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THE BAIRE CATEGORY THEOREM FOR FRÉCHET GROUPS IN WHICH EVERY NULL SEQUENCE HAS A SUMMABLE SUBSEQUENCE

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A number of mathematicians have studied the efficacy of using the summability property for topological groups mentioned in our title, commonly called property (K), as a substitute for complete metrizability in certain theorems of functional analysis (see for example [1], [2], [5]). From this work it is known that a normed vector space with property (K) is not necessarily completely metrizable [8], [9]; while a metrizable group with property (K) is a Baire space [5]. In this note we will view property (K) as a form of completeness in (non-metrizable) topological groups; especially topological groups in which the topology is determined by the convergent sequences. In fact our main result is that the metrizability in Burzyk's Baire space theorem quoted above may be weakened to "Fréchet" in the sense of Franklin [6] (see definitions below). Although our results apply to general (non-abelian) topological groups, we will employ additive notation throughout this note.

1. *Definitions.* A subset S of a topological space X is sequentially closed if no sequence in S converges to a member of X\S. A topological space X is sequential [6] if every sequentially closed subset of X is closed. A topological space X is Fréchet [6] if for every subset S of

X clS = {x: there is a sequence in S converging to x}. A topological space X is countably bisequential [10] (=strongly Fréchet [13]) if given a decreasing sequence (F_n) of subsets of X accumulating at x in X, there is $x_n \in F_n$ so that the sequence (x_n) converges to x.

A topological group G has property (K) if for every sequence (x_n) in G converging to 0, there is a subsequence (x_{n_k}) so that $\Sigma_{k=1}^{\infty} x_{n_k}$ exists.

We omit the elementary proof of the following lemma.

2. *Lemma. A Fréchet space* X *is countably bisequential if and only if given* $x \in X$ *and, for every* $m \in \mathbb{N}$, *a sequence* $(x_n^m)_{n\in N}$ *converging to* x, *there is an increasing sequence* $(m(k))_{k\in\mathbb{N}}$ *of positive integers and a sequence* $(n(k))_{k\in\mathbb{N}}$ *of positive integers so that* $(x_{n(k)}^{m(k)})_{k\in\mathbb{N}}$ *converges to* x.

To show that our Theorem 5 below is an honest improvement of Burzyk's theorem we briefly discuss Σ-products. Given a family $\{X_\alpha : \alpha \in I\}$ of topological spaces and a point $x = (x_\alpha)_{\alpha\in I}$ in $\Pi_{\alpha\in I} X_\alpha$, the Σ-product of the X_α's with base point x is $\Sigma_x = \{(y_\alpha)_{\alpha\in I} \in \Pi_{\alpha\in I} X_\alpha : x_\alpha \neq y_\alpha$ for only countably many $\alpha \in I\}$ with the relative topology. One easily sees that a Σ-product of completely metrizable groups with base point 0 has property (K) and, moreover, we also have the following.

3. *Theorem* [11]. *Every Σ-product of first countable spaces is a Fréchet space.*

In view of Lemma 2, a Σ-product of first countable spaces is in fact countably bisequential. Of course, a Σ-product of uncountably many non-trivial spaces has no G_δ points.

4. *Lemma* [12]. *A Fréchet topological group is countably bisequential.*

5. *Theorem. A Hausdorff Fréchet topological group* G *with property* (K) *is a Baire space.*

Proof. Let $(U_n: n \in \mathbb{N})$ be a decreasing sequence of dense open sets in G. In light of Proposition 1.27 of [7] it suffices to show that $\cap_{n\in\mathbb{N}} U_n \neq \emptyset$ (i.e. that G is of second category).

Let $(x_n^1: n \in \mathbb{N})$ be a sequence in U_1 converging to 0. Inductively, for every $m \in \mathbb{N}$ we can find a sequence $(x_n^m: n \in \mathbb{N})$ that converges to 0 and is eventually in each of the dense open sets $-\Sigma_{i=1}^{m-1}\delta_i x_{n_i}^i + U_m$, where $\delta_i \in \{0,1\}$ and $n_i \in \mathbb{N}$. This can be done for there are countably many such dense open sets; an intersection of finitely many dense open sets is dense and hence contains a sequence converging to 0; now employ Lemmas 2 and 4.

We wish to define, for every $k \in \mathbb{N}$, a collection $\mathcal{S}_k$ of finite (partial) sequences in $\mathbb{N}$ of length k, and for every $(n_1,n_2,\cdots,n_k) \in \mathcal{S}_k$ an open set $V(n_1,n_2,n_k)$ so that

$$(*)\ \mathrm{cl}V(n_1,n_2,\cdots,n_k) \subset U_k, \text{ and}$$

$$(**)\ \Sigma_{m=1}^{k}\delta_m x_{n_m}^m \in \cap_{m=1}^{k} V(n_1,n_2,\cdots,n_m)^{\delta_m} \text{ for } \delta_m \in \{0,1\};$$

where $V(n_1,n_2,\cdots,n_m)^0 = G$ and $V(n_1,n_2,\cdots,n_m)^1 = V(n_1,n_2,\cdots,n_m)$.

For $k = 1$, let $\mathcal{S}_1 = \{(n_1): n_1 \in \mathbb{N}\}$; for every $(n_1) \in \mathcal{S}_1$ let $V(n_1)$ be an open neighborhood of $x_{n_1}^1$ so that $\mathrm{cl}V(n_1) \subset U_1$.

Let us assume the induction is complete through the kth step. Fix $(n_1,n_2,\cdots,n_k) \in \mathcal{S}_k$. Let

$$V = \cap_{\delta_m \in \{0,1\}} [-\Sigma_{m=1}^{k} \delta_m x_{n_m}^m + (\cap_{m=1}^{k} V(n_1, n_2, \cdots, n_m)^{\delta_m})],$$

which by (**) is an open neighborhood of 0. Further, let

$$U = \cap_{\delta_m \in \{0,1\}} (-\Sigma_{m=1}^{k} \delta_m x_{n_m}^m + U_{k+1}),$$

which is a dense open set in G. Because the sequence $(x_n^{k+1}: n \in \mathbb{N})$ is eventually in $U \cap V$, we may let $f(n_1, n_2, \cdots, n_k)$ be the least integer n^* so that for all $n_{k+1} \geq n^*$ we have $x_{n_{k+1}}^{k+1} \in U \cap V$.

For a given $n_{k+1} \geq f(n_1, n_2, \cdots, n_k)$, let W be an open neighborhood of $x_{n_{k+1}}^{k+1}$ so that $\mathrm{cl}\, W \subset U \cap V$. Let

$$V(n_1, n_2, n_k, n_{k+1}) = \cup_{\delta_m \in \{0,1\}} (\Sigma_{m=1}^{k} \delta_m x_{n_m}^m + W).$$

Since $\Sigma_{m=1}^{k} \delta_m x_{n_m}^m + \mathrm{cl}\, W \subset \Sigma_{m=1}^{k} \delta_m x_{n_m}^m + U \subset U_{k+1}$ for any choice of δ_m's, we have $V(n_1, n_2, n_k, n_{k+1}) \subset U_{k+1}$. As $x_{n_{k+1}}^{k+1} \in V \subset -\Sigma_{m=1}^{k} \delta_m x_{n_m}^m + \cap_{m=1}^{k} V(n_1, n_2, \cdots, n_m)\delta_m$, we see that $\Sigma_{m=1}^{k} \delta_m x_{n_m}^m + x_{n_{k+1}}^{k+1} \in \cap_{m=1}^{k} V(n_1, n_2, \cdots, n_m)\delta_m$; moreover, $x_{n_{k+1}}^{k+1} \in W$ gives us $\Sigma_{m=1}^{k} \delta_m x_{n_m}^m + x_{n_{k+1}}^{k+1} \in \Sigma_{m=1}^{k} \delta_m x_{n_m}^m + W \subset V(n_1, n_2, \cdots, n_k, n_{k+1})$. Consequently, $\Sigma_{m=1}^{k} \delta_m x_{n_m}^m + x_{n_{k+1}}^{k+1} \in \cap_{m=1}^{k} V(n_1, n_2, \cdots, n_m)\delta_m \cap V(n_1, n_2, \cdots, n_k, n_{k+1})$. Thus if we let $\mathcal{S}_{k+1} = \{(n_1, n_2, \cdots, n_k, n_{k+1}): (n_1, n_2, \cdots, n_k) \in \mathcal{S}_k$ and $n_{k+1} \geq f(n_1, n_2, \cdots, n_k)\}$ it is clear how to complete the induction.

Let $\mathcal{S}$ be the set of all (infinite) sequences $(n_m: m \in \mathbb{N})$ in $\mathbb{N}$ so that

(1) for every $k \in \mathbb{N}$ we have $(n_1, n_2, \cdots, n_k) \in \mathcal{S}_k$; and

(2) there is a $K \in \mathbb{N}$ so that for all $k \geq K$ we have $n_{k+1} = f(n_1, n_2, \cdots, n_k)$.

Since the collection $\mathcal{S}$ is countably infinite, we can find a sequence $(n_m^*: m \in \mathbb{N})$ in $\mathbb{N}$ that dominates every member of $\mathcal{S}$; which is to say that for every $(n_m: m \in \mathbb{N}) \in \mathcal{S}$, there is a $M \in \mathbb{N}$ so that for all $m \geq M$ we have $n_m \leq n_m^*$.

The convergence of the sequences $(x_n^m: n \geq n_m^*)$ (for $m \in \mathbb{N}$) to 0 enables us to employ Lemmas 2 and 4 to find an increasing sequence $(m(k): k \in \mathbb{N})$ and a sequence $(n(k): k \in \mathbb{N})$ in $\mathbb{N}$, with $n(k) \geq n_{m(k)}^*$ for every $k \in \mathbb{N}$, so that $(x_{n(k)}^{m(k)}: k \in \mathbb{N})$ converges to 0. Further, we may assume $m(1) = 1$.

We inductively define an increasing sequence $(k(j): j \in \mathbb{N})$ in $\mathbb{N}$ and pick, for each $j \in \mathbb{N}$, a $\sigma(j) \in \mathcal{S}_{k(j)}$; $\tau(j)$ will denote the natural extension of $\sigma(j)$ to a member of $\mathcal{S}$. First, let $k(1) = 1$ and $\sigma(1) = (n(1)) \in \mathcal{S}_1$. Now assume that $k(j)$ and $\sigma(j)$ have been given. Since $(n_m^*: m \in \mathbb{N})$ dominates $\tau(j)$, we can find a $k(j+1) > k(j)$ so that the $m(k(j+1))$ term of $\tau(j) \leq n_{m(k(j+1))}^* \leq n(k(j+1))$. Let $\sigma(j+1)$ be a finite sequence of length $k(j+1)$ which agrees with $\tau(j)$ until the $k(j+1)$ term; the $k(j+1)$ term of $\sigma(j+1)$ being $n(k(j+1))$. Because $\tau(j) \in \mathcal{S}$, the truncated sequence $\tau(j)|_{k(j+1)} \in \mathcal{S}_{k(j+1)}$; consequently $\sigma(j+1) \in \mathcal{S}_{k(j+1)}$. Moreover, $k(j+1) > k(j)$ implies that $\sigma(j+1)|_{k(j)} = \sigma(j)$.

Evidently the sequence $(x_{n(k(j))}^{m(k(j))}: j \in \mathbb{N})$ converges to 0. Since G satisfies property (K), there is an increasing sequence $(j(i): i \in \mathbb{N})$ in $\mathbb{N}$ so that $\Sigma_{i=1}^{\infty} x_{n(k(j(i)))}^{m(k(j(i)))}$ converges to an $x \in G$; we may assume that $j(1) = 1$.

From (**) we see that for every $I \in \mathbb{N}$

$$\Sigma_{i=1}^{I} x_{n(k(j(i)))}^{m(k(j(i)))} \in \cap_{i=1}^{I} V(\sigma(k(j(i))));$$

and thus $x \in \cap_{i=1}^{\infty} \mathrm{cl}V(\sigma(k(j(i)))) \subset \cap_{i=1}^{\infty} U_{k(j(i))} \subset \cap_{r\in\mathbb{N}} U_r$, as desired.

We will use the next theorem to show that a Hausdorff sequential topological group with property (K) need not be Baire and, incidentally, answer Problem 1 of [12].

6. *Theorem. A direct sum* X *of a countable family* $\{X_n\colon n \in \mathbb{N}\}$ *of regular, first countable spaces with base point* $x = (x_n\colon n \in \mathbb{N})$ *(where* $x_n \in X_n$*) with the box product topology is sequential if and only if all of the* X_n*'s are locally countably compact or for all but finitely many* $n \in \mathbb{N}$, x_n *is isolated in* X_n. *(By "direct sum with base point* $(x_n\colon n \in \mathbb{N})$*" we mean* $X = \{(y_n\colon n \in \mathbb{N}) \in \Pi_{n\in\mathbb{N}} X_n\colon y_n \neq x_n$ *for only finitely many* $n \in \mathbb{N}\}$.*)*

Proof. To get the sufficiency of the first condition, we could employ the fact that the product of finitely many countably compact k-spaces is countably compact to modify the proof that Example 1 of [12] is sequential. The second condition implies that X is first countable and hence sequential.

For the necessity, we may assume that X_1 is not locally countably compact and that for infinitely many $n \in \mathbb{N}$, x_n is not isolated. Let X' denote the direct sum of the spaces $\{X_n\colon n \geq 2\}$ with base point $x' = (x_n\colon n \geq 2)$. One can easily find a closed subspace S of X' that is homeomorphic to S_2 (see [4]) which is countable, sequential and not Fréchet. It follows from [14] (Theorem 1.1) that $X_1 \times S$ is not sequential. As a result, $X = X_1 \times X'$ is not sequential.

7. *Example*. A sequential topological group with property (K) that is not Baire.

Let $\{G_n: n \in \mathbb{N}\}$ be a family of non-discrete locally compact metric groups and let G be the direct sum of the G_n's with the box product topology. For each $m \in \mathbb{N}$ let $H_m = \Pi_{n=1}^{m} G_n$, to be viewed as a subspace of G in the natural way. One readily verifies that $G = \cup_{i=1}^{\infty} H_m$; further, the H_m's are closed and have void interior in G, so that G is not Baire. On the other hand, any sequence in G converging to $0 = (0_n: n \in \mathbb{N})$ must eventually be in one of the H_m's and, since the H_m's are completely metrizable, the sequence has a summable subsequence.

As pointed out by Arhangel'skii [3], a topological group of countable tightness, in particular, a Fréchet group, which is Cech-complete is metrizable and hence satisfies property (K). As we mentioned, the example of [8], [9] shows that the converse fails even for metrizable spaces. While an example in [5] shows that a metrizable Baire topological vector space need not have property (K). Among topological groups not satisfying tightness or sequential conditions, even compactness does not insure property (K). Indeed, the product of c copies of the factor group $\mathbb{R}/\mathbb{Z}$ does not possess property (K).

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