L_k to L_{k+2n+2}.

Let $N_a = \{a_1, a_2, a_3, \cdots\}$ be an infinite sequence of nonnegative integers. A *schema* $P_a = \{S_{a_1}, S_{a_2}, S_{a_3}, \cdots\}$ is an infinite sequence of subschemata S_{a_i} arranged so that $S_{a_{i+1}}$ begins where S_{a_i} ends. That is, if $k = 1 + 2(a_1+1) + 2(a_2+1) + \cdots + 2(a_i+1)$, then $S_{a_{i+1}}$ begins at L_k and ends at $L_{k+2(a_{i+1}+1)}$. Executing each subschema of P_a in order results in a ray R_a which contains every L_i and R_i and converges to $[p,q]$.

We coordinatize R_a by identifying the lower endpoint of R_1 with 0, the midpoint of each semicircle above y = 1 with an odd integer, and the midpoint of each semicircle below y = -1 with an even integer, in an order-preserving manner on R_a. The order for the correspondence is, in fact, built into the order in which the connection instructions of S_n are stated. Figure 2 illustrates the embedding M_0 of the sin 1/x continuum produced by executing the schema $P_0 = \{S_0, S_0, S_0, \cdots\}$. Figure 3 illustrates an embedding M_a produced by executing the schema $P_a = \{S_2, S_1, \cdots\}$.

Note that for n > 0, the first (last) semicircle of schema S_n "skips over" 2n vertical segments to connect L_k

to L_{k+2n+1} (R_{k+1} to R_{k+2n+2}). The segments skipped over are connected in adjacent pairs on each side of [p,q]. We call the first (last) semicircle of S_n, for $n \geq 0$, a *bend toward* p; we call the n intervening semicircles on each side of [p,q] *bends away from* p. By an *outer loop* we mean a loop of R_a which contains a bend toward p, and by an *inner loop* we mean a loop of R_a which contains a bend away from p.

Observe that in E^2 outer loops *locally separate* inner loops from p. By this we mean that for a sufficiently small ε-ball $S(p,\varepsilon)$ about p, each arc in $S(p,\varepsilon)$ from p to an inner loop must meet some outer loop. (In fact, it must meet an outer loop of the subschema that contains the inner loop.)

2.2 *Definition.* Let N represent the set of nonnegative integers. Two infinite sequences A and B selected from N (with replacement) are *distinct* iff after removing any finite (possibly null) initial subsequence from A and any finite (possibly null) initial subsequence from B, the remaining infinite sequences A' and B' are not identical. For example, $\{1,3,5,\cdots\}$ and $\{2,4,6,\cdots\}$ are distinct; $\{0,0,0,\cdots\}$ and $\{1,1,1,\cdots\}$ are distinct; $\{1,2,3,\cdots\}$ and $\{4,5,6,\cdots\}$ are not distinct. The definition of distinct sequences generalizes to infinite sequences selected from any set indexed by N.

2.3 *Lemma. There exist uncountably many distinct infinite sequences selected from* N.

2.4 *Lemma. There exist uncountably many distinct schemata for embedding the* sin 1/x *continuum.*

Proof. We use the notation of Section 2.1 and Definition 2.2. Let $\{S_0, S_1, S_2, \cdots\}$ be the set of subschemata and let $\{N_a\}_{a \in A}$ be the set of distinct infinite sequences selected from N. Suppose that $N_a = \{a_1, a_2, a_3, \cdots\}$ and let $P_a = \{S_{a_1} S_{a_2}, S_{a_3}, \cdots\}$ be a schema for embedding the sin 1/x continuum. It follows that $\{P_a\}_{a \in A}$ is uncountable, and that for all $a \neq b \in A$, P_a and P_b are distinct lists of subschemata.

2.5 *The prime end structures of* K *and* M_0. Before proceeding to construct our uncountably many embeddings of the sin 1/x continuum, we illustrate some of the concepts involved in prime end theory by applying them to K and M_0. Definitions and further references may be found in [Br].

Prime ends are a way of looking at and classifying the approaches to the boundary of a simply connected domain with nondegenerate boundary. The complement in S^2 of a nonseparating nondegenerate plane continuum X, denoted $S^2 - X$, is a simply connected domain. While $E^2 - X$ is not simply connected, as $E^2 \cup \{\infty\}$, the one-point compactification of E^2, is S^2, we can refer to the prime end structure of $E^2 - X$ by associating it with the prime end structure of $S^2 - X$, where the embedding of X misses the point at infinity.

A prime end of $E^2 - X$ is defined by a *chain of crosscuts* converging to a point of X, where a *crosscut* is an open arc in $E^2 - X$ whose endpoints lie in X. If Q is a crosscut of $E^2 - X$, then $Q \cup X$ separates E^2. A sequence of crosscuts

$\{Q_i\}_{i=1}^{\infty}$ is a *chain* provided that Q_i converges to a point, that no two crosscuts have a common endpoint, and that Q_i separates Q_{i-1} and Q_{i+1} in E^2-X. So, for example, the prime ends E and F of E^2-K in Figure 1 are defined by chains of crosscuts $\{Q_i\}_{i=1}^{\infty}$ and $\{Q_i'\}_{i=1}^{\infty}$, respectively, while in Figure 2, $\{T_i\}_{i=1}^{\infty}$ defines prime end H of E^2-M_0.

The *impression* of a prime end E, denoted I(E), is the intersection of the closures of the bounded domains cut off by the crosscuts in a chain defining E. For example, in Figure 1 it can be noted that I(E) = [p,q] = I(F), while in Figure 2, I(H) = [p,q].

The *set of principal points* of a prime end E, denoted P(E), is the collection of all points in X to which some chain of crosscuts defining E converges. For example, in Figure 1, P(E) = {p}, and P(F) = {q}, while in Figure 2, P(H) = {p}.

A prime end E is of the *first kind* if I(E) = P(E), both degenerate, of the *second kind* if I(E) $\neq$ P(E), only P(E) degenerate, of the *third kind* if I(E) = P(E), both nondegenerate, and of the *fourth kind* if I(E) $\neq$ P(E), both nondegenerate. It can be shown that P(E) $\subseteq$ I(E) in any case, and that both are continua in X. Thus prime ends E, F, and H of Figures 1 and 2 are all of the second kind. Any other prime end G, of either E^2-K or E^2-M_0, will be of the first kind, or *trivial*. Thus we can say that the prime end structure of E^2-K consists of two prime ends of the second kind and all other prime ends trivial.

A more pricise description of prime end structure is afforded by tne notion of a C-*map*. A C-map ϕ is a homeomorphism of S^2-X onto Ext B, where Ext B is the complementary domain of the unit disk B in S^2 which contains the point at infinity, which (map) satisfies the conditions:

(1) if Q is a crosscut of S^2-X, then $\phi(Q)$ is a crosscut of Ext B, and

(2) the endpoints of images of crosscuts are dense in Bd B, the boundary of B.

If we require, as we may, that ϕ take the point at infinity in S^2-X to the point at infinity in Ext B, then we can regard the restriction of ϕ to $S^2-\{\infty\} = E^2$ as a C-map also.

Suppose a chain of crosscuts defines a prime end E of E^2-X. Then the images of the crosscuts under a C-map ϕ will converge to a single point e in Bd B. We say e *corresponds* to E. In fact, there is a one-to-one correspondence between the prime ends of E^2-X and the points of Bd B. For example, in Figure 1, points e and f in Bd B correspond to prime ends E and F, respectively. In Figure 2, h corresponds to H. There is no homeomorphism g: Bd B $\twoheadrightarrow$ Bd B such that both g(e) and g(f) are points corresponding to prime ends of the second kind, since h corresponds to the only such prime end in E^2-M_0. Hence the prime end structures of K and M_0 are not identical [B-M, Definition 2.10]. Consequently, K and M_0 are inequivalently embedded [B-M, Theorem 2.11].

In the general result which follows, each embedding of the sin 1/x continuum will have the same prime end structure as M_0. Note that the *accessible* points of M_0 are the ray R and point p of [p,q]. A point of a plane continuum X is accessible if it can be reached by a half open arc in the complement whose closure adds exactly that point. Such a half open arc is called an *endcut*. Each of our uncountably many embeddings of the sin 1/x continuum will have the same set of accessible points as M_0. The significance of this result is the contrast it provides to the usual procedure for producing inequivalent embeddings of a plane continuum: produce embeddings with different points accessible. Such a procedure is a sufficient, but not a necessary, condition for producing inequivalent embeddings of a continuum in E^2.

2.6 *Proof of Theorem* 1.1. Let $\{P_a\}_{a \in A}$ be the uncountable set of distinct schemata demonstrated in Lemma 2.4. For each $a \in A$, let M_a be the embedding of the sin 1/x continuum according to schema P_a. We claim the prime end structures of M_a and M_b for all $a \neq b \in A$ are identical. Any chain of crosscuts converging to p in M_a (M_b) defines a prime end E_a (E_b) of the second kind, for which [p,q] is the impression and p is the only principal point. Any other prime end of M_a (M_b) is trivial. Moreover, the set of accessible points of M_a (M_b) is the ray R_a (R_b) and the point p of [p,q]. Hence M_a and M_b have the same prime end structure and the same set of accessible points.

We claim that M_a and M_b are inequivalent embeddings. By way of contradiction, suppose that M_a and M_b are equivalently embedded. Then there is a homeomorphism $h: E^2 \to E^2$ onto the plane such that $h(M_b) = M_a$. Then $h(R_b) = R_a$, and h must be order-preserving on R_b.

In M_b let $\{p_j\}_{j=1}^{\infty}$ be those even integer points on R_b which lie on *outer* loops; that is, those integer points of R_b for which a sequence of crosscuts $\{Q_j\}_{j=1}^{\infty}$, each Q_j joining p_j to p, can be chosen so that diam $Q_j \to 0$ as $j \to \infty$. (These points are on the bends toward p in each subschema. Note that this sequence of crosscuts is not a chain.) For odd j, each pair p_j, p_{j+1} of outer loop points is separated on R_b by a finite set $\{p_j(1), p_j(2), \cdots, p_j(n_j)\}$ of even integer points of R_b which lie on *inner* loops. (These points are on the bends away from p of each subschema.) For even j, p_j and p_{j+1} are in succession on R_b, with no inner loops between them. Note that any crosscut joining $p_j(i)$ to p has diameter no less than that of [p,q].

For odd j, let $A_j = \{p_j, p_j(1), p_j(2), \cdots, p_j(n_j), p_{j+1}\}$ be the sequence of even integer points in order on R_b described in the preceding paragraph. Consider $h(A_j)$. Since diam $Q_j \to 0$, diam $h(Q_j) \to 0$, as well. Hence outer loop points p_j, p_{j+1} cannot infinitely often be carried to loop ends of R_a in the neighborhood of p which contain inner loop points. Similarly, inner loop points of R_b cannot infinitely often be carried onto loop ends of R_a that contain outer loop points. As the even integer points converge to p, and h is order-preserving on R_b, there is a positive odd integer n, such that for all $j \geq n$, the set A_j, consisting of two outer loop points and n_j inner loop points between them, must be carried into a set of loop ends of R_a in the close neighborhood of p consisting of two outer loop ends and n_j inner loop ends between them.

Each loop end of R_a in the neighborhood of p contains an even integer point of R_a. For simplicity we may assume that h carries, from p_n onward, even integer points of R_b to even integer points of R_a, one-to-one and order-preserving. If $\{p'_k\}_{k=1}^{\infty}$ is the set of even integer points of R_a on outer loops, then our preceding remarks require that there be a positive odd integer m such that there is a set of even integer points $B_m = \{p'_m, p'_m(1), p'_m(2), \cdots, p'_m(n_n), p'_{m+1}\}$ that corresponds one-to-one to the set A_n, including $p'_m(i)$ being an inner loop point. Thereafter, B_{m+2} corresponds one-to-one to A_{n+2}, B_{m+4} to A_{n+4}, etc.

Each subschema contains only two outer loops. Hence A_n and B_m each correspond to a listing of the even integer points in a single subschema, $S_{a_{(n+1)/2}}$ and $S_{b_{(m+1)/2}}$, respectively. Likewise for A_{n+2} and B_{m+2}, etc. This implies that in P_a and P_b, $S_{a_{(n+r)/2}} = S_{b_{(m+r)/2}}$, for all odd $r \geq 1$. Thus P_a is identical to P_b after the removal of some finite initial subsequence from one and some finite initial subsequence from the other. This contradicts our choice of P_a and P_b distinct.

Therefore, no such homeomorphism h can exist. Hence, M_a and M_b are inequivalently embedded, but have the same prime end structure and same set of accessible points.

2.7 *Remark.* The uncountable class above does not exhaust the inequivalent embeddings of the sin 1/x continuum. All the embeddings we have considered have the prime end structure of Figure 2. There are also uncoutably many inequivalent embeddings of the sin 1/x continuum with the prime end structure of Figure 1, and every point accessible. Additional embeddings with the prime end structure of Figure 1 are illustrated in Figure 4. In Figure 5(a) we show an embedding with a prime end G defined by any chain of cross-cuts converging to some point of [p,q]; for example, chain $\{P_i\}_{i=1}^{\infty}$. Note that I(G) = P(G) = [p,q]. Figure 5(b) is another embedding with the same prime end structure. In these embeddings, only points of the ray R are accessible.

3. The Knaster U-Continuum

The *Knaster* U-*continuum* K can be represented abstractly as the inverse limit system $K = \varprojlim\{I_i, f_i\}_{i=1}^{\infty}$, where $I_i = [0,2^i]$ and $f_i\colon [0,2^{i+1}] \twoheadrightarrow [0,2^i]$ is the *rooftop function* defined by

$$f_i(x) = \begin{cases} x & , \text{ if } 0 \le x \le 2^i \\ 2^{i+1}-x, & \text{if } 2^i \le x \le 2^{i+1} \end{cases}$$

This representation, used by Watkins in [W], has the advantage of providing a convenient coordinatization of the endpoint composant of K. The endpoint of K is $(0,0,0,\cdots)$, and the composant of the endpoint is that set of points in K whose coordinates, from some i^{th} coordinate onward, are constant.

3.1 *Standard embedding* K_0 *of* K *in* E^2. We construct the *standard embedding* K_0 of K in E^2 below. Let $\mathcal{C}$ be the middle third Cantor set constructed on the unit segment I between (0,0) and (1,0) in the xy-plane. By a *deleted interval* we mean a complementary interval of $I-\mathcal{C}$. By a *subinterval of* $\mathcal{C}$ we mean a first or last third interval at the k^{th} stage in the construction of $\mathcal{C}$ intersected with $\mathcal{C}$, that is any $[m/3^k, (m+1)/3^k] \cap \mathcal{C}$ with non-empty interior in $\mathcal{C}$.

Connect the points of $[0,1/3] \cap \mathcal{C}$ to the points of $[2/3,1] \cap \mathcal{C}$ by a Cantor set of semicircles lying above the x-axis (except for their endpoints in $\mathcal{C}$) and centered at $(1/2,0)$. For each $k = 2,3,\cdots$, connect the points of subinterval $[6/3^k,7/3^k] \cap \mathcal{C} = D_{k-1,2}$ with the points of subinterval $[8/3^k,9/3^k] \cap \mathcal{C} = D_{k-1,1}$ by a Cantor set of semicircles lying below the x-axis (except for their endpoints in $\mathcal{C}$) and centered at the point midway between these subintervals. The union of all semicircles thus constructed is a continuum $K_0 \subset E^2$, with we call the *standard embedding* of K. In fact it is Knaster's original construction. See Figure 6.

In this and subsequent sections, semicircles lying above the x-axis, except for their endpoints, we term *upper semicircles*, and semicircles lying below the x-axis, except for their endpoints, we term *lower semicircles*. Concentric semicircles are said to be *parallel* semicircles. We extend this term to apply to a collection of arcs which are piecewise composed of concentric semicircles. Among a closed collection of concentric semicircles, we call that with largest radius an *outer semicircle*, and that with smallest radius an *inner semicircle*.

3.1.1 *Topological equivalence of* K *and* K_0. We claim that K and K_0 are homeomorphic (so that K_0 is indeed an embedding of K). First note that there is a ray $C_0 \subset K_0$ whose endpoint is (0,0) and which includes (1,0) and the endpoints of every deleted interval. Now C_0 is the composant of (0,0) in K_0. We will coordinatize C_0 by $[0,\infty)$ in such a way that 0 corresponds to (0,0), each odd positive integer corresponds to the midpoint of an upper semicircle, and each even positive integer corresponds to the midpoint of a lower semicircle. Not all upper and lower semicircles are involved, but only those in C_0. We will also show that there is a retraction $r_k\colon K_0 \twoheadrightarrow [0,2^k] \subset C_0$, for each $k = 1,2,3,\cdots$, so that

(1) r_k moves no point more than $1/2^k$, and

(2) $r_k = f_k \circ r_{k+1}$.

It then follows that $\varprojlim r_k = r\colon K_0 \twoheadrightarrow K$ is a homeomorphism.

For each $k = 1,2,3,\cdots$, let a_k denote the midpoint of the outer semicircle in the collection joining $D_{k,1}$ to $D_{k,2}$, and let d_k denote the midpoint of the inner semicircle. Let L_k be the vertical segment joining a_k to d_k. Note that $L_k \cap K_0$ is a *cut* of K_0. That is, $L_k \cap K_0$ is an irreducible separator of K_0. Necessarily, a cut of K_0 is a Cantor set.

For nonnegative reals x and y, $x < y$ denotes, in the usual order, that x is less than y. For any ray R, for $x,y \in R$, $x < y$ denotes that $f^{-1}(x) < f^{-1}(y)$ for any coordinatization f of R by $[0,\infty)$. For subsets A,B of a ray R (or of $[0,\infty)$), $A < B$ denotes that $x < y$ for all $x \in A$ and $y \in B$. The symbols $>$, $\leq$, and $\geq$ are similarly defined. For any ray R, for any $x < y \in R$, let $[x,y]$ denote that interval in R between x and y, inclusive; that is, $\{z \in R | x \leq z \leq y\}$.

If S is a continuum separating E^2 into exactly two components, we let Int(S) denote the bounded complementary domain of S and we let Ext(S) denote the unbounded complementary domain of S.

Let 0 denote the point $(0,0) \in C_0$. Note that $L_k \cup [a_k,d_k]$ is a simple closed curve. Define the following closed subsets of K_0 which meet only in the cut L_k:

$$[0,L_k] = \mathrm{Cl}(\mathrm{Ext}(L_k \cup [a_k,d_k])) \cap K_0$$

$$[L_k,L_{k+1}] = \mathrm{Cl}(\mathrm{Int}(L_k \cup [a_k,d_k])) \cap K_0$$

The notation is meant to suggest that the first set contains 0, the second set contains the *next* cut, and that the cut L_k lies between them.

Observe that $[0,a_k] \subset [0,L_k]$ and that $[a_k,a_{k+1}] \subset [L_k,L_{k+1}]$. It is not hard to see that we should coordinatize C_0 by identifying 2^k with a_k and $2^{k+1}+2^k$ with d_k, and that we can extend the coordinatization to intervening points in such a way that a retraction r_k can be defined satisfying (1) and (2). We omit details as we shall carry out a similar process in Section 3.2.5 for more complicated embeddings. Note, however, that the order of points on C_0 is $0 < a_1 < a_2 < d_1 < a_3 < d_2 < \cdots < a_k < d_{k-1} < a_{k+1} < \cdots$.

3.2 *Embeddings of the Knaster* U-*continuum*. For each sequence $N_a = \{a_1,a_2,\cdots\}$ of integers such that $a_k \geq 2$, we will define an inverse limit system $\hat{K}_a = \varprojlim\{A_k,g_k\}_{k=1}^{\infty}$, and we will construct what we call the *natural embedding* K_a of $\hat{K}_a$ in E^2.

Let $n_1 = 1$, and let $n_k = 1 + \sum_{j=1}^{k-1} a_j$, for $k = 2,3,\cdots$. For $k = 1,2,\cdots$, let $A_k = [0,2^{n_k}]$. Henceforth, unless otherwise stated, let k range over $1,2,3,\cdots$. Recall the definition of the rooftop functions f_k: $[0,2^{k+1}] \twoheadrightarrow [0,2^k]$, and define g_k: $A_{k+1} \twoheadrightarrow A_k$ by $g_k = f_{n_k} \circ f_{n_k+1} \circ \cdots \circ f_{n_{k+1}-1}$. Since each g_k is a composition of the next a_k unused f_j's, it is clear that $\hat{K}_a$ is homeomorphic to K.

We define the embedding K_a of $\hat{K}_a$ in E^2 below, coordinatize the composant C_a of the endpoint of K_a, and at the same time construct small retractions r_k from K_a onto subarcs of C_a that correspond to the factor spaces of $\hat{K}_a$ so that the retractions commute with the g_k's. Thus we show that K_a and $\hat{K}_a$ are homeomorphic. We also develop notation and concepts which will be used in subsequent sections to show that there are uncountably many inequivalent embeddings of K in E^2, each the natural embedding of an inverse limit system based upon a sequence of integers like N_a. It turns out that if two sequences of integers are distinct, the natural embeddings of the corresponding inverse limit systems are inequivalent.

3.2.1 *Constructing the embedding* K_a *of* $\hat{K}_a$ *in* E^2. Let $\mathcal{C}$ be the middle third Cantor set on the unit segment $I = [0,1]$ from $(0,0)$ to $(1,0)$ in the xy-plane. Connect points of $[0,1/3] \cap \mathcal{C}$ with points of $[2/3,1] \cap \mathcal{C}$ by a Cantor set of upper semicircles centered at $(1/2,0)$. We connect the points of certain subintervals of $\mathcal{C}$ by Cantor sets of lower semicircles by executing the recursive procedure below for each $a_k \in N_a$.

Subintervals of $\mathcal{C}$ are said to be *equal* provided that the corresponding intervals of I have equal length. Initially, let $D_2^1 = \mathcal{C} \cap [0,1/3]$ and $D_1^1 = \mathcal{C} \cap [2/3,1]$ be equal subintervals of $\mathcal{C}$. Iterate for each $a_k \in N_a$ the following steps:

(1) Assume $D_1^k > D_2^k$ are equal subintervals of $\mathcal{C}$, as yet unconnected by lower semicircles. Let $m_k = 2^{a_k-1}$.

(2) In D_1^k identify $2m_k$ equal subintervals of $\mathcal{C}$: $D_{1,1}^k > D_{1,2}^k > \cdots > D_{1,2m_k}^k$ (ordered from right-to-left on the x-axis).

(3) Connect by lower semicircles centered midway between them, the following pairs of subintervals of $\mathcal{C}$: $D_{1,1}^k$ to $D_{1,2}^k$, $D_{1,3}^k$ to $D_{1,4}^k, \cdots, D_{1,2m_k-1}^k$ to $D_{1,2m_k}^k$.

(4) In D_2^k identify $2m_k$ equal subintervals of $\mathcal{C}$: $D_{2,1}^k > D_{2,2}^k > \cdots > D_{2,2m_k}^k$.

(5) Connect by lower semicircles centered midway between them, the following pairs of subintervals of $\mathcal{C}$: $D_{2,2}^k$ to $D_{2,3}^k$, $D_{2,4}^k$ to $D_{2,5}^k, \cdots, D_{2,2m_k-2}^k$ to $D_{2,2m_k-1}^k$.

(6) Let $D_1^{k+1} = D_{2,1}^k$ and $D_2^{k+1} = D_{2,2m_k}^k$.

The union of all upper and lower semicircles thus constructed is a continuum $K_a \subset E^2$. Sig Figure 7 for K_a based upon the sequence $N_a = \{2,2,\cdots\}$.

We make the following observations about the way in which points of $\mathcal{C}$ are connected by arcs consisting piecewise of concentric upper and lower semicircles:

(a) D_1^k and D_2^k are connected by a Cantor set of parallel arcs consisting of upper semicircles, and concentric lower semicircles constructed in previous stages, in alternation.

(b) The partitions of D_1^k and D_2^k in steps (2) and (4) induce a partition of the Cantor set of arcs in (a) into $2m_k$ Cantor sets of (piecewise semicircular) arcs.

(c) $D_{1,m}^k$ is connected to $D_{2,2m_k-m+1}^k$ by the m^{th} element of the induced partition in (b).

(d) The concentric lower semicircles constructed in steps (3) and (5) result in $D_{2,1}^k$ being connected to $D_{2,2m_k}^k$ by a Cantor set of piecewise semicircular arcs in which elements of the partition in (b) alternate with the sets of concentric lower semicircles constructed in (3) and (5).

(e) The order in which the $D_{i,m}^k$'s appear in the Cantor set of arcs connecting $D_{2,2m_k}^k$ to $D_{2,1}^k$ is:

$$D_{2,2m_k}^k, D_{1,1}^k, D_{1,2}^k, D_{2,2m_k-1}^k, D_{2,2m_k-2}^k, \ldots,$$
$$D_{2,2m_k-m}^k, D_{1,m+1}^k, D_{1,m+2}^k, D_{2,2m_k-m-1}^k, \ldots,$$
$$D_{2,2}^k, D_{1,2m_k-1}^k, D_{1,2m_k}^k, D_{2,1}^k$$

Our conclusions in Section 3.2.2 below concerning the order in which certain points lie on C_a follow from the above observations.

3.2.2 *Composant* C_a *of* K_a. Composant C_a of K_a is a ray whose endpoint is (0,0) and which includes (1,0) and the endpoints of all deleted intervals, but does not include any other point of $\mathcal{C}$. (The other points of $\mathcal{C}$ are in the uncountably many other composants of K_a, each of which is a one-to-one continuous image of the reals.) We will coordinatize C_a by $[0,\infty)$ by identifying (0,0) with 0, midpoints of upper semicircles in C_a with odd positive integers, and midpoints of lower semicircles in C_a with even positive integers, in a manner to be described more fully below.

Let π denote projection from E^2 into the x-axis. For $x,y \in E^2$, we say $x < y$ *in projection order*, meaning that $\pi(x) < \pi(y)$.

Note that all inner and outer semicircles are contained in C_a. We identify the midpoints of inner and outer semicircles as follows:

Let $a_{k,m}$ $(m = 1,\cdots,m_k)$ denote the midpoint of the outer semicircle in the collection of concentric lower semicircles joining $D^k_{1,2m-1}$ to $D^k_{1,2m}$. Call such points collectively *a-points*. Let $c_{k,m}$ $(m = 1,\cdots,m_k-1)$ denote the midpoint of the inner semicircle in the concentric collection joining $D^k_{2,2m_k-2m}$ to $D^k_{2,2m_k-2m+1}$. Call such points collectively *c-points*. Note that the projection order of a- and c-points consists (after $a_{1,1} > \cdots > a_{1,m_1}$) of repeated sequences of the form:

$$\cdots > a_{k,1} > a_{k,2} > \cdots > a_{k,m_k} > c_{k-1,m_{k-1}-1} >$$
$$c_{k-1,m_{k-1}-2} > \cdots > c_{k-1,1} > a_{k+1,1} > \cdots >$$
$$a_{k+1,m_{k+1}} > c_{k,m_k-1} > \cdots > c_{k,1} > a_{k+2,1} > \cdots$$

in which the c-points of stage k-1 are to the left (in the projection order) of the a-points of stage k, and therefore to the left of the a-points of stage k-1. However, in the order on C_a we have the alternating sequence:

$$\cdots < a_{k,1} < c_{k,1} < a_{k,2} < c_{k,2} < \cdots <$$
$$c_{k,m_k-1} < a_{k,m_k} < a_{k+1,1} < \cdots$$

Let $b_{k,m}$ $(m = 1,\cdots,m_k-1)$ denote the midpoint of the outer semicircle in the collection joining $D^k_{2,2m}$ to $D^k_{2,2m+1}$.

Call such points collectively b-*points*. Note that the b-point $b_{k,m}$ lies directly below the c-point c_{k,m_k-m}. Let $d_{k,m}$ $(m = 1,\cdots,m_k)$ denote the midpoint of the inner semicircle in the collection joining $D^k_{1,2m_k-2m+1}$ to $D^k_{1,2m_k-2m+2}$. Call such points collectively d-*points*. Note that $d_{k,m}$ lies directly above a_{k,m_k-m+1}. The projection order of b- and d-points consists (after $d_{1,m_1} > \cdots > d_{1,1}$) of repeated sequence of the form:

$$\cdots > d_{k,m_k} > d_{k,m_k-1} > \cdots > d_{k,1} > b_{k-1,1} >$$
$$\cdots > b_{k-1,m_{k-1}-1} > d_{k+1,m_{k+1}} > \cdots > d_{k+1,1} >$$
$$b_{k,1} > \cdots > b_{k,m_k-1} > d_{k+2,m_{k+2}} > \cdots$$

in which the d-points of stage k are to the right (in projection order) of the b-points of stage k-1, and therefore to the right of the b-points of stage k.

Observe that the b- and d-points of stage k lie *between* $a_{k+1,m_{k+1}}$ and $a_{k+2,1}$ on C_a. Consequently, in the order on C_a, we have repeated sequences of the form:

$$\cdots < a_{k+1,1} < c_{k+1,1} < a_{k+1,2} < \cdots <$$
$$a_{k+1,m_{k+1}-1} < c_{k+1,m_{k+1}-1} < a_{k+1,m_{k+1}} < d_{k,1} <$$
$$b_{k,1} < d_{k,2} < \cdots < d_{k,m_k-1} < b_{k,m_k-1} < d_{k,m_k} <$$
$$a_{k+2,1} < \cdots$$

3.2.3 *Pockets in* K_a. Let A be an arc in E^2. We say that A is an ε-*pocket* iff there is an endcut $B \subset E^2 - A$ to a point $a \in A$ such that

(1) For each $b \in B$, there is a crosscut Q to A, *transverse* to B at b, with $\operatorname{diam}(Q) < \varepsilon$.

(2) For each $x \in A$, there is a $y \in A$ such that $x < a < y$ in some natural order on A, $d(x,y) < \varepsilon$, and there is a crosscut Q from x to y such that Q is transverse to B at some point $b \in B$ and $\operatorname{diam}(Q) < \varepsilon$.

A crosscut Q is *transverse* to B at b iff $Q \cap B = \{b\}$, and for a sufficiently small open disk neighborhood U of b, each component of $U-Q$ contains exactly one component of $(U \cap B) - \{b\}$. By *pocket* we mean an ε-pocket for some ε, usually small compared to $\operatorname{diam}(A)$.

Suppose that X is a connected subset of E^2 and $A \subset X$ is an ε-pocket. Suppose there is a crosscut Q of X joining the endpoints of A, with $\operatorname{diam}(Q) < \varepsilon$. Then $Q \cup A$ is a simple closed curve. If $X \subset \operatorname{Cl}(\operatorname{Ext}(Q \cup A))$ for each sufficiently small crosscut Q, we say that A is a *pocket in* X. If A is a pocket in X, Q is a crosscut of X joining the endpoints of A so that $X \subset \operatorname{Cl}(\operatorname{Ext}(Q \cup A))$ and $\operatorname{diam}(Q) < \varepsilon$, and B is an endcut to A at point a satisfying conditions (1) and (2), then we call $\operatorname{Int}(Q \cup A)$ the *inside* of the pocket and $\operatorname{Ext}(Q \cup A)$ the *outside* of the pocket. We call the endpoints of the pocket the *open end* of the pocket, and we call the point $a \in B \cap A$ the *closed end* of the pocket. We call the components of $A - \{a\}$ the *sides* of the pocket. (Strictly speaking, the inside, outside, sides, and closed end of a pocket depend upon the choices made for Q and B. In practice, this ambiguity will make little difference as ε is usually very small compared to $\operatorname{diam}(A)$.)

As we shall show in Section 3.3, the accessible points of K_a are precisely the points in C_a. Thus pockets in K_a are subarcs of ray C_a. We will show that the following four types of pockets occur in C_a:

aca-*pockets*: a subarc of C_a between two consecutive a-points of a given stage of construction, and the c-point between them as the closed end.

adb-*pockets*: the subarc of C_a between the last a-point of stage k+1 and the first b-point of stage k, and the d-point between them as closed end.

bdb-*pockets*: a subarc of C_a between two consecutive b-points of a given stage of construction, and the d-point between them as closed end.

bda-*pockets*: the subarc of C_a between the last b-point of stage k and the first a-point of stage k+2, and the d-point between them as closed end.

Note that given k, there are m_k-1 aca-pockets between successive a-points of stage k; there are m_k-2 bdb-pockets between successive b-points of stage k; there is one adb-pocket, namely the subarc $[a_{k+1,m_{k+1}}, d_{k,1}, b_{k,1}]$; there is one bda-pocket, namely $[b_{k,m_k-1}, d_{k,m_k}, a_{k+2,1}]$. We show below that each of the above-mentioned pockets is a $2/2^{n_k}$-pocket in K_a.

Let $\mathcal{D}$ be the collection of midpoints of deleted intervals. Connect the points of $\mathcal{D}$ in pairs by a countable collection of upper semicircles centered at (1/2,0). At the k^{th} stage in the construction of K_a *via* the recursive procedure of Section 3.2.1, if $D^k_{i,m}$ is connected to $D^k_{i,m+1}$

by lower semicircles, then connect the points of $\mathcal{D}$ that lie between points of $D^k_{i,m}$ to those that lie between points of $D^k_{i,m+1}$ by lower semicircles centered midway between $D^k_{i,m}$ and $D^k_{i,m+1}$. Let $r_{k,m}$ be that point in $\mathcal{D}$ which lies midway between $\pi(a_{k,m})$ and $\pi(a_{k,m+1})$ and let $s_{k,m} = \pi(c_{k,m})$, which is in $\mathcal{D}$, for $m = 1,\cdots,m_k-1$. We obtain, for each k, and each m, an arc $A'_{k,m} \subset E^2 - K_a$ from $r_{k,m}$ to $s_{k,m}$ which is the union of $2^{n_k-1}-1$ semicircles, alternating upper and lower. Let $A_{k,m}$ be the endcut from $r_{k,m}$ to $c_{k,m}$ obtained by extending $A'_{k,m}$ by a vertical segment straight down to $c_{k,m}$. Then $A_{k,m}$ is clearly an endcut which shows that subarc $[a_{k,m},c_{k,m},a_{k,m+1}]$ is a $2/2^{n_k}$-pocket in K_a, for $A_{k,m}$ is parallel to the parallel arcs which form the sides of the pocket (the sides are parallel except for small segments containing $a_{k,m}$, $a_{k,m+1}$, and $c_{k,m}$). We call $A_{k,m}$ the *midline* of the aca-pocket whose closed end is $c_{k,m}$.

In a similar fashion we construct the *midline* $B_{k,m}$ of the pocket $[b_{k,m-1},d_{k,m},b_{k,m}]$ as an endcut from a point of D midway between the projections of $b_{k,m-1}$ and $b_{k,m}$ to $d_{k,m}$ at the closed end of the pocket, for $m = 2,\cdots,m_k-1$. *Midlines* $B_{k,1}$ to $d_{k,1}$ at the closed end of pocket $[a_{k+1,m_{k+1}}, d_{k,1},b_{k,1}]$, and B_{k,m_k} to d_{k,m_k} at the closed end of pocket $[b_{k,m_k-1},d_{k,m_k},a_{k+2,1}]$ are special cases only because the endpoints of $B_{k,1}$ and B_{k,m_k} that lie on the x-axis are the points of $\mathcal{D}$ midway between $D^k_{2,1}$ and $D^k_{2,2}$, and between $D^k_{2,2m_k}$ and $D^k_{2,2m_k-1}$, respectively.

Observe that in projection order, the open end of an aca-pocket is closer to 0 than its closed end, while the open end of an adb-, bdb-, or bda-pocket is further from 0 than its closed end. Of course, both the open and closed ends of a pocket in K_a are close to 0 compared to the diameter of the pocket.

Henceforth, for each p in the collection of c- and d-points, let M(p) denote the midline of the pocket whose closed end is the point p.

3.2.4 *Cuts of* K_a. A *cut* of K_a is a Cantor set in K_a which separates K_a. For example, each $D^k_{i,m}$ in our construction of K_a is a cut of K_a. We identify below a convenient collection of cuts of K_a which will be useful in showing the topological equivalence of K_a and $\hat{K}_a$, as well as showing the inequivalence of K_a (based on sequence N_a) and K_b (based on sequence N_b, distinct from N_a) in subsequent sections.

Let $L_{k,m}$ be the vertical segment joining $a_{k,m}$ to d_{k,m_k-m+1}, for each k and each $m = 1,\cdots,m_k$. Let $M_{k,m}$ be the vertical segment joining $c_{k,m}$ to b_{k,m_k-m}, for each k and each $m = 1,\cdots,m_k-1$. Then $L_{k,m}(M_{k,m})$ *cuts* K_a; that is each of $L_{k,m} \cap K_a$ and $M_{k,m} \cap K_a$ is a cut of K_a. Let $L_k = L_{k,1} \cup \cdots \cup L_{k,m_k}$, and let $M_k = M_{k,1} \cup \cdots \cup M_{k,m_k-1}$.

For distinct points p and q in the collection of all a-points, b-points, and (0,0) (we will refer to (0,0) as 0 hereafter), let Q(p,q) denote a crosscut of small ($<2d(p,q)$) diameter from p to q lying below the x-axis. It is clear that such a crosscut exists, since a- and b-points are on outer semicircles. As before, for $p < q \in C_a$, [p,q] denotes the interval in C_a between p and q. Note that for fixed p and q, the continuum in K_a irreducible between the endpoints p and q of Q(p,q) is [p,q]. Moreover, Q(p,q) ∪ [p,q] is a simple closed curve. For fixed p and q, $Cl(Int(Q(p,q) \cup [p,q])) \cap K_a$ is independent of the choice of crosscut Q(p,q), and is either [p,q] (if neither p nor q is 0) or is K_a (if either p or q is 0).

Observe that the two segments $L_{k,m}$ and $M_{k,m}$ together with the subarcs of C_a between their endpoints, namely $[a_{k,m}, c_{k,m}]$ and $[b_{k,m_k-m}, d_{k,m_k-m+1}]$, form a simple closed curve. A similar statement can be made for $M_{k,m}$ and $L_{k,m+1}$. Define the following closed subsets of K_a which meet only in the indicated cuts (their boundaries relative to K_a), and which form a closed chain covering K_a:

$$[0,L_{k,1}] = Cl(Int(Q(0,a_{k+2,1}) \cup [0,a_{k,1}] \cup L_{1,1} \cup [d_{k,m_k}, a_{k+2,1}])) \cap K_a,$$

$$\vdots$$

$$[L_{k,m}, M_{k,m}] = Cl(Int(L_{k,m} \cup [a_{k,m}, c_{k,m}] \cup M_{k,m} \cup [b_{k,m_k-m}, d_{k,m_k-m+1}])) \cap K_a,$$

$$[M_{k,m}, L_{k,m+1}] = Cl(Int(M_{k,m} \cup [c_{k,m}, a_{k,m+1}] \cup L_{k,m+1} \cup [b_{k,m_k-m}, d_{k,m_k-m}])) \cap K_a,$$

$$\vdots$$

$$[L_{k,m_k}, L_{k+1}] = Cl(Int(L_{k,m_k} \cup [a_{k,m_k}, a_{k+1,1}] \cup Q(a_{k+1,1}, a_{k+1,m_{k+1}}) \cup [a_{k+1,1}, d_{k,1}])) \cap K_a.$$

The notation is meant to suggest that the subset $[L_{k,m}, M_{k,m}]$ lies *between* cuts $L_{k,m} \cap K_a$ and $M_{k,m} \cap K_a$, while the first subset in the chain contains 0 and the last contains L_{k+1}. Actually, $[0,L_{k,1}]$ also contain $\cup_{j=k+2}^{\infty} L_j \cap K_a$ and $\cup_{j=k+1}^{\infty} M_j \cap K_a$. The interiors (relative to K_a) of the above closed subsets constitute a partition of K_a into $2m_k$ sets which meet only in their boundaries.

3.2.5 *Topological equivalence of* K_a *and* $\hat{K}_a$. We use the closed covers of K_a defined in Section 3.2.4 to define a sequence of retractions $r_k: K_a \twoheadrightarrow [0,a_{k,1}] \subset C_a$, and we coordinate C_a with the nonnegative reals so that g_k is considered as a function with domain a subset of C_a so that we satisfy the conditions

(1) r_k moves no point more than $2/2^{n_k}$, and

(2) $r_k = g_k \circ r_{k+1}$.

It follows that $\varprojlim r_k = r: K_a \twoheadrightarrow \hat{K}_a$ is a homeomorphism.

Observe that $[0,a_{k,1}] \subset [0,L_{k,1}],\cdots,[a_{k,m},c_{k,m}] \subset [L_{k,m},M_{k,m}],[c_{k,m},a_{k,m+1}] \subset [M_{k,m},L_{k,m+1}],\cdots,[a_{k,m_k},a_{k+1,1}] \subset [L_{k,m_k},L_{k+1}]$. Since the $2/2^{n_k}$-neighborhood of each arc above contains the corresponding subset of K_a, it is easy to see that there is a retraction $r_{k+1}: K_a \twoheadrightarrow [0,a_{k+1,1}]$ such that r_{k+1} moves no point more than $2/2^{n_k}$, and such that $r_{k+1}([0,L_{k,1}]) = [0,a_{k,1}],\cdots,r_{k+1}([L_{k,m},M_{k,m}]) = [a_{k,m},c_{k,m}],r_{k+1}([M_{k,m},L_{k,m+1}]) = [c_{k,m},a_{k,m+1}],\cdots, r_{k+1}([L_{k,m_k},L_{k+1}]) = [a_{k,m_k},a_{k+1,1}]$. However, in order to satisfy condition (2), we will need to define the retractions inductively, while coordinatizing C_a so that condition (2) makes sense. It will turn out that with respect to a-, b-, c-, and d-points, the coordinatization of C_a is

(3) $a_{k,m} = (2m-1)(2^{n_k})$, for $1 \leq m \leq m_k$

$c_{k,m} = (2m)(2^{n_k})$, for $1 \leq m \leq m_k-1$

(4) $b_{k,m} = (2m_{k+1}-1)(2^{n_{k+1}}) + (2m)(2^{n_k})$,

for $1 \leq m \leq m_k-1$

$d_{k,m} = (2m_{k+1}-1)(2^{n_{k+1}}) + (2m-1)(2^{n_k})$,

for $1 \leq m \leq m_k$

We define the sequence $\{r_k\}_{k=1}^{\infty}$ as follows: Recall that $L_k = L_{k,1} \cup \cdots \cup L_{k,m_k}$ and $M_k = M_{k,1} \cup \cdots \cup M_{k,m_k-1}$. Let $L = \cup_{k=1}^{\infty} L_k$ and let $M = \cup_{k=1}^{\infty} M_k$. Then $L \cup M$ cuts K_a into a Cantor set α of arcs, each arc, except for $[0,a_{1,1}]$, being composed of an upper semicircle and half of each of two lower semicircles. Let $r_1: K_a \twoheadrightarrow [0,a_{1,1}]$ be a retraction so that r_1 is one-to-one on each arc in α, r_1 moves no point more than 1/2, $r_1^{-1}(0) = ((L-L_1) \cup M) \cap K_a$, and $r_1^{-1}(a_{1,1}) = L_1 \cap K_a$. We may take r_1 to be radial projection from (1/2,0) as center on upper semicircles, except near lower semicircles, and suitably modified so as to be one-to-one on the lower semicircles at each end of the upper semicircles. Coordinatize $[0,a_{1,1}]$ by $[0,2]$ so that the midpoint of the upper semicircle is identified with 1, $a_{1,1}$ is identified with 2, and extend linearly in between (in terms of the length of $[0,a_{1,1}]$ as a curve in E^2).

In order to simplify notation, let $L_{k,m}$, $M_{k,m}$, L_k, M_k, L, and M denote the cuts of K_a for the remainder of this section, rather than the arcs which do the cutting. We now define $r_2: K_a \twoheadrightarrow [0,a_{2,1}]$ and extend our coordinatization so that $[0,a_{2,1}]$ is identified with $[0,2^{n_2}]$, conditions

(1) and (2) are satisfied for k = 1,2, and (3) is satisfied for k = 1,2.

Observe that $r_1^{-1}(2)$ is naturally partitioned into m_1 cuts, namely $L_{1,1}, L_{1,2}, \cdots, L_{1,m_1}$, and that $r_1^{-1}(0)$ is partitioned into m_k+1 cuts, namely $(L-(L_1 \cup L_2)) \cup (M-M_1)$, $M_{1,1}, M_{1,2}, \cdots, M_{1,m_1-1}, L_2$. Moreover, for each $t \in (0,2)$, $r_1^{-1}(t)$ is naturally partitioned into $2m_1$ cuts, namely $[0,L_{1,1}] \cap r_1^{-1}(t), [L_{1,1},M_{1,1}] \cap r_1^{-1}(t), \cdots, [L_{1,m_1},L_2] \cap r_1^{-1}(t)$. For each $t \in [0,2] = [0,a_{1,1}]$, $r_1^{-1}(t)$ is a Cantor set, and the union over all t of such Cantor sets is K_a. Each such Cantor set is partitioned as noted above, and r_2 is defined as follows: For each $t \in [0,2]$,

for $x \in [0,L_{1,1}] \cap r_1^{-1}(t)$, $r_2(x) = [0,a_{1,1}] \cap r_1^{-1}(t)$,

for $x \in [L_{1,1},M_{1,1}] \cap r_1^{-1}(t)$, $r_2(x) = [a_{1,1},c_{1,1}] \cap r_1^{-1}(t)$,

⋮

for $x \in [L_{1,m},M_{1,m}] \cap r_1^{-1}(t)$, $r_2(x) = [a_{1,m},c_{1,m}] \cap r_1^{-1}(t)$,

for $x \in [M_{1,m},L_{1,m+1}] \cap r_1^{-1}(t)$, $r_2(x) = [c_{1,m},a_{1,m+1}] \cap r_1^{-1}(t)$,

⋮

for $x \in [L_{1,m_1},L_2] \cap r_1^{-1}(t)$, $r_2(x) = [a_{1,m_1},a_{2,1}] \cap r_1^{-1}(t)$.

Since r_2 carries $[0,a_{1,1}] = [0,2]$ to itself by the identity, carries $[a_{1,1},c_{1,1}]$ to $[0,a_{1,1}]$ one-to-one in reverse order, carries $[c_{1,1}, a_{1,2}]$ to $[0, a_{1,1}]$ one-to-one in order, etc., the action of r_2 on $[0, a_{2,1}]$ is similar to the projection of the graph of g_2 onto the second coordinate. We can extend our coordinatization by identifying $[0, a_{2,1}]$ with $[0, 2^{n_2}]$ in such a way that $r_1 = g_1 \circ r_2$, and (3) is satisfied for $k = 1,2$. For example, if $x \in [a_{1,1}, c_{1,1}]$ and $r_1(x) = t$, then identify x with (2)(2)-t; if $x \in [c_{1,1} a_{1,2}]$ and $r_1(x) = t$, then identify x with (2)(2)+t; if $x \in [a_{1,m_1}, a_{2,1}]$ and $r_1(x) = t$, then identify x with $2^{n_2} - t$.

We now define $r_3\colon K_a \twoheadrightarrow [0, a_{3,1}]$, extend our coordinatization so that $[0, a_{3,1}]$ is identified with $[0, 2^{n_3}]$, satisfy (1), (2), and (3) for $k = 1,2,3$, and satisfy (4) for $k = 1$. Observe that $r_2^{-1}(0) = (L-(L_1 \cup L_2)) \cup (M-M_1)$ and is partitioned into m_2+1 cuts, namely, $(L - \cup_{j=1}^{3} L_j) \cup (\cdot(M-(M_1 \cup M_2))$, $M_{2,1}, M_{2,2}, \cdots, M_{2,m_2-1}, L_3$. Also, $r_2^{-1}(2^{n2}) = L_2$, and is partitioned into m_2 cuts, namely, $L_{2,1}, \cdots, L_{2,m_2}$. Moreover, for each $t \in (0, 2^{n_2}) = (0, a_{2,1})$, $r_2^{-1}(t)$ is naturally partitioned into $2m_2$ cuts, namely, $[0, L_{2,1}] \cap r_2^{-1}(t), [L_{2,1}, M_{2,1}] \cap r_2^{-1}(t), \cdots, [L_{2,m_2}, L_3] \cap r_2^{-1}(t)$. Therefore, we define r_3 in a fashion similar to the way we defined r_2 above. For each $t \in [0, a_{2,1}] = [0, 2^{n_2}]$,

for $x \in [0, L_{2,1}] \cap r_2^{-1}(t)$, $r_3(x) = [0, a_{2,1}] \cap r_2^{-1}(t)$,

for $x \in [L_{2,1}, M_{2,1}] \cap r_2^{-1}(t)$, $r_3(x) = [a_{2,1}, c_{2,1}] \cap r_2^{-1}(t)$,

$\vdots$

$$\text{for } x \in [L_{2,m_2}, L_3] \cap r_2^{-1}(t),\ r_3(x) = [a_{2,m_2}, a_{3,1}] \cap r_2^{-1}(t).$$

We extend our coordinatization by identifying $[0,a_{3,1}]$ with $[0,2^{n_3}]$ in such a way that (1), (2), and (3) are satisfied for k = 3, just as we did for k = 2. Moreover, since the b- and d-points in $[0,a_{3,1}]$ all occur in the interval $[a_{2,m_2},a_{3,1}]$, and since each d-point lies above an a-point of stage k = 1, and each b-point lies below a c-point of stage k = 1, we have that (4) is satisfied for k = 1. Note that the $(2m_2-1)(2^{n_2})$-term of the coordinatization of $d_{1,m}$ and $b_{1,m}$ is in fact the coordinate of a_{2,m_2}.

For each k, $k \geq 4$, we define r_k in terms of r_{k-1} and the previously defined cuts of K_a just as we defined r_3 in terms of r_2 above.

We have therefore shown that for each sequence $N_a = \{a_1, a_2, \cdots\}$ of integers $a_k \geq 2$, there is an embedding K_a of the Knaster U-continuum K in E^2, with K_a the natural embedding of an inverse limit system $\hat{K}_a$ topologically equivalent to K.

3.3 *The prime end structure of* K_a. Suppose that K_a is the embedding of K constructed in Section 3.2 on the basis of the sequence N_a of integers each greater than 2. We use the notation of Section 3.2 below. Let A denote the collection of a- and b-points of C_a, ordered as on C_a, and let C denote the collection of c- and d-points of C_a, ordered as on C_a. For each p ∈ A, recall that Q(0,p) denotes a small lower crosscut from 0 to p. For each $q \in C$, recall that $M(q)$ denotes the midline of the pocket of which q is the closed end. Let $M(0,q)$ denote the union of $M(q)$ and a lower semicircle from 0 to the endpoint of $M(q)$ on the x-axis. Then $M(0,q)$ is a crosscut from 0 to q.

3.3.1 *Lemma. The set of accessible points of* K_a *is* C_a.

Proof. Every point of $[0,a_{1,1}] \cup A$ is clearly accessible. Every point of $C_a - ([0,a_{1,1}] \cup A)$ lies in one of the pockets identified in Section 3.2.3. Hence, every point $x \in C_a - ([0,a_{1,1}] \cup A)$ can be reached by a crosscut from 0 that follows $M(0,x)$, if x is a c- or d-point, or that follows $M(0,q)$, for some $q \in C$, to a point $y \in M(0,q)$, then follows half of a crosscut which is transverse to $M(0,q)$ at y and has x as an endpoint. Hence C_a is accessible at every point.

Suppose x is an accessible point of K_a. We claim $x \in C_a$. Suppose not. For each $p \in A$, $Q(0,p) \cup [0,p]$ is a simple closed curve. Moreover, $K_a \subset \mathrm{Cl}(\mathrm{Int}(Q(0,p) \cup [0,p]))$. Evidently, for $p > q \in A$, $\mathrm{Cl}(\mathrm{Int}(Q(0,p) \cup [0,p])) \subset \mathrm{Cl}(\mathrm{Int}(Q(0,q) \cup [0,q]))$, since we can chose that the crosscuts meet only in 0. In fact, $K_a = \cap_{p \in A} \mathrm{Cl}(\mathrm{Int}(Q(0,p) \cup [0,p]))$. Recall that $\mathrm{diam}(Q(0,p)) < 2d(0,p)$.

Let R be an endcut to x. Since $x \notin C_a$, and in virtue of the towering of the simple closed curves above, with K_a being their intersection, either there is a first $Q(0,p)$ that R doesn't meet, or R meets every $Q(0,p)$. In the first case, clearly $x \in [0,p]$. In the second case $x = 0$. Hence $x \in C_a$, a contradiction.

3.3.2 *Lemma. The prime ends of* E^2-K_a *are all trivial (of the first kind), except for that prime end* E *defined by any chain of crosscuts converging to* $0 \in C_a$.

Proof. If Q is a crosscut of E^2-K_a, then by Lemma 3.3.1, both endpoints of Q lie in C_a. Let J denote the arc of C_a irreducible between the endpoints of Q. Then $Q \cup J$ is a simple closed curve, and either $K_a \subset Cl(Int(Q \cup J))$ or $Cl(Int(Q \cup J)) \cap K_a = J$.

Let $\{Q_i\}_{i=1}^{\infty}$ be a chain of crosscuts defining a prime end F and let J_i be the arc in C_a irreducible between the endpoints of Q_i. If for some j, $Cl(Int(Q_j \cup J_j))$ fails to contain K_a, then for all $i \geq j$, $J_i \supset J_{i+1}$. Hence F is trivial. So suppose $K_a \subset Cl(Int(Q_i \cup J_i))$ for all i. We claim that $\{Q_i\}_{i=1}^{\infty}$ converges to 0. Suppose not. Then the endpoints of the Q_i's lie in K_a-$S(0,\varepsilon)$, where $S(0,\varepsilon)$ is an ε-ball about 0 and ε is sufficiently small, for all but finitely many Q_i. There is a j such that $diam(Q_i) < \varepsilon/2$ for all $i \geq j$. So we may assume that $Q_i \subset E^2 - Cl(S(0,\varepsilon/2))$, for all $i \geq j$. But then Q_j misses a ray R from 0 to ∞. Hence $K_a \not\subset Cl(Int(Q_j \cup J_j))$, a contradiction.

3.4 *Proof of Theorem* 1.2. Let $N_a = \{a_1, a_2, \cdots\}$ and $N_b = \{b_1, b_2, \cdots\}$ be sequences of integers such that for each $a_k \in N_a$ and each $b_j \in N_b$, $a_k \geq 2$ and $b_j \geq 2$. Let K_a (K_b) be the embedding of K following the procedures of Section 3.2 for N_a (N_b). Let C_a (C_b) denote the endpoint composant of K_a (K_b) coordinatized by $[0,\infty)$ as previously described. It follows that the set of accessible points of K_a (K_b) is C_a (C_b) by Lemma 3.3.1. Since the endpoint composant of K is unique, K_a and K_b have the same set of accessible points. By Lemma 3.3.2, K_a and K_b have the same prime end structure.

To prove Theorem 1.2, it suffices to show that if K_a and K_b are equivalent embeddings, then N_a and N_b are *not* distinct sequences (Definition 2.2 and Lemma 2.3). Suppose that $h: E^2 \twoheadrightarrow E^2$ is a homeomorphism such that $h(K_a) = K_b$. Since our proof relies on constructions of crosscuts and midlines always in a compact neighborhood of K_a or K_b, we lose no generality by assuming that h and h^{-1} are uniformly continuous.

Before proceeding further, we need to set up some notation. Let $n_1 = l_1 = 1$, let $n_k = 1 + \sum_{i=1}^{k-1} a_i$, and let $l_j = 1 + \sum_{i=1}^{j-1} b_i$. Let $m_k = 2^{a_k - 1}$ and let $i_j = 2^{b_j - 1}$. The a-, b-, c-, and d-points of C_a and C_b, with their coordinatizations, are then designated as follows:

	C_a	C_b
a-points:	$a_{k,m} = (2m-1)(2^{n_k})$	$\alpha_{j,i} = (2i-1)(2^{l_j})$
c-points:	$c_{k,m} = (2m)(2^{n_k})$	$\gamma_{j,i} = (2i)(2^{l_j})$
b-points:	$b_{k,m} = (2m_{k+1}-1)(2^{n_{k+1}}) + (2m)(2^{n_k})$	$\beta_{j,i} = (2i_{j+1}-1)(2^{l_{j+1}}) + (2i)(2^{l_j})$
d-points:	$d_{k,m} = (2m_{k+1}-1)(2^{n_{k+1}}) + (2m-1)(2^{n_k})$	$\delta_{j,i} = (2i_{j+1}-1)(2^{l_{j+1}}) + (2i-1)(2^{l_j})$

For a- and d-points of C_a, $1 \leq m \leq m_k$; for b- and c-points of C_a, $1 \leq m \leq m_k - 1$. For a- and d-points of C_b, $1 \leq i \leq i_j$; for b- and c-points of C_b, $1 \leq i \leq i_j - 1$.

Let $A = \{a_{1,1}, a_{1,2}, \cdots, a_{1,m_1}, a_{2,1}, \cdots, a_{2,m_2}, b_{1,1}, \cdots, b_{1,m_1-1}, a_{3,1}, \cdots\}$ and $B = \{\alpha_{1,1}, \cdots, \alpha_{1,i_1}, \alpha_{2,1}, \cdots, \alpha_{2,i_2}, \beta_{1,1}, \cdots, \beta_{1,i_1-1}, \alpha_{3,1}, \cdots\}$ be the sequences of a- and b-points of C_a and C_b, respectively, ordered as they appear on C_a and C_b. For $p \in A$ let $A(p)$ denote the infinite sequence that remains after removing the initial finite subsequence of points preceding p in A. Similarly define $B(q)$ for $q \in B$.

The aca-, adb-, bdb-, and bda-pcokets of C_a and C_b are as described in Section 3.2.3. The critical step in our proof will be to show that aca-pockets of C_a must be carried by h to aca-pockets of C_b, after removing some initial finite subsequence of pockets from each, and that a similar correspondence exists between each of the remaining types of pockets. This will be shown to imply that N_a and N_b are not distinct.

3.4.1 *Lemma. The homeomorphism* h *has the following properties:*

(1) $h(C_a) = C_b$, *with* $h(0) = 0$.

(2) h *is order-preserving from* C_a *onto* C_b.

(3) *There are points* $p_0 \in A$ *and* $q_0 \in B$ *such that we may assume* $h(A(p_0)) = B(q_0)$; *that is,* $h|A(p_0): A(p_0) \twoheadrightarrow B(q_0)$ *is one-to-one and order-preserving onto* $B(q_0)$.

Proof. Since C_a (C_b) is the only composant of K_a (K_b) which is a ray, properties (1) and (2) follow from the fact that $h|K_a$ is a homeomorphism onto K_b. Property (3), however, follows from the fact that h is defined on E^2, as we shall show.

For each $p \in A$, chose a short lower crosscut $Q(0,p)$ (Section 3.2.4) so that for $p \neq q \in A$, $Q(0,p) \cap Q(0,q) = \{0\}$. We can do this because the reversed projection order on A duplicates the order of A (induced by C_a). Observe that $\mathrm{diam}(Q(0,p)) \to 0$. In a sufficiently small ball $S(0,\varepsilon)$ about $0 \in C_b$, the components of $C_b \cap S(0,\varepsilon)$ that contain an outer semicircle separate all other components of $C_b \cap S(0,\varepsilon)$ from 0. Since $\mathrm{diam}(Q(0,p)) \to 0$ implies that $\mathrm{diam}(h(Q(0,p)))$ $\to 0$, it follows that there is a $p_1 \in A$ such that for all $p \geq p_1 \in A$, $h(p)$ lies on an outer semicircle of C_b. (More precisely, $h(p)$ lies on a component of $C_b \cap S(0,\varepsilon)$ that contains an outer semicircle, for sufficiently small ε.)

Suppose that infinitely often h carries a pair of consecutive points of $A(p_1)$ to the same outer semicircle of C_b. That is, suppose there is a sequence $\{\{r_i,t_i\}\}_{i=1}^{\infty}$ of consecutive points of $A(p_1)$ and a sequence $\{C_i\}_{i=1}^{\infty}$ of outer semicircles of C_b, such that $h(r_i),h(t_i) \in C_i$. Then $h([r_i,t_i]) \subset C_i$, since h is order-preserving onto C_b. But between each r_i and t_i on C_a there is a point s_i such that $s_i \to 1 \in C_a$. Hence $h(s_i) \to h(1) \in C_b$. However, $C_i \to 0 \in C_b$, and $0 \neq h(1)$. In view of this contradiction, we may suppose that there is a $p_2 \in A$, $p_2 \geq p_1$, such that for all $p \neq q \in A(p_2)$, $h(p)$ and $h(q)$ lie on different outer semicircles of C_b.

Since each outer semicircle of C_b contains exactly one point of B, we lose no generality by assuming that the image of each point in $A(p_2)$ is a point of B. Hence we have that $h|A(p_2): A(p_2) \to B$ is one-to-one and order-preserving with respect to the orders of C_a and C_b, into B. By repeating the preceding argument for $h^{-1}: E^2 \twoheadrightarrow E^2$ such that $h^{-1}(K_b) = K_a$, we may assume that there is a $q_1 \in B$ such that the image of each point of $B(q_1)$ under h^{-1} is a point of A. Hence we have that $h^{-1}|B(q_1): B(q_1) \to A$ is one-to-one and order-preserving into A.

Since A converges to $0 \in C_a$ and B converges to $0 \in C_b$, there is a $q_2 \in B$, $q_2 \geq q_1$, such that $h^{-1}(B(q_2)) \subset A(p_2)$. Let $p_3 = h^{-1}(q_2)$ in $A(p_2)$. Then $A(p_3) \subset A(p_2)$. If $p \geq p_3 \in A$, then $h(p) \geq h(p_3) = q_2$. Hence $h(p) \in B(q_2)$. We claim that $h|A(p_3): A(p_3) \to B(q_2)$ is onto. Let $q \in B(q_2)$. Then $q \geq q_2$, and since $h^{-1}|B(q_2)$ is one-to-one and order-preserving into $A(p_2)$, $h^{-1}(q) \geq h^{-1}(q_2) = p_3$. But then $h^{-1}(q) \in A(p_3)$. Let $p_0 = p_3$ and $q_0 = q_2$, and property (3) is established.

Recall that on C_a (C_b), between each pair of consecutive points of A (B), there is a unique c- or d-point of C_a (C_b) which is the closed end of the pocket in C_a (C_b) of which the consecutive pair of points is the open end.

3.4.2 *Lemma. There are points* $p_0 \in A$ *and* $q_0 \in B$ *such that if* $p_1 < p_2 \in A(p_0)$ *are consecutive points and* r *is the* c- *or* d-*point that lies between them on* C_a, *then we may assume that pocket* $[p_1,r,p_2] \subset C_a$ *is carried by* h *onto a pocket* $[q_1,s,q_2] \subset C_b$, *where we may assume that* $h(p_1) = q_1$ *and* $h(p_2) = q_2$ *are consecutive in* $B(q_0)$ *and that* $s = h(r)$

is the c- *or* d-*point of* C_b *that lies between* q_1 *and* q_2.

Proof. By Lemma 3.4.1, we can find $p_0 \in A$ and $h(p_0) = q_0 \in B$ satisfying the lemma, except possibly for the condition that h(r) be the c- or d-point of C_b that lies between q_1 and q_2. However, since h is order-preserving onto C_b, and since the c- and d-points of C_a converge to 0, we may assume that h(r) lies in *some* lower semicircle $C_1 \subset [q_1,q_2] \subset C_b$. Let s be the c- or d-point of C_b contained in $[q_1,q_2]$. There is only one such point, and it lies on the inner semicircle $C_2 \subset [q_1,q_2]$, of which there is only one contained in $[q_1,q_2]$. So, if we can show that C_1 and C_2 are actually the same lower semicircle, then we may assume that $h(r) = s$.

Suppose that C_1 and C_2 are distinct lower semicircles. Then C_2 contains the closed end of the pocket, and C_1 is on one side of the pocket $[q_1,s,q_2]$. Hence there are upper semicircles C_3 and C_4 in $[q_1,q_2]$ such that C_3 lies between C_1 and C_2 on C_b and C_4 is parallel to C_3 but on the opposite side of the pocket. Let $x \in C_2$ and $y \in C_1$. We claim that any crosscut from x to y has $\text{diam}(Q) > 1/4$. We may assume that $p_0 \in A$ and $q_0 \in B$ have been chosen so that for all consecutive $q_1 < q_2 \in B(q_0)$, $[q_1,q_2]$ is an ε-pocket with $0 < \varepsilon < 1/16$ and $1/3-2\varepsilon > 1/4$, since $2/2^{l_j} \to 0$. For each crosscut R from C_3 to C_4 that is transverse to the midline M(s) of pocket $[q_1,q_2]$ and has $\text{diam}(R) < \varepsilon$, Q must meet R. Since $\text{diam}(C_3) \geq 1/3$, because C_3 is an upper semicircle, we have $\text{diam}(Q) \geq \text{diam}(C_3)-2\varepsilon \geq 1/3-2\varepsilon > 1/4$.

In the order on C_b either $C_2 < C_1$ or $C_2 > C_1$. We assume that $C_2 < C_1$; the case for $C_2 > C_1$ is symmetric.

Since h is uniformly continuous, chose $\delta > 0$ so that for all $x,y \in E^2$, if $d(x,y) < \delta$, then $d(h(x),h(y)) < \varepsilon$. We may assume that $p_0 \in A$ and $h(p_0) = q_0 \in B$ have been chosen so that for all consecutive $p_1 < p_2 \in A(p_0)$, $[p_1,p_2]$ is a δ-pocket, since $2/2^{n_k} \to 0$. Let $M(r)$ be the midline of pocket $[p_1,r,p_2]$. Then for each $x \in [p_1,r)$, there is a $y \in (r,p_2]$ and a crosscut Q from x to y transverse to $M(r)$ with $\text{diam}(Q) < \delta$. Thus $\text{diam}(h(Q)) < \varepsilon$.

Since we have assumed that $C_2 < C_1$, we have $q_1 < s < h(r) < q_2$. Consequently, we have $p_1 < h^{-1}(s) < r < p_2$. There is an $x \in C_2$ so that $x < s$. Then $p_1 < h^{-1}(x) < h^{-1}(s) < r$. Since $[p_1,p_2]$ is a δ-pocket, there is a $y \in (r,p_2]$ and a crosscut Q from $h^{-1}(x)$ to y such that Q is transverse to $M(r)$ and $\text{diam}(Q) < \delta$. Then $h(Q)$ is a crosscut from x to $h(y)$ and $\text{diam}(h(Q)) < \varepsilon$.

Since $[q_1,q_2]$ is an ε-pocket, and since $s < h(r) < q_2$, there is a $z \in [q_1,s)$ and a crosscut R from z to $h(r)$ such that R is transverse to $M(s)$ and $\text{diam}(R) < \varepsilon$. If it were the case that $z \in [x,s)$, then z would be in C_2. Hence R would be a crosscut from $z \in C_2$ to $h(r) \in C_1$, and so would have $\text{diam}(R) > 1/4$. Hence we must have $q_1 < z < x < s < h(r) < h(y) < q_2$.

We may assume that R and $h(Q)$ are inside pocket $[q_1,q_2]$. Now $R \cup [z,h(r)]$ is a simple closed curve with $h(y)$ contained in $\text{Ext}(R \cup [z,h(r)])$. Moreover, $\text{Int}(R \cup [z,h(r)])$ is inside the pocket $[q_1,q_2]$ and its closure contains x. Hence crosscut $h(Q)$ from x to $h(y)$ must meet crosscut R from z to $h(r)$. Thus $R \cup h(Q)$ contains a crosscut

T from x to h(r), and $\text{diam}(T) \leq \text{diam}(h(Q)) + \text{diam}(R) < \varepsilon + \varepsilon < 1/8$. However, T is a crosscut from $x_1 \in C_2$ to $h(r_1) \in C_1$, so $\text{diam}(T) > 1/4$. In view of this contradiction, it follows that C_1 and C_2 are the same lower semicircle.

It follows from Lemma 3.4.2 that we can find $p_0 \in A$ and $q_0 = h(p_0) \in B$, so that $h|[p_0,\infty): [p_0,\infty) \twoheadrightarrow [q_0,\infty)$ carries pockets in $[p_0,\infty) \subset C_a$ onto pockets in $[q_0,\infty) \subset C_b$ in a one-to-one, order-preserving correspondence. However, given, say, an aca-pocket in $[p_0,\infty)$, we have not yet determined what *type* of pocket in $[q_0,\infty)$ its image under h must be. The next lemma establishes that h must preserve types of pockets from $[p_0,\infty)$ onto $[q_0,\infty)$.

3.4.3 *Lemma. Let* $p_0 \in A$ *and* $q_0 = h(p_0) \in B$ *be chosen so as to satisfy Lemmas* 3.4.1 *and* 3.4.2. *Then* $h|[p_0,\infty): [p_0,\infty) \twoheadrightarrow [q_0,\infty)$ *carries each* aca-, adb-, bdb-, *and* bda-*pocket in* $[p_0,\infty) \subset C_a$ *onto a pocket in* $[q_0,\infty) \subset C_b$ *of precisely the same type, in a one-to-one, order-preserving correspondence.*

Proof. We argue in this first paragraph that the proof reduces to showing that the image under h of each aca-pocket of $[p_0,\infty)$ is an aca-pocket of $[q_0,\infty)$. Assume that is the case. Then by symmetry, the assumption also holds for aca-pockets of $[q_0,\infty)$ and $h^{-1}|[q_0,\infty): [q_0,\infty) \twoheadrightarrow [p_0,\infty)$. We may assume that $p_0 \in A$ has been chosen so that $p_0 = a_{k,1}$ for some k. Consequently, by the order of a- and b-points on C_a, the consecutive pockets of $[p_0,\infty)$ appear in the following *groupings*: a finite number of aca-pockets, one adb-pocket, a finite number of bdb-pockets, one bda-pocket, followed by the first aca-pocket of the next grouping. By our assumption, the image of each aca-pocket in $[p_0,\infty)$ is an aca-pocket in $[q_0,\infty)$. Consider the image of an adb-pocket of $[p_0,\infty)$. It cannot be an aca-pocket of $[q_0,\infty)$, because this would contradict our assumption regarding h^{-1}. It could not be a bdb-pocket in $[q_0,\infty)$, since we know that its first endpoint must be an a-point of C_b because this endpoint is the last endpoint of the previous aca-pocket. Similarly, it cannot be a bda-pocket. So the image of an adb-pocket of $[p_0,\infty)$ must be an adb-pocket of $[q_0,\infty)$. By similar reasoning, the image of each bdb- and bda-pocket of $[p_0,\infty)$ must be, respectively, a bdb- and bda-pocket of $[q_0,\infty)$. Since the above argument holds for each grouping in turn on $[p_0,\infty)$, it follows that $h|[p_0,\infty)$ preserves types of pockets.

Let $p_1 < p_2 \in A(p_0)$ be consecutive, so that $[p_1,r,p_2]$ is an aca-pocket in $[p_0,\infty)$. It suffices, by the above argument, to show that $[q_1,s,q_2] \subset [q_0,\infty)$ is also an aca-pocket, where $h(p_1) = q_1$, $h(p_2) = q_2$, and $h(r) = s$, r being the c-point between p_1 and p_2 and s being the c-point between q_1 and q_2. (It follows from Lemma 3.4.2 that we may assume $h(r) = s$ and that s is either a c-point *or* a d-point. We aim to show s must be a c-point.)

By way of contradiction, suppose that $[q_1,s,q_2]$ is not an aca-pocket of $[q_0,\infty)$. It certainly is a pocket, so we may assume, without loss of generality, that it is a bdb-pocket. The proof for the remaining two cases is almost exactly the same.

We may suppose that $p_0 \in A$ and $h(p_0) = q_0 \in B$ have been chosen to satisfy not only Lemmas 3.4.1 and 3.4.2, but also the conditions:

(1) $1/16 > \eta > 0$ and $\eta > \varepsilon > 0$ are chosen, by uniform continuity of h^{-1}, so that for all $x,y \in E^2$ with $d(x,y) < \varepsilon$, $d(h^{-1}(x),h^{-1}(y)) < \eta$.

(2) $\varepsilon > \delta > 0$ is chosen, by uniform continuity of h, so that for all $x,y \in E^2$ with $d(x,y) < \delta$, $d(h(x),h(y)) < \varepsilon$.

(3) The outer and inner semicircles containing all a-, b-, c-, and d-points of $[p_0,\infty)$ are contained in $S(0,\delta)$.

(4) The outer and inner semicircles containing all a-, b-, c-, and d-points of $[q_0,\infty)$ are contained in $S(0,\varepsilon)$.

As r is a c-point of C_a, it lies on an inner semicircle C(r) and there is a b-point t of C_a directly below r and lying on an outer semicircle C(t), parallel to C(r). Since the closed end of an aca-pocket lies to the right of the open end in projection order, since t is directly below r, and since reverse projection order duplicates the order on C_a for points of A, we have $p_0 \leq p_1 < p_2 < t$ on $[p_0,\infty)$. Moreover, C(r), C(t), and all the lower semicircles parallel to them are contained in $S(0,\delta)$.

Let L(r,t) be the vertical segment joining r to t. Note that $L(r,t) \cap K_a$ is one of the cuts of K_a previously defined. Also note that $L(r,t) \subset S(0,\delta) \subset S(0,\eta)$. Now C(t), together with the short arcs in $S(0,\eta)$ of the two upper semicircles which meet C(t) at its endpoints, locally separates $L(r,t) - \{t\}$ from 0; that is, t is the only point of L(r,t) that can be reached by a crosscut from 0 of diameter less than η. (See Figure 8.) This holds because outer semicircles, together with short arcs of of the upper semicircles which meet them, separate all lower semicircles parallel to them from 0 in a sufficiently small neighborhood of 0.

Since $h|[p_0,\infty)$ is order-preserving, we have $q_0 \leq q_1 < q_2 < h(t)$ on $[q_0,\infty)$. We have assumed that $s = h(r)$ is the d-point at the closed end of the bdb-pocket $[q_1,s,q_2]$. Hence there is an a-point u of C_b directly below s. Let C(s) denote the inner semicircle containing s and C(u) denote the outer semicircle containing u. Since the closed end of a bdb-pocket is to the right of the open end in projection order, since u is directly below s, and since reverse projection order duplicates the order on C_b for points of B, we have $0 < u < q_1 < q_2$ on C_b.

Let $C(u)^+$ denote C(u) union the short arcs of the two upper semicircles that meet C(u) contained in $S(0,\varepsilon)$. For any $x \in C(u)^+$, we have $0 < x < q_1 < q_2$. Moreover, x can be reached by a crosscut from 0 contained in $S(0,\varepsilon)$.

Since $q_1 < q_2 < h(t)$, it follows that we have $h(t) \notin C(u)^+$. However, h(t) lies on *some* outer semicircle C(h(t)) in $[q_0,\infty)$, since $h(t) \in B(q_0)$. Now h(L(r,t)) is an arc in $S(0,\varepsilon)$ from s to h(t), and h(L(r,t)) must cut K_b. Since $C(u)^+$ separates all other outer semicircles in $S(0,\varepsilon)$ from s, we have that $h(L(r,t)) \cap C(u)^+ \neq \emptyset$.

Let $x \in h(L(r,t)) \cap C(u)^+$. Obviously, $x \neq h(t)$. Let Q be a crosscut from 0 to x, $Q \subset S(0,\varepsilon)$. Then $h^{-1}(Q)$ is a crosscut from 0 to $h^{-1}(x) \in L(r,t) - \{t\}$ with $h^{-1}(Q) \subset S(0,\eta)$.

But this contradicts the fact that no point of $L(r,t) - \{t\}$ can be reached in $S(0,\eta)$ by a crosscut from 0. In view of this contradiction, $[q_1,s,q_2]$ cannot be a bdb-pocket.

3.4.4 *Lemma. There are positive integers* K *and* J *such that the sequence* $N_a - \{a_1,a_2,\cdots,a_{K-1}\}$ *is identical to the sequence* $N_b - \{b_1,b_2,\cdots,b_{J-1}\}$. *Consequently,* N_a *and* N_b *are not distinct.*

Proof. Let $p_0 \in A$ and $h(p_0) = q_0 \in B$ be chosen so as to satisfy Lemma 3.4.3. We may assume that for some K, $p_0 = a_{K,1} \in A$. Then $A(p_0) = \{a_{K,1},a_{K,2},\cdots,a_{K,m_K},b_{K-1},\cdots,b_{K-1,m_{K-1}-1},a_{K+1,1},\cdots\}$. Lemma 3.4.3 implies that $h(a_{K,1}) = \alpha_{J,I}$, for some J and some I, $1 \leq I \leq i_J$. This holds because $h|[p_0,\infty)$ preserves types of pockets. By similar reasoning, it follows that $h(a_{K,2}) = \alpha_{J,I+1},\cdots$, $h(a_{K,m_K}) = \alpha_{J,I+m_K-1},h(b_{K-1,1}) = \beta_{J-1,1}$, and so forth. In particular, it follows that h carries $A(a_{K+1,1})$ onto $B(\alpha_{J+1,1})$ in a one-to-one order-preserving correspondence that, moreover, preserves types of points (a-points go to a-points, b-points go to b-points). It follows that $m_{K+1} = i_{J+1}$ (from the a-points) and that $m_K = i_J$ (from the b-points).

By the same reasoning applied to each successive grouping $\{a_{K+r,1},\cdots,a_{K+r,m_{K+r}},b_{K+r-1,1},\cdots,b_{K+r-1,m_{K+r-1}},a_{K+r+1,1}\}$ carried by h onto the grouping $\{\alpha_{J+r,1},\cdots,\alpha_{J+r,i_{J+r}},\beta_{J+r-1,1},\cdots,\beta_{J+r-1,i_{J+r-1}},\alpha_{J+r+1,1}\}$ for each $r \geq 2$, it follows that for each $r \geq 0$, $m_{K+r} = i_{J+r}$. Since

$m_{K+r} = 2^{a_{K+r}-1}$ and $i_{J+r} = 2^{b_{J+r}-1}$, it follows that for all $r \geq 0$, $a_{K+r} = b_{J+r}$ for $a_{K+r} \in N_a$ and $b_{J+r} \in N_b$. Therefore, $N_a - \{a_1, a_2, \cdots, a_{K-1}\}$ is identical to $N_b - \{b_1, b_2, \cdots, b_{J-1}\}$. Hence N_a and N_b are not distinct.

In view of Lemma 3.4.4, if N_a and N_b are distinct sequences of integers greater than or equal to 2, then the corresponding embeddings K_a and K_b of K in E^2 are inequivalent. Since there are uncountably many such distinct sequences, Theorem 1.2 is established.

4. Uncountabley Many Inequivalent Embeddings of Uncountably Many Knaster Continua

In [W] Will Watkins has shown that there are uncountably many nonhomeomorphic Knaster continua, answering a question posed by J. W. Rogers, Jr. in [R]. We have considered only the simplest of these continua, namely K^2, which is produced as the inverse limit of arcs under two-to-one bonding maps. If we were to use instead six-to-one bonding maps, the resulting *six-fold* Knaster continuum K^6 would not be homeomorphic to K^2, though, like K^2, it would have only one endpoint.

For the sake of uniformity in describing a variety of Knaster continua, we alter our inverse limit notation slightly, so that the i^{th} bonding map will have domain the i^{th} space and range the $(i-1)^{th}$ space. We will henceforth use $[0,1]$ as the 0^{th} space, where the bonding functions are indexed by $i = 1, 2, \cdots$.

Let k, m, and n be positive integers such that $n = km$. Define a *k-to-one rooftop function* $f_m^n\colon [0,n] \twoheadrightarrow [0,m]$ by

$$f_m^n(rm) = \begin{cases} 0, & \text{for even } r \\ m, & \text{for odd } r \end{cases}, \quad r = 0,1,2,\cdots,k,$$

and extend linearly between rm and (r+1)m, for $0 \le r \le k-1$.

For positive integers $n_1, n_2, \cdots, n_i$, let $\prod_{j=1}^{i} n_j = n_1 n_2 \cdots n_i$ and let $\prod_{j=1}^{0} n_j = 1$.

4.1 *Embeddings of the six-fold Knaster continuum.* Let the *six-fold Knaster continuum* K^6 be defined by the inverse limit system $K^6 = \varprojlim\{[0,6^i], g_i\}_{i=1}^{\infty}$, where $g_i = f_{6^{i-1}}^{6^i}$ is a six-to-one rooftop function. The *standard embedding* K_0^6 of K^6 in E^2 is illustrated in Figure 9. The Cantor set $\mathcal{C}^6$ used in the construction is *not* a middle third Cantor set, but rather, after the first stage, is a 5/11-Cantor set. (In constructing a 5/11-Cantor set, five of eleven equal subintervals are deleted from each subinterval of stage k, so that there remain six disjoint intervals of stage k+1 in each interval of stage k.)

For each sequence $N_a = \{a_1, a_2, \cdots\}$ of positive integers, there is an inverse limit system

$$\hat{K}_a^6 = \varprojlim\{[0,6^{n_i}], g_{a_i}\}_{i=1}^{\infty}$$

where $n_0 = 0$, $n_i = \sum_{j=1}^{i} a_j$, and $g_{a_i} = f_{6^{n_{i-1}}}^{6^{n_i}}$ is a 6^{a_i}-to-one rooftop function. Since a 6^{a_i}-to-one rooftop function is the composition of a_i six-to-one rooftop functions, $\hat{K}_a^6$ is homeomorphic to K^6.

It can be shown, by techniques similar to those of Section 3, that there is a *natural embedding* K_a^6 of $\hat{K}_a^6$ in E^2. The recursive procedure for constructing the embedding is the same as that in Section 3.2.1, except that $\mathcal{C}^6$ replaces

$\mathcal{C}$, and 6^{a_i} replaces 2^{a_i}, while $6^{a_i}/2$ replaces m_i. Using the notion of pockets, it can be shown that if K_a^6 and K_b^6 are equivalent embeddings, then N_a and N_b are not distinct. Thus we obtain a result analogous to Theorem 1.2 for K^6. All prime ends of K_a^6 are trivial, except for one of the second kind, and the set of accessible points is the unique endpoint composant.

4.2 *Uncountably many nonhomeomorphic Knaster continua.* Following Watkins [W], we define an uncountable collection of nonhomeomorphic Knaster continua. Let $N = \{n_1, n_2, n_3, \cdots\}$ be a sequence of positive integers. Define the inverse limit system

(1) $K^N = \varprojlim\{[0, \prod_{j=1}^{i} n_j], g_{n_i}\}_{i=1}^{\infty}$

where $g_{n_i} = f_{\prod_{j=1}^{i-1} n_j}^{\prod_{j=1}^{i} n_j}$ is an n_i-to-one rooftop function.

If infinitely many of the n_i's are even, then K^N is a Knaster continuum with exactly one endpoint, called a *U-type* Knaster continuum. Continua K^6 and K^2 are both U-type Knaster continua, based, respectively, on the sequences $\{2,2,2,\cdots\}$ and $\{6,6,6,\cdots\}$. If only finitely many n_i's are even, then K^N is a continuum with exactly two endpoints, called an *S-type* Knaster continuum. The simplest S-type Knaster continuum is K^3 based on the sequence $\{3,3,3,\cdots\}$. Obviously, no U-type Knaster continuum is homeomorphic to any S-type Knaster continuum.

Watkins constructs the uncountable collection of nonhomeomorphic Knaster continua as follows: Let

$P' = \{p_1, p_2, p_3, \cdots\}$ be a (finite or infinite) subset of the primes, ordered so that $p_i < p_{i+1}$. Form the infinite sequence

(2) $P = \{p_1, p_1, p_2, p_1, p_2, p_3, p_1, p_2, p_3, p_4, \cdots\}$

in which each $p_i \in P'$ appears infinitely often. Re-index P without changing the order in which the primes appear so that $P = \{p_1', p_2', p_3', \cdots\}$. Let K^P be the inverse limit system defined by

(3) $K^P = \varprojlim\{[0, \prod_{j=1}^{i} p_j'], g_{p_i'}\}_{i=1}^{\infty}$

as defined in (1) above, but with $N = P$.

Watkins shows that if P' and Q' are different subsets of primes (that is, there is a prime $p \in P' - Q'$ or $q \in Q' - P'$), then, with P and Q formed from P' and Q' as in (2), the continua K^P and K^Q are *not* homeomorphic [W, Theorem 3a]. The conclusion that there are uncountably many nonhomeomorphic Knaster continua follows from the fact that there are uncountably many different subsets of the primes.

If the prime 2 should appear in P', then 2 will appear infinitely often in the sequence P. Hence K^P will be a U-type Knaster continuum. However, if $2 \notin P'$, then every element of P will be odd. Hence K^P will be an S-type Knaster continuum. Since there are uncountably many subsets of the primes which include (do not include) 2, there are uncountably many nonhomeomorphic U-type (S-type) Knaster continua.

The techniques of Section 3 can be adapted to S-type Knaster continua, but we restrict our attention to the U-type continua.

4.3 *Uncountably many inequivalent embeddings of each of uncountably many U-type Knaster continua.* Let $P' = \{p_1, p_2, p_3, \cdots\}$ be a subset of the primes with $p_1 = 2$. We define an inverse limit system as follows: Let $m_i = \prod_{j=1}^{i} p_j$. Let $P^* = \{m_1, m_2, m_3, \cdots\}$. Let

(4) $K^{P^*} = \varprojlim\{[0, \prod_{j=1}^{i} m_j], g_{m_i}\}_{i=1}^{\infty}$

as defined in (1) above with $N = P^*$. Given P', K^P (see (3)) and K^{P^*} are homeomorphic, since $g_{m_i} = g_{p_1} \circ g_{p_2} \circ \cdots \circ g_{p_i}$. The advantage of K^{P^*} is that each m_i is even. This makes the standard embedding of K^{P^*} easier to construct.

The *standard embedding* $K_0^{P^*}$ of K^{P^*} in E^2 is constructed much like the standard embedding of K^6. We start with a Cantor set $\mathcal{C}^{P^*}$ in which the number of intervals deleted at each stage of construction depends upon m_i. Thus, we first delete a middle third interval in $I_0 = [0,1]$, leaving two intervals $I_{0,1}$ and $I_{0,2}$. We divide each $I_{0,j}$ into $2m_1-1$ equal intervals, and delete the "middle" m_1-1 of them, leaving m_1 disjoint intervals $I_{0,j,k}$ contained in each $I_{0,j}$. We divide each $I_{0,j,k}$ into $2m_2-1$ equal intervals, delete the "middle" m_2-1 of them, leaving m_2 disjoint intervals $I_{0,j,k,l}$ in each $I_{0,j,k}$. We continue this process for each m_i, constructing the Cantor set

$$\mathcal{C}^{P^*} = I_0 \cap (I_{0,1} \cup I_{0,2}) \cap (\cup_{j=1}^{2} \cup_{k=1}^{m_1} I_{0,j,k}) \cap (\cup_{j=1}^{2} \cup_{k=1}^{m_1} \cup_{l=1}^{m_2} I_{0,j,k,l}] \cap \cdots.$$

The recursive procedure for constructing $K_0^{P^*}$, given $\mathcal{C}^{P^*}$, is the same as the procedure in Section 3.2.1, except that m_i replaces 2^{a_i} and $m_i/2$ replaces the m_i of 3.2.1.

Let $N_a = \{a_1, a_2, a_3, \cdots\}$ be a sequence of positive integers. Let $n_0 = 0$, and let $n_i = \sum_{j=1}^{i} a_j$. Define the maps

$$g_{a_i} = g_{m_{n_{i-1}+1}} \circ \cdots \circ g_{m_{n_i-1}} \circ g_{m_{n_i}}:$$

$$[0, \prod_{j=1}^{n_i} m_j] \twoheadrightarrow [0, \prod_{j=1}^{n_{i-1}} m_j]$$

Let $\hat{K}_a^{P*}$ be the inverse limit system

$$\hat{K}_a^{P*} = \varprojlim\{[0, \prod_{j=1}^{n_i} m_j], g_{a_i}\}_{i=1}^{\infty}$$

Since g_{a_i} is the composition of the next a_i unused g_{m_j}'s, $\hat{K}_a^{P*}$ is homeomorphic to K^{P*}.

The *natural embedding* K_a^{P*} of $\hat{K}_a^{P*}$ in E^2 starts with the Cantor set $\mathcal{C}^{P*}$, constructed above for the standard embedding. However, the recursive procedure for constructing K_a^{P*} replaces 2^{a_i} in Section 3.2.1 with $r_i = \prod_{j=n_{i-1}+1}^{n_i} m_j$, and replaces the m_i of Section 3.2.1 with $r_i/2$.

Techniques similar to those of Sections 3.2 through 3.4 can be used to prove an analog of Theorem 1.2 for each K^{P*}. That is, K_a^{P*} has as its set of accessible points the unique ray-like composant of its endpoint; the prime end structure of K_a^{P*} consists of trivial prime ends, except for one of the second kind; and, if K_a^{P*} and K_b^{P*} are equivalently embedded in E^2, then N_a and N_b are not distinct. Hence there are uncountably many inequivalent embeddings of K^{P*}. Moreover, there are uncountably many U-type Knaster continua for which the above statements are true.

4.4 *Questions.* We conclude with several questions concerning extensions of Theorem 1.2 to additional continua.

4.4.1 *Question.* Are there uncountably many inequivalent embeddings of the pseudo arc with the same prime end structure?

4.4.2 *Question.* Are there uncountably many inequivalent embeddings of every indecomposable chainable continuum (with the same prime end structure)?

4.4.3 *Question.* Are there uncountably many inequivalent embeddings of every (of some) indecomposable nonchainable continuum (with the same prime end structure)?

4.4.4 *Question.* Does Ingram's [I] atriodic nonchainable tree-like continuum have uncountably many inequivalent embeddings (with the same prime end structure)?

4.4.5 *Question.* Does the atriodic nonchainable tree-like continuum of [M-1] have uncountably many inequivalent embeddings with the same prime end structure, specifically, all with a simple dense canal? Do all embeddings of that continuum have a simple dense canal?

We assume all the embeddings referred to above are into the plane. Answers to Questions 4 and 5, in particular may be of some relevance to the fixed point problem for nonseparating plane continua.

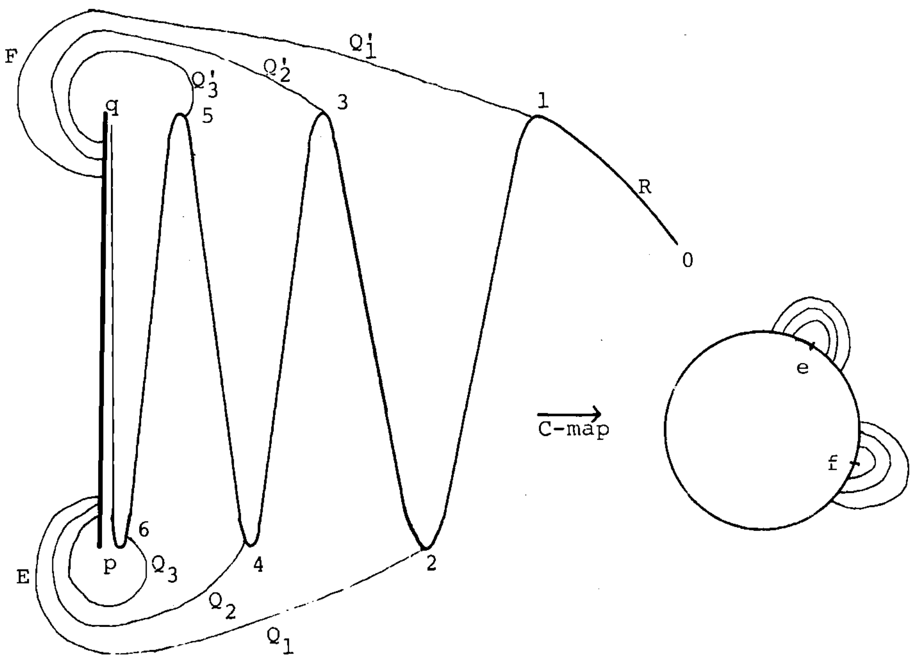


Figure 1. Standard embedding K of sin 1/x continuum.

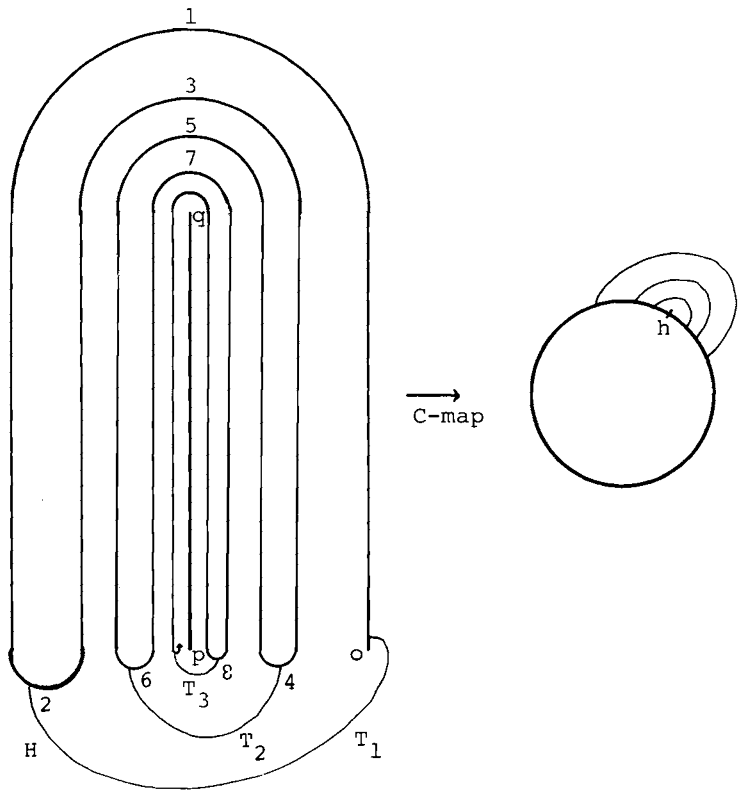


Figure 2. Embedding M_o of sin 1/x continuum based on schema $\{S_o, S_o, S_o, \cdots\}$.

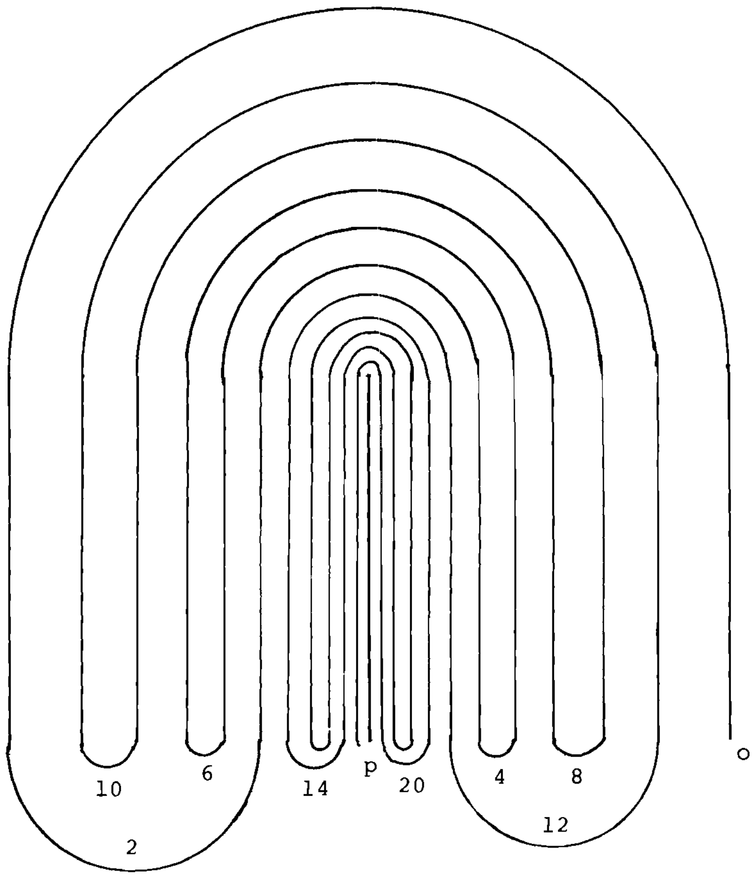
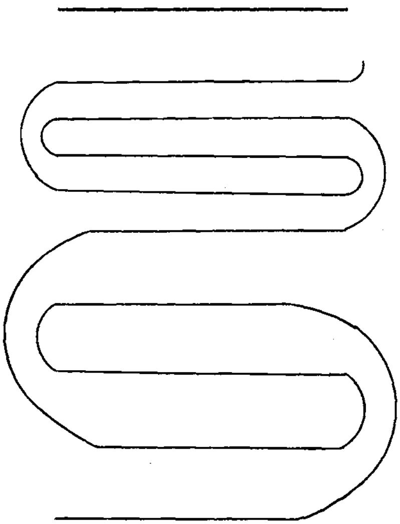
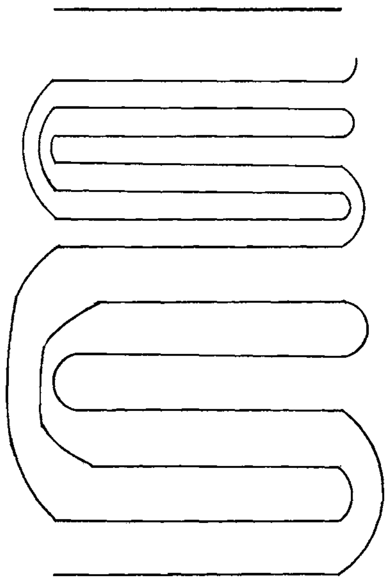


Figure 3. Embedding of sin 1/x continuum based on schema $\{S_2, S_1, \cdots\}$.

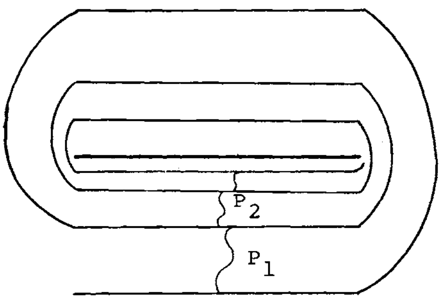


(a)

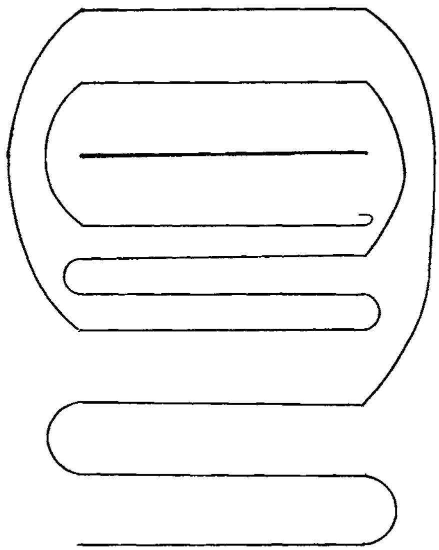


(b)

Figure 4. Embeddings of sin 1/x continuum with same prime end structure as K.



(a)



(b)

Figure 5. Embeddings of sin 1/x continuum with a prime end of the third kind.

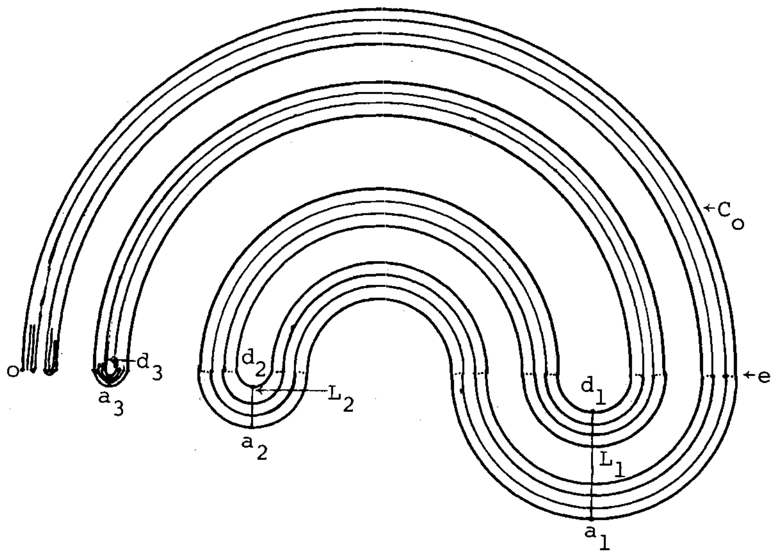


Figure 6. Standard embedding K_o of Knaster U-continuum (bucket handle).

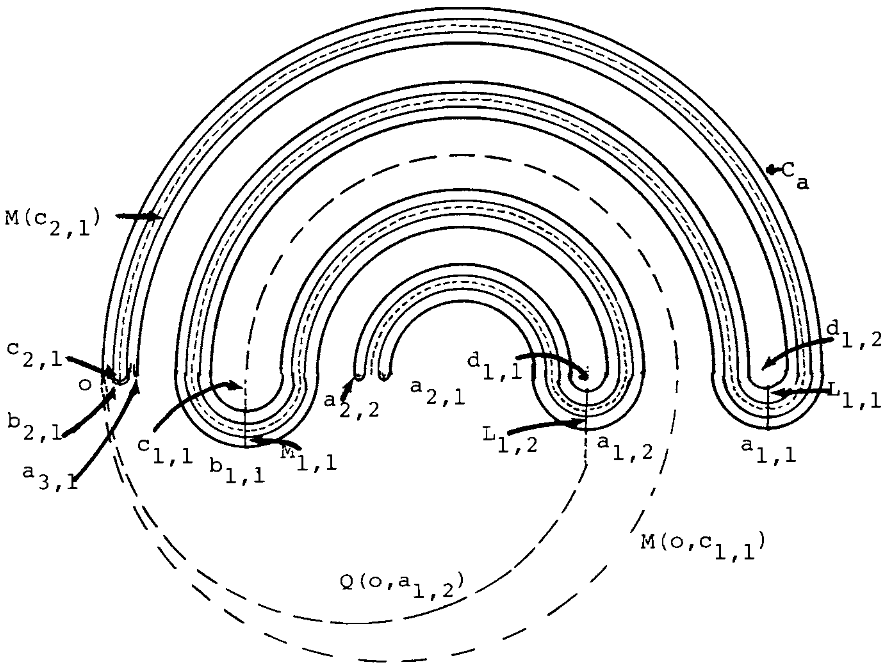


Figure 7. Natural embedding K_a of Knaster U-continuum K, defined by inverse limit $\hat{K}_a$ based on $N_a = \{2,2,\cdots\}$.

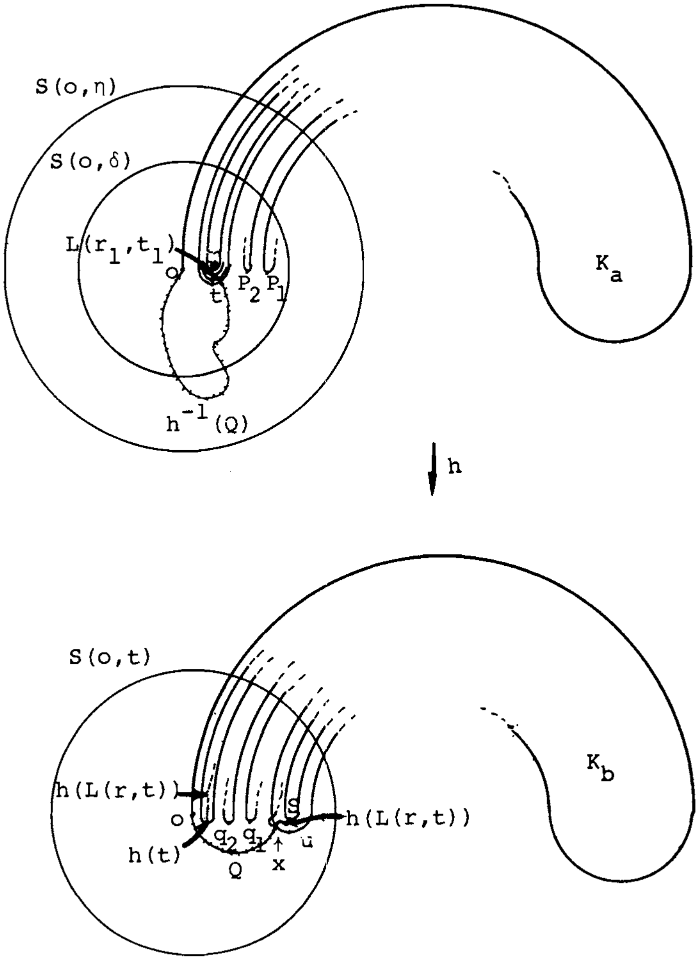


Figure 8. Impossibility of h carrying aca-pocket to bdb-pocket.

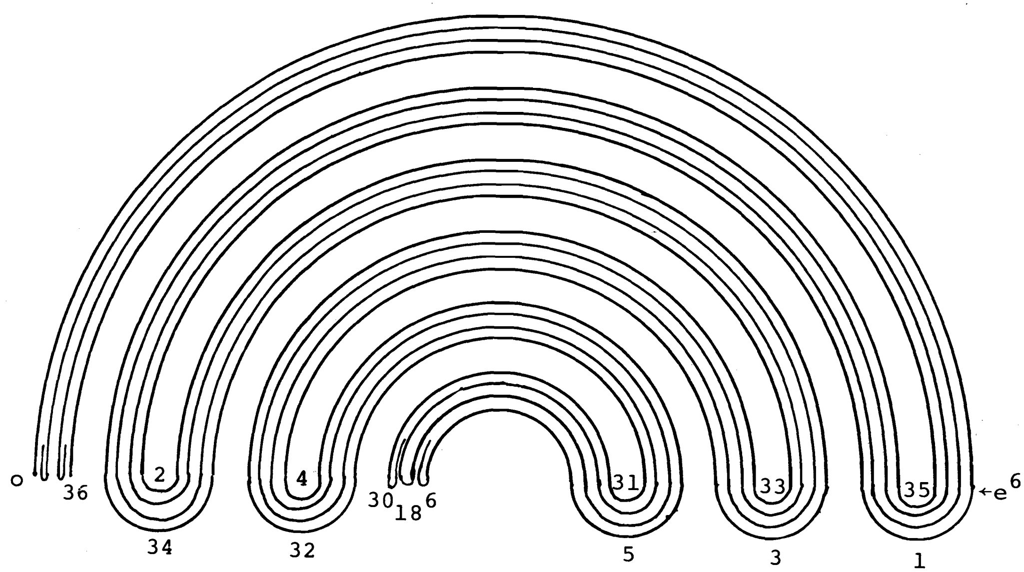


Figure 9. Standard embedding K_o^6 of six-fold Knaster (U-type) continuum K^6, defined by six-to-one bonding maps.

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