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AN ADMISSIBLE CONDITION FOR CONTRACTIBLE HYPERSPACES

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Let X be a nonvoid metric continuum. Denote by 2^X and C(X) the hyperspaces of nonempty closed subsets and subcontinua of X respectively and endow each with the Hausdorff metric H.

In 1938 Wojkyslawski proved that 2^X is contractible if X is locally connected [11]. In 1942 Kelley [1] proved that the contractibility of 2^X is equivalent to the contractibility of C(X). Furthermore, he introduced a sufficient condition, namely property (3.2), for the contractibility of the hyperspaces of metric continua. In [5], a necessary condition, namely admissibility, is given for a space whose hyperspace is contractible. It was also proven that the contractibility of the hyperspace C(X) is equivalent to the existence of a continuous fiber map on X into the hyperspace $C^2(X)$ of subcontinua of C(X) for the class of metric continua with property c (abbreviated as c-space). In the present paper we show that if $f: X \to Y$ and $g: Y \to X$ are continuous such that $f \circ g$ is homotopic to the identity map id_Y on Y, and if X is a c-space then Y is also a c-space. Hence if X and Y are homotopically equivalent, then X is a c-space if and only if Y is. We also show that the product space $X \times Y$ is a c-space if and only if both X and Y are c-spaces. Many corollaries to the above results are also given which are generalizations of results in [4].

Throughout the paper, the symbols I and μ will be reserved for the closed interval and a Whitney map [10] with $\mu(X) = 1$ respectively. Note that $\mu(X) = 1$ necessarily requires X to be nondegenerate, a condition which we will assume whenever required without explicitly stating so.

1. Preliminaries

We collect in this section some definitions and known facts and prove a new lemma. Let X be a nonvoid metric continuum.

A map H: $X \times I \to C(X)$ is *increasing* if $h(x,t) \subset h(x,t')$ for $t \leq t'$ and $x \in X$. A *contraction of* X *in* C(X) is a continuous homotopy h: $X \times I \to C(X)$ such that, for each $x \in X$, $h(x,0) = \{x\}$ and $h(x,1) = A$. A contraction of X in 2^X is analogously defined.

Theorem 1.1 [1]. *The following statements are equivalent.*

1. *A contraction of* X *in* C(X) *exists.*
2. 2^X *is contractible.*
3. C(X) *is contractible.*

Theorem 1.2 [1]. *If* C(X) *is contractible then an increasing contraction of* X *in* C(X) *exists.*

The contractibility of C(X) implies the contractibility of $C^2(X)$, by Theorem 1.1. Also, since the union may from $C^2(X)$ onto C(X) is a retraction, the contractibility of $C^2(X)$ implies the contractibility of C(X). Thus we have the following.

Theorem 1.3 [1]. *C(X) is contractible if and only if the hyperspace* $C^2(X)$ *of subcontinua of* C(X) *is contractible.*

We recall the definition of the Hausdorff metric H on 2^X. For $A,B \in 2^X$,

$$H(A,B) = \max\{\max_{a\in A} d(a,B), \max_{b\in B} d(b,A)\},$$

where d(x,A) is the distance from x to A.

Lemma 1.4. *If* $A,B,C,D \in 2^X$ *then*

$$H(A \cup B, C \cup D) \leq \max\{H(A,C),H(B,D)\}.$$

Proof. Let $\eta > \max\{H(A,C),H(B,D)\}$. Then $C \cup D \subset \{x|d(x,A) < \eta\} \cup \{x|d(x,B) < \eta\} = \{x|d(x,A \cup B < \eta\}$. Also $A \cup B \subset \{x|d(x,C \cup D) < \eta\}$.

Let X be a nonvoid continuum. We now define an admissibility condition [5] and prove some propositions. For $x \in X$, let $F(x) = \{A \in C(X)|x \in A\}$, and for $(x,t) \in X \times I$, $F_t(x) = F(x) \cap \mu^{-1}(t)$. An element $A \in F(x)$ is said to be *admissible at* x if, for each $\varepsilon > 0$, there is $\delta > 0$ such that each y in the δ-neighborhood of x has an element $B \in F(y)$ such that $H(A,B) < \varepsilon$. For each $x \in X$, the collection $\mathcal{A}(x) = \{A \in F(x)|A$ is admissible at x$\}$ is called the *admissible fiber at* x. We say that X is *admissible* if $\mathcal{A}_t(x) = \mathcal{A}(x) \cap \mu^{-1}(t)$ is nonempty for each $(x,t) \in X \times I$.

Proposition 1.5. *If* $A \in \mathcal{A}(\xi)$ *and* $B \in \mathcal{A}(x)$ *and* $\xi \in A \cap B$ *then* $A \cup B \in \mathcal{A}(x)$. *Hence, if* $A_i \in \mathcal{A}(x)$, $i = 1,2,\cdots,n$, *then* $\cup_{i=1}^n A_i \in \mathcal{A}(x)$.

Proof. Let $\varepsilon > 0$. Since $A \in \mathcal{A}(\xi)$, there is $\tau < \varepsilon$ where each point y of the τ-neighborhood V of ξ has an element $C \in F(y)$ such that $H(A,C) < \varepsilon$. Since $B \in \mathcal{A}(x)$ there is

$\delta > 0$ such that each point z of the δ-neighborhood W of x has an element $D \in F(z)$ such that $H(B,D) < \tau$. One sees that $\xi \in B$ and $H(B,D) < \tau$ imply $V \cap D \neq \emptyset$. Hence, for each $z \in W$ there are $D \in F(z)$, $y \in V \cap D$ and $C \in F(y)$ such that $H(A,C) < \varepsilon$, $H(B,D) < \varepsilon$ and $C \cup D \in F(z)$. By Lemma 1.4, we have $H(A \cup B, C \cup D) \leq \max\{H(A,C),H(B,D)\} < \varepsilon$, and the proposition is proved.

Proposition 1.6. *For each* $x \in X$ *its admissible fiber* $\mathcal{A}(x)$ *is closed in* $C(X)$, $\{x\} \in \mathcal{A}(x)$ *and* $X \in \mathcal{A}(x)$.

Proof. Suppose A_n, $n = 1,2,\cdots$, is a sequence in $\mathcal{A}(x)$ which converges to A in $C(X)$. Obviously, $A \in F(x)$. Let $\varepsilon > 0$. There is a positive integer N such that $H(A,A_N) < \varepsilon/2$. Since $A_N \in \mathcal{A}(x)$, there is a δ-neighborhood V of x such that each point y of V has an element $B \in F(y)$ such that $H(A_N,B) < \varepsilon/2$. From $H(A,B) \leq H(A,A_N) + H(A_N,B) < \varepsilon$, we have $A \in \mathcal{A}(x)$ and hence $\mathcal{A}(x)$ is closed. The remaining parts of the proposition are obvious.

We note that since $C(X)$ is compact [1], $\mathcal{A}(x)$ is compact.

Proposition 1.7. *Let* $B \in F(x)$ *and* $C = \cup\{A \in \mathcal{A}(x) \mid A \subset B\}$ *then* $C \in \mathcal{A}(x)$.

Proof. First we prove C is a subcontinuum of X. Clearly C is connected and $x \in C$. Let x_n, $n = 1,2,\cdots$, be a sequence in C converging to x_0. For each $n \geq 1$ choose $A_n \in \mathcal{A}(x)$ such that $x_n \in A_n \subset B$. Since $\mathcal{A}(x)$ is compact in $C(X)$, we may assume that the sequence A_n, $n = 1,2,\cdots$, also converges to an element $A_0 \in \mathcal{A}(x)$. Obviously, $x_0 \in A_0 \subset B$. Hence $x_0 \in A_0 \subset C$. We conclude that C is closed in X.

Now suppose $\varepsilon > 0$. Since C is compact in X, there are points $c_1, c_2, \cdots, c_n$ in C such that C is contained in the ε-neighborhood of the finite set $\{c_1, c_2, \cdots, c_n\}$. For each i, let $A_i \in \mathcal{A}(x)$ such that $c_i \in A_i \subset B$ and let $B_0 = \cup_{i=1}^n A_i$. Since $C \supset B_0 \supset \{c_1, c_2, \cdots, c_n\}$, we have $H(C, B_0) < \varepsilon$. By Proposition 1.5, $B_0 \in \mathcal{A}(x)$. Since $\mathcal{A}(x)$ is compact in C(X) we have $C \in \mathcal{A}(x)$.

Proposition 1.8 [7]. *If* $h: X \times I \to C(X)$ *is a continuous increasing map such that* $x \in h(x,0)$ *for* $x \in X$ *then* $h(x,t) \in \mathcal{A}(x)$ *for* $(x,t) \in X \times I$.

Theorem 1.9 [7]. *If* X *is a nondegenerate metric continuum and* C(X) *is contractible, then* X *is an admissible space.*

2. Fiber Maps

In [5] it was shown that the contractibility of C(X) is equivalent to an existence of a set-valued map $\alpha: X \to C(X)$ possessing a certain property. In this section we prove that this property is preserved by the homotopy equivalence relation. Hence, we obtain generalizations of many of the results in [4] and [8].

Definition 2.1 [5]. A set-valued map $\alpha: X \to C(X)$ is said to be a c-*map* if, for each $x \in X$, $\alpha(x)$ is a closed subset of the admissible fiber $\mathcal{A}(x)$ such that

(1) $\{x\}, X \in \alpha(x)$.

(2) For each pair A_0, A_1 in $\alpha(x)$ with $A_0 \subset A_1$, there is an ordered segment [2, p. 57] in $\alpha(x)$ from A_0 to A_1.

(3) For each $A \in \alpha(x)$, and $\varepsilon > 0$, there is a neighborhood W of x such that each point y of W has an element $B \in \alpha(y)$ such that $H(A,B) < \varepsilon$.

We say that the space X is a c-*space* if there is a set-valued c-map $\alpha: X \to C(X)$. Clearly every c-space is an admissible space.

Proposition 2.2 [5]. *Every set-valued* c-*map* $\alpha: X \to C(X)$ *is lower semicontinuous. Furthermore, if* $\hat{\alpha}(x,t) = \alpha(x) \cap \mu^{-1}(t)$, *then* $\hat{\alpha}$ *is lower semicontinuous on* $X \times I$.

Theorem 2.3 [5]. *Let* X *be a metric continuum. Then* C(X) *is contractible if and only if there is continuous set-valued* c-*map on* X *into* C(X).

In [1] Kelley defined a property (subsequently named property K in Nadler [2]) and proved that the hyperspaces of a space having property K are always contractible. The class of metric continua having property K includes locally connected continua and the hereditarily indecomposable continua. We now restate the result of Kelley.

Proposition 2.4 [1]. *If* X *has property* K, *then there is a continuous* c-*map* $\alpha: X \to C(X)$.

Proof. Since X has property K, $F(x) = \mathcal{A}(x)$ by [5, Proposition 2.4] and $F: X \to C(X)$ is continuous by [9, Theorem 2.2]. The existence of ordered segments in F(x) for every pair $A_0 \subset A_1$ is given in [1, p. 24]. Hence the admissible fiber map $\mathcal{A}$ is a continuous set-valued c-map.

Let X be a metric continuum. A function $\alpha: X \to C^2(X)$ is called *admissible* if, for each $x \in X$,

(1)' $\{x\} \in \alpha(x)$,

(2)' $\alpha(x) \subset \mathcal{A}(x)$ and $\alpha(x)$ is closed in $\mathcal{A}(x)$,

(3)' $\alpha(x)$ contains a maximal element A_x, i.e., $A \subset A_x$ for all $A \in \alpha(x)$,

(4)' $\alpha(x)$ is segmentwise connected, i.e., for each pair A_0, A_1 in $\alpha(x)$ with $A_0 \subset A_1$, there is an ordered segment [2, p. 57] in $\alpha(x)$ from A_0 to A_1.

Let $\mathcal{N}_\alpha = \{A_x | A_x$ is a maximal element in $\alpha(x)$, $x \in X\}$ and $\mathcal{N}_\alpha^2 = \{\{A_x\} | A_x \in \mathcal{N}_\alpha\} \subset C^2(X)$.

Proposition 2.5. *The following statements are equivalent.*

(1) $C(X)$ *is contractible.*

(2) *There is a continuous admissible function* $\alpha: X \to C^2(X)$ *such that* $\cap \mathcal{N}_\alpha \neq \emptyset$.

(3) *There is a continuous admissible function* $\alpha: X \to C^2(X)$ *such that the set* $\mathcal{N}_\alpha^2$ *is contractible in* $C^2(X)$.

Proof. (1) ⇒ (2). Suppose $h: X \times I \to C(X)$ is an increasing contraction. Let $\alpha(x) = \{h(x,t) | t \in I\}$. Then the continuity of h provides the continuity of α and it is obvious that α satisfies the admissible conditions (1')-(4') with $A_x = h(x,1) = \cap \mathcal{N}_\alpha$.

(2) ⇒ (3). Suppose $\alpha: X \to C^2(X)$ is a continuous admissible function such that $\cap \mathcal{N}_\alpha \neq \emptyset$. Let $x_0 \in \cap \mathcal{N}_\alpha$ and let $\gamma: I \to C(X)$ be an ordered segment from $\{x_0\}$ to X. Define $\beta: \mathcal{N}_\alpha^2 \times I \to C^2(X)$ by

$$\beta(\{A_x\},t) = \{A_x \cup \gamma(t)\}.$$

Then β is continuous and $\beta(\{A_x\},1) = \{X\}$ for each $\{A_x\} \in \mathcal{N}_\alpha^2$.

(3) $\Rightarrow$ (1). Suppose $\alpha\colon X \to C^2(X)$ is a continuous admissible function and $\beta\colon \mathcal{N}_\alpha^2 \times I \to C^2(X)$ is a contraction. We may assume β is increasing. Let $\sigma\colon C^2(X) \to C(X)$ be the function defined by $\sigma(T) = \cup T$. Then σ is continuous.

We now observe that for each $x \in X$ the maximal element A_x of $\alpha(x)$ is unique and $A_x = \sigma \circ \alpha(x)$. Therefore the function $x \to \{A_x\}$ is continuous from X to $C^2(X)$. Hence we define a function $\tau\colon X \to C^2(X)$ by $\tau(x) = \alpha(x) \cup \{\sigma\beta(\{A_x\},t) \mid t \in I\}$. Then τ is continuous. Since $\sigma\beta(\{A_x\},1) = A$, for some A, and for all $x \in X$, we may define an ordered segment $\gamma\colon I \to C(X)$ from A to X and join it to τ, that is $\phi(x) = \tau(x) \cup \{\gamma(t) \mid t \in I\}$. Then it is not difficult to check that ϕ satisfies the definition of a set-valued c-map and ϕ is continuous. Hence by Theorem 2.3, $C(X)$ is contractible.

Suppose X and Y are metric continua.

Theorem 2.6. *Suppose* $f\colon X \to Y$ *and* $g\colon Y \to X$ *are continuous functions such that* $f \circ g\colon Y \to Y$ *is homotopic to the identity* id_Y. *If* X *is a* c-*space then* Y *is a* c-*space.*

Proof. Let $h\colon Y \times I \to Y$ be a homotopy such that $h(y,0) = y$ and $h(y,1) = f \circ g(y)$ for each $y \in Y$. Let $\bar{h}(y,t) = \cup\{h(y,s) \mid 0 \le s \le t\}$. Then $\bar{h}\colon Y \times I \to C(Y)$ is a continuous homotopy such that $\bar{h}(y,t) \subset \bar{h}(y,t')$ whenever $t \le t'$. Let $\beta_1(y) = \{\bar{h}(y,t) \mid t \in I\}$. Then the continuity of $\bar{h}$ implies the continuity of $\beta_1\colon Y \to C^2(Y)$ and each element $\bar{h}(y,t)$ of the set $\beta_1(y)$ is an admissible element at y such that for each pair B_0, B_1 in $\beta_1(y)$ with $B_0 \subset B_1$, there is an ordered segment in $\beta_1(y)$ from B_0 to B_1.

Let $\alpha: X \to C(X)$ be a set-valued c-map. For $y \in Y$, let $x = g(y)$ and $\beta_2(y) = \{\overline{h}(y,1) \cup f(A) \mid A \in \alpha(x)\}$. Since $f(x) = f \circ g(y) \in \overline{h}(y,1) \cap f(A)$, we have $\overline{h}(y,1) \cup f(A) \in C(Y)$. Now we will show that $\beta_2: Y \to C^2(Y)$ is lower semicontinuous.

Let $\varepsilon > 0$. Since f is continuous, there is $\varepsilon' > 0$ such that if $A, A' \in C(X)$ such that A and A' are less than ε' apart, then $H(f(A), f(A')) < \varepsilon$. Since α is lower semicontinuous and $\alpha(x)$ is compact, there exists $\delta > 0$ such that if $d(x,x') < \delta$ and $A \in \alpha(x)$, there is an element $A' \in \alpha(x')$ such that A and A' are less than ε' apart. Now the continuity of g implies that there is $\delta_0 > 0$ such that if $d(y,y') < \delta_0$ then $d(g(y), g(y')) < \delta$. Also, by the continuity of $\overline{h}$, we choose $\delta_1 > 0$ such that if $d(y,y') < \delta_1$, then $H(\overline{h}(y,1), \overline{h}(y',1)) < \varepsilon'$. Let $\overline{\delta} = \min\{\delta_0, \delta_1\}$, $x = g(y)$, $x' = g(y')$. Then if $d(y,y') < \overline{\delta}$ and $\overline{h}(y,1) \cup f(A) \in \beta_2(y)$ then there is $\overline{h}(y',1) \cup f(A') \in \beta_2(y')$ such that $H(\overline{h}(y,1) \cup f(A), \overline{h}(y',1) \cup f(A')) \leq \max\{H(\overline{h}(y,1), \overline{h}(y',1)), H(f(A), f(A'))\} < \varepsilon$ by Lemma 1.4. Hence the elements of $\beta_2(y)$ are admissible at y and β_2 is lower semicontinuous. Since f preserves ordered segments, we see that β_2 satisfies the condition of ordered segment. Now let $\gamma: I \to C(Y)$ be an ordered segment from f(X) to Y and let $\beta_3(y) = \{\gamma(t) \cup \overline{h}(y,1) \mid t \in I\}$ and $\beta(y) = \beta_1(y) \cup \beta_2(y) \cup \beta_3(y)$. The maximal element of $\beta_1(y)$ is $\overline{h}_1(y,1)$ which is also the minimal element of $\beta_2(y)$, and the maximal element of $\beta_2(y)$ is $\overline{h}(y,1) \cup f(X)$, and the minimal element of $\beta_3(y)$ is $f(X) \cup \overline{h}(y,1)$. So the continuity of β_1

and β_3 together with the lower semicontinuity of β_2 provide the lower semicontinuity of β, and hence (3) is verified for β. It is easy to verify that β also satisfies the condition (1) and (2) of Definition 2.1.

Corollary 2.7. *Suppose* X *and* Y *are homotopically equivalent metric continua. Then* X *is a* c-*space if and only if* Y *is.*

Let id_Y denote the identity map of Y onto itself and [f] the homotopy class of continuous maps of Y into itself which contains f.

Theorem 2.8. *A metric continuum* Y *is a* c-*space if and only if for some* g *in* $[id_Y]$ *it is true that* g(Y) *is a* c-*space.*

Proof. If Y is a c-space then let $g = id_Y$. Conversely, suppose for some $g \in [id_Y]$, g(Y) is a c-space. Let X = g(Y) and f: X → Y be the inclusion map. Then $f \circ g = g \in [id_Y]$. So Theorem 2.6 gives the conclusion.

Corollary 2.9. *Suppose* X *is a deformation retract of* Y. *If* X *is a* c-*space, so is* Y.

Proof. Let γ: Y → X be a retraction which is homotopic to the identity map id_Y. Then Theorem 2.8 provides the conclusion.

Corollary 2.10. *If* Y *is a retract of* X *and* X *is a* c-*space, then so is* Y.

Proof. Let f: $X \to Y$ be a retraction and g: $Y \to X$ the inclusion map. Then $f \circ g = id_Y$. So by Theorem 2.6, Y is a c-space.

Theorem 2.11. *Let* $X = X_1 \cup X_2$ *where* X *and* X_1 *are subcontinua and* X_2 *is a closed subset such that* $X_1 \cap X_2$ *is a strong deformation retract of* X_2. *If* X_1 *is a* c-*space so is* X.

Proof. X_1 is a deformation retract of X.

Theorem 2.12. *The product space* $X \times Y$ *is a* c-*space if and only if both* X *and* Y *are* c-*spaces.*

Proof. Each factor space is a retract of $X \times Y$. Therefore by Corollary 2.10, $X \times Y$ is a c-space.

Conversely, if α_X: $X \to C(X)$ and α_Y: $Y \to C(Y)$ are set-valued c-maps, then $\alpha_X \times \alpha_Y$: $X \times Y \to C(X \times Y)$ defined by $\alpha_X \times \alpha_Y(x,y) = \alpha_X(x) \times \alpha_Y(y) = \{A \times B | A \in \alpha_X(x), B \in \alpha_Y(y)\}$ is a set-valued c-map.

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