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1. Introduction

Throughout this paper the term mapping will mean a continuous function and a continuum will be a compact, connected metric space. Suppose X is a continuum, K is a subcontinuum of X, and f is mapping of a continuum onto X. The statement that f *is weakly confluent with respect to* K means some component of $f^{-1}(K)$ is thrown by f onto K. The statement that f *is weakly confluent* means f is weakly confluent with respect to each subcontinuum of X.

Any mapping of a continuum onto a tree is weakly confluent with respect to each arc which does not contain a junction point in its interior. Many people, such as Read [7, Lemma p. 236], Ingram [3, Lemma 1], and Marsh [5, Lemma 4.7] have given a proof of some version of this, using the fact that the interior of such an arc separates the tree. Feurerbacher [2, Lemma 9] showed that if K is an arc in a circle S then any mapping of a continuum onto S must be weakly confluent with respect to K or $\overline{S-K}$.

In Theorem 4 of this paper we show that if $K_1,\cdots,K_n$ is a collection of subcontinua of a one-dimensional polyhedron X whose interiors are mutually exclusive and contain no junction points, then the following are equivalent.

(1) Any mapping of a continuum onto X is weakly confluent with respect to one of $K_1,\cdots,K_n$,
and

(2) The union of the interiors of $K_1, \cdots, K_n$ separate X. In Theorem 5 we give conditions on the polyhedron which insure the separation in (2) above. We then use inverse limit representations of one-dimensional polyhedra to give conditions under which any mapping of a continuum onto a one-dimensional polyhedron X must be weakly confluent with respect to one of a given collection of subcontinua of X.

The theorems in this paper can be used to show that certain one-dimensional continua are in Class(W), where Class(W) is the class of continua which are images of weakly confluent mappings only. We give an example of how these theorems may be used.

2. Weak Confluence and Separation of One-Dimensional Polyhedra

In this section we establish the main theorems of the paper.

Theorem 1. *Suppose* X *is a one-dimensional connected polyhedron and* $K_1, K_2, \cdots, K_n$ *are mutually exclusive non-degenerate subcontinua of* X, *no one of which contains a junction point or an endpoint of* X. *Then the following are equivalent:*

(1) *If* f *is a mapping of a continuum onto* X *then* f *is weakly confluent with respect to one of* $K_1, K_2, \cdots, K_n$, *and*

(2) $X - \cup_{i=1}^n K_i$ *is not connected.*

Proof. (1) ⇒ (2): Suppose $X - \cup_{i=1}^n K_i$ is connected. Let $A_1, A_2, \cdots, A_n$ be a mutually exclusive collection of arcs in X such that $A_i \subseteq \text{Int } K_i$ for $i = 1, 2, \cdots, n$. Then

$X - \cup_{i=1}^n A_i$ is connected and we denote by M *the continuum* $\overline{X - \cup_{i=1}^n A_i}$.

We define a mapping f of M onto X which is not weakly confluent with respect to any K_i. For $i = 1,2,\cdots,n$, $\overline{K_i - A_i}$ is the union of two mutually exclusive arcs α_i and β_i each of which has one endpoint which is an endpoint of K_i and one endpoint which is an endpoint of A_i. We define $f|\alpha_i$ and $f|\beta_i$ so that $f|\alpha_i$ is a homeomorphism which maps α_i onto $\alpha_i \cup A_i$ and $f|\beta_i$ is a homeomorphism which maps β_i onto $\beta_i \cup A_i$, and so that the endpoint of K_i belonging to α_i is a fixed point of $f|\alpha_i$ and the endpoint of K_i belonging to β_i is a fixed point of $f|\beta_i$. We define $f|(X - \cup_{i=1}^n K_i)$ to be the identity mapping on $X - \cup_{i=1}^n K_i$.

For $i = 1,2,\cdots,n$ $f^{-1}(K_i)$ has two components, α_i and β_i. Neither $f(\alpha_i)$ nor $f(\beta_i)$ is K_i, hence f is not weakly confluent with respect to K_i.

(2) ⇒ (1): Suppose $X - (\cup_{i=1}^n K_i)$ is not connected. Let f be a mapping of a continuum M onto X.

Case 1. $X - K_1$ is not connected. Let A be an arc in X containing no junction point or endpoint of X such that $K_1 \subseteq \text{Int } A$. Then $X - A$ is not connected and has only two components, C_1 and C_2. Let $a_1 \in \overline{C_1} \cap A$ and $a_2 \in \overline{C_2} \cap A$. Let g be the mapping of X onto A defined by

$$g(x) = \begin{cases} a_1 & \text{if } x \in \overline{C_1} \\ a_2 & \text{if } x \in \overline{C_2} \\ x & \text{if } x \in A. \end{cases}$$

The composition $g \circ f$ is a mapping of M onto the arc A and so by [7, Lemma p. 236] $g \circ f$ is weakly confluent. Thus, there is a subcontinuum H of M such that $g \circ f(H) = K_1$. Since $f(H)$ is a continuum in X which is thrown by g onto K, $g|A$ is a homeomorphism, and $g^{-1}(K_1) = K_1$, then $f(H) = K_1$. Therefore, f is weakly confluent with respect to K_1.

Case 2. $X - K_1$ is connected. Let m be a positive integer less than n such that $X - \cup_{i=1}^{m} K_i$ is connected and $X - \cup_{i=1}^{m+1} K_i$ is not connected. Let $A_1, A_2, \cdots, A_{m+1}$ be mutually exclusive arcs in X, no one of which contains a junction point or an endpoint of X, such that $K_i \subseteq \text{Int } A_i$ for $i = 1, \cdots, m+1$. Then $X - \cup_{i=1}^{m} A_i$ is connected and $X - \cup_{i=1}^{m+1} A_i$ is not connected. Since $X - \cup_{i=1}^{m+1} A_i = (X - \cup_{i=1}^{m} A_i) - A_{m+1}$, then $X - \cup_{i=1}^{m+1} A_i$ has only two components, C_1 and C_2.

Let $a_1 \in A_1 \cap \overline{C_1}$ and $a_2 \in A_2 \cap \overline{C_2}$. Let g be the mapping of X onto A_1 defined by

$$g(x) = \begin{cases} x & \text{if } x \in A_1 \\ a_1 & \text{if } x \in \overline{C}_1 \\ a_2 & \text{if } x \in \overline{C}_2 \end{cases},$$

and for $i = 2, \cdots, m+1$, we define $g|A_i$ to be a homeomorphism which throws A_i onto A_1 in such a way that $g(K_i) = K_1$, $g(A_i \cap \overline{C}_1) = \{a_1\}$, and $g(A_i \cap \overline{C}_2) = \{a_2\}$.

The composition $g \circ f$ is a mapping of M onto the arc A, and so by [7, Lemma p. 236] $g \circ f$ is weakly confluent. Thus, there is a subcontinuum H of M such that $g \circ f(H) = K_1$. Since $f(H)$ is a continuum in X which is thrown by g onto K_1, $g^{-1}(K_1) = \cup_{i=1}^{m+1} K_i$, and $g|A_i$ is a homeomorphism for $i = 1, \cdots, m+1$, then $f(H)$ is one of $K_1, \cdots, K_{m+1}$. Therefore, f is weakly confluent with respect to one of $K_1, \cdots, K_{m+1}$.

The next theorem gives conditions which insure the separation in (2) of Theorem 1.

To each metric space X there corresponds a non-negative integer $b_1(X)$ (see [4, p. 409]). If X is a polyhedron, $b_1(X)$ is the one-dimensional Betti number of X.

Theorem 2. *Suppose* X *is a one-dimensional connected polyhedron and* n *is non-negative integer such that* $b_1(X) = n$, *and* $K_1, K_2, \cdots, K_{n+1}$ *are mutually exclusive subcontinua of* X, *no one of which contains a junction point or an endpoint of* X. *Then* $X - \cup_{i=1}^{n+1} K_i$ *is not connected.*

Proof. Suppose $X - \cup_{i=1}^{n+1} K_i$ is connected. Let $A_1, A_2, \cdots, A_{n+1}$ be mutually exclusive arcs in X, no one of which contains a junction point or an endpoint of X, such that $K_i \subseteq \text{Int } A_i$, for $i = 1, 2, \cdots, n+1$. Then $X - \cup_{i=1}^{n+1} A_i$ is connected.

Let a be an endpoint of A_1 and g be the mapping of X into X defined by

$$g(x) = \begin{cases} a \text{ if } x \in \overline{X - \cup_{i=1}^{n+1} A_i} \\ x \text{ otherwise} \end{cases}.$$

Since g is a monotone mapping, it follows from [4, Theorem 4, p. 433] that $b_1(X) \geq b_1(g(X))$. But g[X] has only n+1 simple closed curves and one junction point; hence, $b_1(g(X)) = n + 1$. This yields a contradiction.

The next theorem follows from Theorem 1 and 2.

Theorem 3. *Suppose* X *is a one-dimensional connected polyhedron,* n *is a non-negative integer such that* $b_1(X) = n$, *and* $K_1, K_2, \cdots, K_{n+1}$ *are mutually exclusive non-degenerate subcontinua of* X*, no one of which contains a junction point or an endpoint of* X*. If* f *is a mapping of a continuum onto* X *then* f *is weakly confluent with respect to one of* $K_1, \cdots, K_{n+1}$.

In Theorems 4, 5, and 6 we relax the conditions regarding junction points imposed on the collections of subcontinua in the hypotheses of Theorems 1, 2 and 3.

Theorem 4. *Suppose* X *is a one-dimensional polyhedron,* $K_1, \cdots, K_n$ *are non-degenerate subcontinua of* X *whose interiors are mutually exclusive, and no one of* $K_1, K_2, \cdots K_n$ *contains a junction point of* X *in its interior. Then the following are equivalent.*

(1) *If* f *is a mapping of a continuum onto* X *then* f *is weakly confluent with respect to one of* $K_1, K_2, \cdots, K_n$*, and*

(2) $X - \cup_{i=1}^{n} \text{Int } K_i$ *is not connected.*

Proof. (1) ⇒ (2): Suppose $X - \cup_{i=1}^{n} \text{Int } K_i$ is not connected. Let f be a mapping of a continuum M onto X. For each $i = 1, 2, \cdots, n$, let $A_1^i, A_2^i, \cdots$ be a sequence of arcs such that $A_j^i \subseteq \text{Int } K_i$, $A_j^i \subseteq A_{j+1}^i$, and $\lim_{j\to\infty} A_j^i = K_i$. Then for each positive integer j, $A_j^1, A_j^2, \cdots, A_j^n$ are mutually exclusive subcontinua of X, no one of which contains a junction point or an endpoint of X. Since $X - \cup_{i=1}^{n} \text{Int } K_i$ is not connected, then $X - \cup_{i=1}^{n} A_j^i$ is not connected. Then, by Theorem 1, f is weakly confluent with respect to one of $A_j^1, A_j^2, \cdots, A_j^n$.

There exists a positive integer i such that f is weakly confluent with respect to infinitely many of $A_1^i, A_2^i, \cdots$. Thus, there is a sequence $L_1, L_2, \cdots$ of subcontinua of M such that $f(L_1), f(L_2), \cdots$ is a subsequence of $A_1^i, A_2^i, \cdots$. We choose a subsequence $L_{m_1}, L_{m_2}, \cdots$ of $L_1, L_2, \cdots$ which converges to a subcontinuum L of M. Then $f(L) = \lim_{j\to\infty} f(L_{m_j}) = \lim_{j\to\infty} A_j^i = K_i$. Therefore, f is weakly confluent with respect to K_i.

(2) ⇒ (1): Suppose that $X - \cup_{i=1}^n \text{Int } K_i$ is connected. Let $A_1, A_2, \cdots, A_n$ be n arcs in X such that $A_i \subseteq \text{Int } K_i$ for $i = 1,2,\cdots,n$. Then $X - \cup_{i=1}^n A_i$ is connected, and so, by Theorem 1, there exists a continuum M and a mapping f of M onto X such that f is not weakly confluent with respect to A_i, for each $i = 1,2,\cdots,n$. Since, for each i, $K_i - A_i$ is not connected, it follows from Theorem 1 that f is not weakly confluent with respect to K_i, for each $i = 1,2,\cdots,n$.

Theorem 5. *Suppose* X *is a one-dimensional connected polyhedron and* n *is a non-negative integer such that* $b_1(X) = n$ *and* $K_1, K_2, \cdots, K_{n+1}$ *are subcontinua of* X *whose interiors are mutually exclusive, and no one of* $K_1, K_2, \cdots, K_{n+1}$ *contains a junction point of* X *in its interior. Then* $X - \cup_{i=1}^{n+1} \text{Int } K_i$ *is not connected.*

Proof. Suppose $X - \cup_{i=1}^{n+1} \text{Int } K_i$ is connected. Let $A_1, A_2, \cdots, A_{n+1}$ be subcontinua of X such that $A_i \subseteq \text{Int } K_i$, for $i = 1,2,\cdots,n+1$. Then $A_1, A_2, \cdots, A_{n+1}$ are mutually exclusive subcontinua of X, no one of which contains a junction point or an endpoint of X, and $X - \cup_{i=1}^{n+1}$ is connected. This contradicts Theorem 2.

The next theorem follows from Theorems 4 and 5.

Theorem 6. *Suppose that* X *is a one-dimensional connected polyhedron,* n *is a non-negative integer such that* $b_1(X) = n$*, and* $K_1, K_2, \cdots, K_{n+1}$ *are non-degenerate subcontinua of* X *whose interiors are mutually exclusive, and no one of* $K_1, K_2, \cdots, K_{n+1}$ *contains a junction point of* X *in its interior. If* f *is a mapping of a continuum onto* X *then* f *is weakly confluent with respect to one of* $K_1, K_2, \cdots, K_{n+1}$.

Theorem 5 shows that in a one-dimensional connected polyhedron X, any collection of at least $b_1(X) + 1$ subcontinua of X which satisfy certain conditions must separate X. The following theorem shows that it is necessary to require this many subcontinua to assure separation.

Theorem 7. *Suppose* X *is a one-dimensional connected polyhedron and* n *is a positive integer such that* $b_1(X) = n$. *Then there exist* n *mutually exclusive subcontinua* $K_1, K_2, \cdots, K_n$ *of* X*, no one of which contains a junction point or an endpoint of* X*, such that* $X - \cup_{i=1}^{n} K_i$ *is connected.*

Proof. Since $b_1(X) \geq 1$ then X contains a simple closed curve. Let K_1 be an arc in this simple closed curve which contains no junction point of X. Then $X - K_1$ is connected, so by the Euler-Poincaré formula [6, Theorem 9, p. 32] $b_1(\overline{X - K_1}) = b_1(X) - 1$. (One can see this by noting that $\overline{X - K_1}$ has one more 1-simplex and two more 0-simplexes than X.)

We define, inductively, arcs $K_2, \cdots, K_n$ in X such that for $j = 2, \cdots, n$ K_j is in a simple closed curve in $\overline{X - \cup_{i=1}^{j-1} K_i}$,

K_j contains no junction point of X, and $\overline{X - \cup_{i=1}^{j-1} K_i}$ is connected. By the Euler-Poincaré formula,

$$b_1(\overline{X - \cup_{i=1}^{j} K_i}) = b_1(\overline{X - \cup_{i=1}^{j-1} K_i}) - 1 = b_1(X) - \Sigma_{i=1}^{j} i.$$

Therefore, $\overline{X - \cup_{i=1}^{n} K_i}$ is connected.

In the next theorem, we show that the conditions regarding junction points imposed on the collection of subcontinua in Theorem 4 may not be weakened.

Theorem 8. *Suppose* X *is a one-dimensional connected polyhedron and* $K_1, K_2, \cdots, K_n$ *are mutually exclusive proper subcontinua of* X *such that each of* $K_1, K_2, \cdots, K_n$ *contains a junction point of* X *in its interior. Then there exists a continuum* M *and a mapping* f *of* M *onto* X *such that* f *is not weakly confluent with respect to* K_i *for each* $i = 1, 2, \cdots, n$.

Proof. We show there is a continuum M and a mapping f of M onto X which is not weakly confluent with respect to K_1. There is a point x in $X - \cup_{i=1}^{n} K_i$ and an arc $\alpha = [x,a]$ such that $a \in K_1$, $[x,a) \cap (\cup_{i=1}^{n} K_i) = \phi$, and $[x,a)$ contains no junction point of X. Let J be a junction point of X in Int K_1 and let $\beta = [a,J]$ be an arc in K_1 joining a and J.

Case 1. There is an arc $[t,J]$ such that $[t,J) \cap (\cup_{i=1}^{n} K_i) = \phi$ and $[t,J)$ contains no junction point of X. Let $[k,J]$ be an arc in β such that $[k,J)$ contains no junction point of X. Let M be the union of the following three subsets of $X \times [0,1]$:

$[X - (k,J)] \times \{0\}$,

$(\alpha \cup \beta \cup [t,J]) \times \{1\}$, and

$\{X,t\} \times [0,1]$.

Let f be the projection mapping of the continuum M onto X.

We show that f is not weakly confluent with respect to K_1. Suppose there is a subcontinuum H of M such that $f(H) = K_1$. Since J is in the interior of K_1 there is a point y in K_1 such that $y \notin \alpha \cup \beta \cup [k,J]$. Now, $f^{-1}(y) = \{(y,0)\}$, $f^{-1}[(k,J)] = (k,J) \times \{1\}$, and $f|[(k,J) \times \{1\}]$ is one to one. Thus, H must contain the point (y,0) and a point of $(k,J) \times \{1\}$. But, any subcontinuum of M which contains (y,0) and a point of $(k,J) \times \{1\}$ must intersect one of $\{t\} \times [0,1]$ and $\{x\} \times [0,1]$. Therefore, the image of such a continuum under f must contain a point not in K_1, and so f is not weakly confluent with respect to K_1.

Case 2. Case 1 does not hold. Then there exist two arcs [r,J] and [s,J] such that $[r,J) \cup [s,J) \subseteq K_1 - \beta$ and neither [r,J) nor [s,J) contains a junction point of X.

We will resolve this case in two parts. First, suppose that X - (r,J) is not connected. Then X - (r,J) has only two components, one containing r and the other containing J, s, and x. Let M be the union of the following three subsets of $X \times [0,1]$:

$[X - (r,J)] \times \{0\}$,

$(\alpha \cup \beta \cup [r,J]) \times \{1\}$, and

$\{x,r\} \times [0,1]$.

Let f be the projection mapping of M onto X.

We show that f is not weakly confluent with respect to K_1. If H is any subset of M such that $K_1 \subseteq f(H)$ then H must contain the point (s,0) and a point of $(r,J) \times \{1\}$. But, any continuum in M containing two such points must intersect $\{x\} \times [0,1]$, and hence f(H) contains points not in K_1. Thus, f is not weakly confluent with respect to K_1.

On the other hand, suppose that X - (r,J) is connected. Let M be the union of the following three subsets of $X \times [0,1]$:

$[X - (r,J)] \times \{0\}$,

$(\alpha \cup \beta \cup [r,J]) \times \{1\}$, and

$\{x\} \times [0,1]$.

Let f be the projection mapping of the continuum M onto X.

We show that f is not weakly confluent with respect to K_1. If H is any subcontinuum of M such that $K_1 \subseteq f(H)$ then H must contain $(r,J) \times \{1\}$ and the point (s,0). But, any continuum in M containing such points must intersect $\{x\} \times [0,1]$ and hence f(H) contains points not in K_1. Thus, f is not weakly confluent with respect to K_1. This concludes Case 2.

In each case, M was constructed by removing an arc from X and building a bridge over it in $X \times [0,1]$. In doing this we were careful to stay away from $\cup_{i=2}^{n} K_i$. This construction can be repeated for each of $K_2, \cdots, K_n$, resulting in a continuum M' in $X \times [0,1]$ such that the projection mapping f' of M' onto X is not weakly confluent with respect to K_i, for each $i = 1,2,\cdots,n$.

Remark. It is interesting to note that with M' so constructed, one can see that an arc can be mapped onto M' in such a way that the composition of this mapping with f' is not weakly confluent with respect to any K_i. Thus, we may assume the continuum M in the statement of Theorem 8 is an arc.

3. Inverse Limits

In this section we use inverse limit representations of one-dimensional polyhedra to describe conditions under which any mapping of a continuum onto a one-dimensional polyhedron X must be weakly confluent with respect to one member of a given collection of subcontinua of X. These results can be used to show that certain one-dimensional continua are in Class(W).

Suppose $X_1, X_2, \cdots$ is a sequence of compact metric spaces each having diameter less than a fixed positive number c, and suppose $f_1, f_2, \cdots$ is a sequence of mappings such that f_i maps X_{i+1} onto X_i for $i = 1,2,\cdots$. The *inverse limit* of the inverse limit sequence $\{X_i, f_i\}$ is the subset of the product $\prod_{i>0} X_i$ to which $(x_1, x_2, \cdots)$ belongs if and only if $f_n(x_{n+1}) = x_n$ for $n = 1,2,\cdots$. We consider $\prod_{i>0} X_i$ metrized by

$$d(x,y) = \prod_{i>0} 2^{-i} d_i(x_i, y_i)$$

where d_i denotes the metric on X_i. For each $i = 1,2,\cdots$, π_i will denote the projection mapping of the inverse limit onto X_i.

The following lemma was essentially proved by Read [7, Theorem 4] although not stated in this way. A proof is included here only for the sake of completeness.

Lemma 1. *Suppose* X *is the inverse limit of the inverse limit sequence* $\{X_i, f_i\}$ *with each* X_i *a continuum,* K *is a subcontinuum of* X, *and* g *is a mapping of a continuum onto* X. *If* $\pi_i \circ g$ *is weakly confluent with respect to* $\pi_i(K)$ *for infinitely many integers* i, *then* g *is weakly confluent with respect to* K.

Proof. Let g be a mapping of a continuum M onto X, $n_1, n_2, n_3, \cdots$ be a sequence of integers, and $H_1, H_2, \cdots$ be a sequence of subcontinua of M such that $\pi_{n_i} \circ g(H_i) = \pi_{n_i}(K_{n_i})$ for $i = 1,2,\cdots$. We can assume that the sequence $H_1, H_2, H_3, \cdots$ converges to a continuum H in M.

We show that $K \subset g(H)$. Suppose $p \in K$ and $\varepsilon > 0$. Let N be a positive integer such that if $k > N$ then $\sum_{i>n_k} 2^{-i} < \varepsilon$. Let $k > N$. Since $x \in K$, $\pi_{n_k}(p) \in \pi_{n_k}(K) = \pi_{n_k}(H_k)$. Let x be a point of $g(H_k)$ such that $\pi_{n_k}(x) = \pi_{n_k}(p)$. Then for $i < n_k$, $\pi_i(x) = \pi_i(p)$. Thus, $d(p,x) = \sum_{i>0} 2^{-i} d_i(\pi_i(x), \pi_i(p)) < \varepsilon$, and so $d(p, g(H_k)) < \varepsilon$ for $k > N$. Hence, $p \in \lim_{k\to\infty} g(H_k) = g(H)$. This shows that $K \subset g(H)$.

We show $g(H) \subset K$. Suppose $t \in g(K)$ and $\varepsilon > 0$. Let N be a positive integer such that if $k > N$ then $\sum_{i>n_k} 2^{-i} < \frac{\varepsilon}{2}$. Choose $k > N$ such that $g(H) \subset B(g(H_k), \frac{\varepsilon}{2})$ and let y be a point of $f(H_k)$ such that $d(t,y) < \frac{\varepsilon}{2}$. Since $y \in f(H_k)$ then

$\pi_{n_k}(y) \in \pi_{n_k} \circ g(H_{n_k}) = \pi_{n_k}(H)$. There is a point s in H such that $\pi_{n_k}(y) = \pi_{n_k}(s)$, and so for $i < n_k$, $\pi_i(y) = \pi_i(s)$. Thus, $d(y,s) = \sum_{i>0} 2^{-i} d_i(\pi_i(y), \pi_i(s)) < \frac{\varepsilon}{2}$, and $d(t,s) \leq d(t,y) + d(y,s) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Since for each $\varepsilon > 0$ there is a point x in H such that $d(r,s) < \varepsilon$, then $s \in \overline{H} = H$. This shows that $g(H) \subset K$.

We have shown that $g(H) = K$, thus g is weakly confluent with respect to K.

In the following lemma, d denotes the Hausdorff metric.

Lemma 2. *Suppose* X *is a continuum,* K *is a subcontinuum of* X, *and* g *is a mapping of a continuum onto* X. *If for each positive number* ε *there is a subcontinuum* L *of* X *such that* g *is weakly confluent with respect to* L *and* $d(K,L) < \varepsilon$ *then* g *is weakly confluent with respect to* K.

Proof. The proof of this lemma is straightforward.

The next two theorems follow easily from the lemmas and Theorems 4 and 6 of section 2.

Theorem 9. *Suppose* X *is the inverse limit of the inverse limit sequence* $\{X_i, f_i\}$ *with each* X_i *a one-dimensional connected polyhedron, and* $K_1, \cdots, K_n$ *are non-degenerate subcontinua of* X *such that for infinitely many integers* i, (1) *the interiors of* $\pi_i K_i, \cdots, \pi_i K_n$ *are mutually exclusive,* (2) *no one of* $\pi_i K_1, \cdots, \pi_i K_n$ *contains a junction point of* X_i *in its interior, and* (3) $X_i - \cup_{j=1}^{n} \mathrm{Int}(\pi_i K_j)$ *is not connected. If* g *is a mapping of a continuum onto* X *then* g *is weakly confluent with respect to one of* $K_1, \cdots, K_n$.

Theorem 10. *Suppose* X *is the inverse limit of the inverse limit sequence* $\{X_i, f_i\}$ *with each* X_i *a one-dimensional connected polyhedron, and* n *is a positive integer such that* $b_1(X_i) \leq n$ *for each* i. *Suppose also that* $K_1, \cdots, K_{n+1}$ *are non-degenerate subcontinua of* X *such that for infinitely many integers* i, (1) *the interiors of* $\pi_i K_1, \cdots, \pi_i K_{n+1}$ *are mutually exclusive and* (2) *no one of* $\pi_i K_i, \cdots, \pi_i K_{n+1}$ *contains a junction point of* X_i *in its interior. If* g *is a mapping of a continuum onto* X *then* g *is weakly confluent with respect to one of* $K_1, \cdots, K_{n+1}$.

A special case of Theorem 9 was proved by Read [7, Theorem 4]. Theorems 9 and 10 may be used to show that certain one-dimensional continua are in Class(W). The following are continua for which Theorem 9 or 10 can be used to show they are in Class(W):

(1) the Class(W) continua defined by Waraszkiewicz in [9] (not all of the continua he described are in Class(W)),

(2) the Case-Chamberlin continuum [1],

(3) Ingram's continua in [3], and

(4) the continuum defined by Sherling in [8].

As an example, we will use Theorem 9 to show that the Case-Chamberlin continuum is in Class(W).

The Case-Chamberlin continuum (see [1]) is an inverse limit on figure eights using one bonding map. Let A and B be two circles tangent at a point J. Assign an orientation to each of A and B. Let f be a mapping which throws A ∪ B onto A ∪ B as follows:

(1) A is thrown onto $A \cup B$ by fixing J, then wrapping around A in the positive direction, then B in the positive direction, and then around each of A and B in the negative direction.

(2) B is thrown onto $A \cup B$ by fixing J, then wrapping around A twice in the positive direction, then B twice in the positive direction, and then around each of A and B twice in the negative direction.

For each i let $X_i = A \cup B$ and $f_i = f$. Let X be the inverse limit of the inverse limit sequence $\{X_i, f_i\}$. One can show that if K is a proper subcontinuum of X then there exists a positive integer n such that (1) for each $i > n$, $J \notin \pi_n K$, or (2) for each $i > n$, $\pi_n K$ is an arc in A having J as an endpoint.

We will show that X is in Class(W). Let g be a mapping of a continuum onto X and let K be a proper subcontinuum of X. We will show that for every positive number ε there is a subcontinuum L of X such that g is weakly confluent with respect to L and $d(K,L) < \varepsilon$ (where d denotes the Hausdorff metric).

We assume $\prod_{i>0} (X_i, d_i)$ metrized by $d(x,y) = \sum_{i\geq 0} 2^{-i} d(\pi_i x, \pi_i y)$. Let $\varepsilon > 0$ and N be a positive integer such that $\sum_{i\geq N} 2^{-i} \frac{\varepsilon}{\operatorname{diam}(A \cup B)}$. There exists an integer $N > J$ such that J is not in the interior of $\pi_n K$.

We can choose mutually exclusive arcs α and β in A such that $f(\alpha) = f(\beta) = \pi_n K$ and J is not in the interior of α or β. There exist subcontinua L_1 and L_2 of X such that

$\pi_{n+1}(L_1) = \alpha$, $\pi_{n+1}(L_2) = \beta$ and for each $i > n+1$, $\pi_i(L_1)$ and $\pi_i(L_2)$ are mutually exclusive arcs in A, neither of which contains J in its interior. Then for $i \geq n$, $X_i - [\pi_i(L_i) \cup \pi_i(L_2)]$ is not connected.

By Theorem 9, g is weakly confluent with respect to L_1 or L_2. Since $\pi_n(L_1) = \pi_n(L_2) = \pi_n(K)$, then

$$d(K,L_1) < \sum_{i<n} 2^{-i}(\text{diam } A \cup B) < \varepsilon, \text{ and}$$

$$d(K,L_2) < \sum_{i<n} 2^{-i}(\text{diam } A \cup B) < \varepsilon.$$

Therefore, for each positive number ε there is a subcontinuum L of X such that g is weakly confluent with respect to L and $d(K,L) < \varepsilon$. By Lemma 2, g is weakly confluent with respect to K. Hence, X is in Class(W).

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