
TOPOLOGY PROCEEDINGS



Volume 9, 1984

Pages 1–6

<http://topology.auburn.edu/tp/>

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Topology Proceedings

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ISSN: 0146-4124

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1. Introduction

Let (X, T) be a topological space. An *extension* of T is a topology T' for X such that $T' \supset T$. If α is a collection of subsets of X , then the *extension of T by α* , denoted by $T(\alpha)$, is the topology for X having $T \cup \alpha$ as a subbase. It has been of interest to find conditions on the members of α in order that a given topological property be preserved under the extension of T by α (see [1], [3]-[9]). In this paper we give a necessary and sufficient condition on the members of α in order that $(X, T(\alpha))$ be weakly equivalent to (X, T) . As a consequence, we have a sufficient condition on the members of α so that dense sets, residual sets, separability, the property of being a Baire space, etc. are preserved under the extension of T by α . We also obtain conditions which imply that a $T(\alpha)$ -continuous function into a metric space is T -continuous at each point of a T -dense set.

2. Weakly Equivalent Extensions

Two topologies T and T^* for a set X are said to be *weakly equivalent* provided that each nonempty member of either of the topologies contains a nonempty member of the other. We say that a collection α of subsets of a set X has *property (W) with respect to a topology T* for X provided that for each finite subcollection α' of α ,

$$\text{cl}_T[\text{int}_T(\mathcal{N}\alpha')] \supset \mathcal{N}\alpha'$$

where cl_T and int_T denote the closure and interior operators with respect to T .

Theorem 2.1. Let (X, T) be a topological space and let α be a collection of subsets of X . Then, the topologies T and $T(\alpha)$ are weakly equivalent if and only if α has property (W) with respect to T .

Proof. First, suppose that α does not have property (W) with respect to T . Then, there is a finite subcollection α' of α such that

$$\text{cl}_T[\text{int}_T(\alpha')] \not\supset \mathcal{N}\alpha'.$$

Let $G = (X - \text{cl}_T[\text{int}_T(\mathcal{N}\alpha')]) \cap (\mathcal{N}\alpha')$. Note that $G \neq \emptyset$ and $G \in T(\alpha)$. Suppose that there is a $V \in T$ such that $V \neq \emptyset$ and $V \subset G$. Since $V \subset G$, $V \subset \mathcal{N}\alpha'$. Thus, since $V \in T$, $V \subset \text{int}_T(\mathcal{N}\alpha')$. However, since $V \subset G$, $V \subset X - \text{int}_T(\mathcal{N}\alpha')$ which, since $V \neq \emptyset$, is a contradiction. Hence, G does not contain any nonempty member of T . Therefore, T and $T(\alpha)$ are not weakly equivalent. Conversely, assume that α has property (W) with respect to T . Let $H \in T(\alpha)$ such that $H \neq \emptyset$. Then, since $T \cup \alpha$ is a subbase for $T(\alpha)$, there is a finite subcollection α' of α and a $U \in T$ such that $U \cap (\mathcal{N}\alpha')$ is a nonempty subset of H . Let $V = \text{int}_T(\mathcal{N}\alpha')$. Since α has property (W) with respect to T , $\text{cl}_T(V) \supset \mathcal{N}\alpha'$. Thus, since $U \in T$ and $U \cap (\mathcal{N}\alpha') \neq \emptyset$, $U \cap V \neq \emptyset$. Clearly, $U \cap V \in T$ and $U \cap V \subset H$. Hence, we have proved that each nonempty member of $T(\alpha)$ contains a nonempty member of T . Therefore, since $T \subset T(\alpha)$, T and $T(\alpha)$ are weakly equivalent.

Let us note that there are collections α such that for any two members A and B of α

$$\text{cl}_T[\text{int}_T(A \cap B)] \supset A \cap B$$

and yet α fails to have property (W) with respect to T .

For example: Let $X = \{(x,0) \in \mathbb{R}^2: -1 \leq x \leq 1\} \cup \{(0,y) \in \mathbb{R}^2: 0 \leq y \leq 1\}$ with the topology T inherited from the usual topology on the plane \mathbb{R}^2 , and let $\alpha = \{A_1, A_2, A_3\}$ where $A_1 = \{(x,0) \in X: -1 \leq x \leq 0\} \cup \{(0,y) \in X: 0 \leq y \leq 1\}$, $A_2 = \{(x,0) \in X: 0 \leq x \leq 1\} \cup \{(0,y) \in X: 0 \leq y \leq 1\}$, and $A_3 = \{(x,0) \in X: -1 \leq x \leq 1\}$.

3. Applications

First, recall the following definitions. Let (X,T) be a topological space and let $A \subset X$. Then: A is T -dense in X provided that $\text{cl}_T(A) = X$; A is T -nowhere dense in X provided that $\text{int}_T[\text{cl}_T(A)] = \emptyset$; A is T -first category in X provided that $A = \{A_i: i = 1, 2, \dots\}$ where each A_i is T -nowhere dense in X ; A is T -second category in X provided that A is not T -first category in X ; A is T -residual in X provided that $X-A$ is T -first category in X . Fort [2] has observed that if two topologies on X are weakly equivalent, then a subset A of X has any of the properties above with respect to one of the topologies if and only if A has the same property with respect to the other topology. Thus, as a consequence of (2.1), we have the following result.

Theorem 3.1. Let (X,T) be a topological space, let α be a collection of subsets of X having property (W) with respect to T , and let $A \subset X$. Then, A is T -dense, T -nowhere

dense, T -first category, T -second category, or T -residual if and only if A is $T(\alpha)$ -dense, $T(\alpha)$ -nowhere dense, $T(\alpha)$ -first category, $T(\alpha)$ -second category, or $T(\alpha)$ -residual, respectively.

Corollary 3.2. Let (X, T) be a topological space and let α be a collection of subsets of X having property (W) with respect to T . If (X, T) is separable (β -separable = contains a T -dense set of cardinality β), then $(X, T(\alpha))$ is separable (β -separable, respectively).

Other results about the preservation of separability under extensions of topologies are in Theorem 8 of [4] and Theorem 5.8 of [1].

Recall that a topological space (X, T) is a *Baire Space* provided that the countable intersection of dense open sets is dense. It is easy to see that (X, T) is a Baire space if and only if every T -residual set is T -dense. Therefore, the following result is an immediate consequence of 3.1.

Corollary 3.3. Let (X, T) be a topological space and let α be a collection of subsets of X having property (W) with respect to T . Then, (X, T) is a Baire space if and only if $(X, T(\alpha))$ is a Baire space.

Fort [2] has shown that if (X, T) and (X, T^*) are weakly equivalent Baire spaces and if a function f from X into a metric space is T^* -continuous at each point of a T^* -dense set, then f is a T -continuous at each point of a T -dense set. Using this theorem, 2.1, and 3.3, we have the following result:

Theorem 3.4. Let (X, T) be a Baire space and let α be a collection of subsets of X having property (W) with respect to T . If a function f from X into a metric space is $T(\alpha)$ -continuous at each point of a $T(\alpha)$ -dense set, then f is T -continuous at each point of a T -dense set.

Let T denote the usual topology on the real line and let $\alpha = \{[a, b) : a, b \in \mathbb{R}^1\}$. We see that a function $f: \mathbb{R}^1 \rightarrow \mathbb{R}^1$ is right continuous if and only if f is $T(\alpha)$ -continuous (with the usual topology on the range). Hence, by 3.4, we have the classical result in real analysis that if f is right continuous, f is continuous at each point of a dense set.

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