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PRE-IMAGES OF θ-SPACES

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A mapping from a topological space X onto a topological space Y is a (quasi-) perfect mapping provided that the mapping is continuous, closed and the inverse images of points are (countably) compact. The inverse images of topological spaces under both of these types of mappings have been of considerable interest since the introduction of such mappings in topology. Early important results were the characterizations of (1) perfect pre-images of completely metrizable spaces as paracompact Čech-complete spaces by Frolík [Fr] and of (2) the perfect pre-images of metrizable spaces by Arhangel'skiǐ [A_2] as paracompact p-spaces. These led to a number of developments in generalized metric space theory. We view the results we have obtained as generalizations of these which explicate some of the fundamentals involved.

We consider here the pre-images of θ-spaces under perfect and quasi-perfect mappings. In particular, we define the class of pre-θ-spaces, which is a more inclusive class than the class of D-spaces of Brandenburg [B] and the *-spaces of Isiwata [I] which characterize the quasi-perfect pre-images of T_1 developable spaces.

We denote the set of natural numbers by N. Any reference given below for a certain concept contains a definition of the concept but is not necessarily the original source.

Definition 1. Suppose that (X,τ) is a topological space and $h: N \times X \to \tau$ is a mapping such that for all $x \in X$ and $n \in N$, $x \in h(n+1,x) \subseteq h(n,x)$ and $g(n,x) = \cup\{h(n,z): z \in h(n,x)$ and $x \in h(n,z)\}$. If for each $x \in X$, $\{g(n,x): n \in N\}$ forms a base at x then X is a θ-*space* ([H], [W]). We will call h a θ-function provided h and the associated function g are described as above.

Condition 2. Assume that (X,τ) is a topological space and that $h: N \times X \to \tau$ is a mapping such that for $x \in X$ and $n \in N$, $x \in h(n+1,x) \subseteq h(n,x)$. Define $g(n,x) = \cup\{h(n,z): z \in h(n,x)$ and $x \in h(n,z)\}$. In Lemmas 2-13, we are assuming that the following properties hold.

(01) For each $x \in X$, $\cap_{n\in N} g(n,x)$ is closed.

(02) For each $x \in X$, if $\langle y_n: n \in N\rangle$ is a sequence in X such that $y_n \in g(n,x)$ for each $n \in N$ then $\langle y_n: n \in N\rangle$ clusters to some point in $\cap_{n\in N} g(n,x)$.

(RI) For each $x \in X$ and $n \in N$ there exists $m \in N$ such that for all $z \in g(m,x)$ there is $k \in N$ such that $g(k,z) \subseteq g(n,x)$.

(RII) For each $x \in X$ and $n \in N$, if $z \in h(n,x)$ and $x \in \cap_{n\in N} g(n,y)$ then there is $k \in N$ such that $g(k,z) \subseteq h(n,y)$.

(EC) If $\cap_{n\in N} g(n,x) \cap \cap_{n\in N} g(n,y) \neq \emptyset$ then $\cap_{n\in N} g(n,x) = \cap_{n\in N} g(n,y)$.

We will call a mapping h a *pre*-θ-*function* provided h and the associated function g satisfy the conditions of 2.

First we will state a series of lemmas concerning a space X with a pre-θ-function which enable us to show that such a space is a quasi-perfect pre-image of a T_1 θ-space.

Lemma 3. *The collection* $\{\cap_{n\in N} g(n,x): x \in X\}$ *is a partition of* X *into countably compact closed subsets.*

The equivalence relation associated with the partition in Lemma 3 may be described by $x^{\sim}y$ if and only if $\cap_{n\in N} g(n,x) = \cap_{n\in N} g(n,y)$. For each $x \in X$ denote the equivalence class associated with x by $x^{\sim}$. We then have for all $x \in X$, $x^{\sim} = \cap_{n\in N} g(n,x)$.

Lemma 4. *For all* $x \in X$, $\{g(n,x): n \in N\}$ *is a base at* $\cap_{n\in N} g(n,x)$.

For each $A \subseteq X$, define INT(A) as $\{x \in X$: there is $n \in N$ such that $g(n,x) \subseteq A\}$.

Lemma 5. *For each* $A \subseteq X$, INT(A) *is open and* INT(A) = $\cup\{x^{\sim}: x \in \text{INT}(A)\}$.

It should be noted here that for all $n \in N$ and $x \in X$, $g(n,x) = \text{INT}(g(n,x))$.

Lemma 6. *For each* $A,B \subseteq X$, $\text{INT}(A \cap B) = \text{INT}(A) \cap \text{INT}(B)$.

Lemma 7. *For each* $A \subseteq X$, INT(A) = INT(INT(A)).

We will denote by $X^{\sim}$ the set of all equivalence classes of X with respect to the relation "$\sim$". Now we will define a map f from X onto $X^{\sim}$ by $f(x) = x^{\sim}$ for each $x \in X$.

Lemma 8. *For each* $A \subseteq X$, $\text{INT}(A) = f^{-1}f(\text{INT}(A))$.

Using Lemmas 8 and 6 we may make X into a topological space by taking $\{f(\mathrm{INT}(A)): A \subseteq X\}$ as a base for the open sets of $X^{\sim}$.

Lemma 9. *The function* $f: X \to X^{\sim}$ *is continuous.*

Lemma 10. *For all* $B \subseteq X^{\sim}$, $\mathrm{int}_{X^{\sim}} B = f(\mathrm{INT}(f^{-1}(B)))$.

It follows from Lemma 4, the remark after Lemma 5 and the definition of the topology that $X^{\sim}$ is first countable. If $x^{\sim}$, $y^{\sim}$ are distinct elements of $X^{\sim}$ then $\cap_{n \in N} g(n,x) \neq \cap_{n \in N} g(n,y)$. Thus by (EC) $\cap_{n \in N} g(n,x) \cap \cap_{n \in N} g(n,y) = \emptyset$. So $y \notin \cap_{n \in N} g(n,x)$ and $x \notin \cap_{n \in N} g(n,y)$ and hence $X^{\sim}$ is T_1.

Lemma 11. *The function* $f: X \to X^{\sim}$ *is closed.*

Remark. The mapping f is a quotient mapping since the mapping is closed and onto. [E, Corollary 2.4.8.]

Lemma 12. *For* $x^{\sim} \in X^{\sim}$, *if* $x,y \in f^{-1}(x^{\sim})$ *then* $\mathrm{INTh}(n,x) = \mathrm{INTh}(n,y)$.

Proof. Let $x^{\sim} \in X^{\sim}$ and $x,y \in f^{-1}(x^{\sim})$. Then $\cap_{n \in N} g(n,x) = \cap_{n \in N} g(n,y)$, so $x \in \cap_{n \in N} g(n,y)$ and $y \in \cap_{n \in N} g(n,x)$. Let $z \in \mathrm{INTh}(n,x)$. Then there is $k \in N$ such that $g(k,z) \subseteq h(n,x)$. Thus $z \in h(n,x)$. By (RII), there is $j \in N$ such that $g(j,z) \subseteq h(n,y)$. Thus $z \in \mathrm{INTh}(n,y)$ and $\mathrm{INTh}(n,x) \subseteq \mathrm{INTh}(n,y)$. By a similar argument $\mathrm{INTh}(n,y) \subseteq \mathrm{INTh}(n,x)$ and thus equality holds.

Let τ' be the quotient topology on $X^{\sim}$ and define $r: N \times X^{\sim} \to \tau'$ by $r(n,x^{\sim}) = f(\mathrm{INTh}(n,x))$ where $x \in f^{-1}(x^{\sim})$. For each $n \in N$, $x \in \mathrm{INTh}(n+1,x) \subseteq \mathrm{INTh}(n,x)$ since

$x \in h(n+1,x) \subseteq h(n,x)$ and $\text{INT}h(n,x) = h(n,x)$ by (RII). So $x^\sim = f(x) \in f(\text{INT}h(n+1,x)) \subseteq f(\text{INT}h(n,x))$ for each $n \in N$. Now define $s(n,x^\sim) = \cup\{r(n,z^\sim): z^\sim \in r(n,x^\sim)$ and $x^\sim \in r(n,z^\sim)\}$. Notice that for each $x^\sim \in X^\sim$ and $n \in N$, $f(g(n,x)) = \cup\{fh(n,z): f(z) \in fh(n,x)$ and $f(x) \in fh(n,z)\} = \cup\{r(n,z^\sim): z^\sim \in r(n,x^\sim)$ and $x^\sim \in r(n,z^\sim)\} = s(n,x^\sim)$.

Lemma 13. *The quotient space* $X^\sim$ *is a* θ*-space.*

Proof. It suffices to show that for each $x^\sim \in X^\sim$, $\{s(n,x^\sim): n \in N\}$ is a base at $x^\sim$. Let $x^\sim \in X^\sim$ and let W be an open set containing $x^\sim$. Now for each $x \in f^{-1}(x^\sim)$, $x \in \cap_{n\in N} g(n,x) \subseteq f^{-1}(W)$ and by Lemma 3, $\{g(n,x): n \in N\}$ is a base at $\cap_{n\in N} g(n,x)$. Thus there is $k \in N$ such that $g(k,x) \subseteq f^{-1}(W)$. Hence $s(k,x^\sim) = fg(k,x) \subseteq ff^{-1}(W) = W$. So $\{s(n,x^\sim): n \in N\}$ is a base at $x^\sim$ and thus $X^\sim$ is a θ-space.

We have shown that if X is a space with a pre-θ-function then there is a quasi-perfect mapping f from X onto a T_1 θ-space $X^\sim$. Now let us consider the reverse implication.

We first state two lemmas which give us a basic component of our characterization. These are essentially known, the earliest reference concerning closely related results is [Mo, Chapter V, Theorems 2 and 3].

Lemma 14. *If* $f: X \to Y$ *is a closed surjection and* $y \in Y$ *has a countable base, then* $f^{-1}(\{y\})$ *has countable character. (i.e.* A *has countable character if there is a collection of open sets* $\{U_n: n \in N\}$ *such that if* $A \subseteq V$ *with* V *open then there is* $n \in N$ *with* $A \subseteq U_n \subseteq V$.)

Lemma 15. *If* f: X → Y *is a quasi-perfect mapping and* Y *is* T_1 *and* $\{D_n: n \in N\}$ *is a decreasing local base at* y, *then if for all* $n \in N$, $x_n \in D_n' = f^{-1}(D_n)$, *the sequence* $\langle x_n: n \in N\rangle$ *has a cluster point in* $f^{-1}(\{y\})$.

Theorem 16. *Suppose* Y *is a* T_1 θ*-space and* f *is a quasi-perfect mapping from a space* X *onto* Y. *Then there exists a pre-*θ*-function for* X.

Proof. Let Y be a T_1 θ-space and f be a quasi-perfect mapping from a space X onto Y. Since Y is a T_1 θ-space there exists a θ-function p for Y. We now define a mapping $h: N \times X \to \tau$, where τ is the topology on X, such that for each $x \in X$ and $n \in N$, $h(n,x) = f^{-1}p(n,f(x))$ and $g(n,x) = \cup\{h(n,z): z \in h(n,x) \text{ and } x \in h(n,z)\}$. Notice that $g(n,x) = \cup\{f^{-1}p(n,f(z)): f(z) \in p(n,f(x)) \text{ and } f(x) \in p(n,f(z))\} = f^{-1}q(n,f(x))$. To see that (01) holds, suppose $x \in X$. Then $f(x) \in Y$ and $\{q(n,f(x)): n \in N\}$ is a base at f(x). Since Y is a T_1-space, $\cap_{n\in N}q(n,f(x)) = \{f(x)\}$ which is closed. The continuity of f implies that $f^{-1}(\cap_{n\in N}q(n,f(x)))$ is closed. But $f^{-1}(\cap_{n\in N}q(n,f(x))) = \cap_{n\in N}g(n,x)$ and thus $\cap_{n\in N}g(n,x)$ is closed.

To see that (02) holds, let $x \in X$ and $\langle y_n: n \in N\rangle$ be a sequence in X such that $y_n \in g(n,x)$ for each $n \in N$. The collection $\{fg(n,x): n \in N\}$ is a decreasing local base at f(x) so by Lemma 15 $\langle y_n: n \in N\rangle$ has a cluster point in $\cap_{n\in N}g(n,x)$.

To show that (RI) holds, let $x \in X$ and $n \in N$. Choose $z \in g(n,x)$. Then $f(z) \in q(n,f(x))$ and $\{q(n,f(z)): n \in N\}$ forms a base at f(z). Hence there is $k \in N$ such that

$q(k,f(z)) \subseteq q(n,f(x))$ and thus $g(k,z) \subseteq g(n,x)$. Hence (RI) holds.

Next we show that (RII) is satisfied. Let $x \in X$ and $n \in N$. Suppose $z \in h(n,x)$ and $x \in \cap_{n\in N} g(n,y)$. Now $z \in h(n,x)$ implies $f(z) \in fh(n,x) = p(n,f(x))$ and $p(n,x)$ is an open set containing $f(z)$. Since Y is a T_1 θ-space there is $k \in N$ such that $q(k,f(z)) \subseteq p(n,f(x))$. Since $x \in \cap_{n\in N} g(n,y)$, $f(x) = f(y)$ and thus $p(n,f(x)) = p(n,f(y))$. Thus $g(k,z) = f^{-1}q(k,f(z)) \subseteq f^{-1}p(n,f(y)) = h(n,y)$. Thus (RII) holds.

Finally we show that (EC) holds. If $\cap_{n\in N} g(n,x) \cap \cap_{n\in N} g(n,y) \neq \emptyset$ then $\cap_{n\in N} q(n,f(x)) \cap \cap_{n\in N} q(n,f(y)) \neq \emptyset$. But Y is a T_1 θ-space, so $\cap_{n\in N} q(n,f(x)) = \{f(x)\}$ and $\cap_{n\in N} q(n,f(y)) = \{f(y)\}$. Thus $f(x) = f(y)$ and hence $\cap_{n\in N} g(n,x) = f^{-1}f(x) = f^{-1}f(y) = \cap_{n\in N} g(n,y)$. Thus (EC) holds and the proof is completed.

In order to summarize what we have just shown, let us make the following definition.

Definition 17. A space (X,τ) having a pre-θ-function will be called a *pre-θ-space*. If X is a pre-θ-space with the additional property that for each $x \in X$, $\cap_{n\in N} g(n,x)$ is compact then we will call X a *strong pre-θ-space*.

We may now restate Lemma 13 and Theorem 16 as follows.

Theorem 18. *A space* X *is a pre-θ-space if and only if there is a quasi-perfect mapping from* X *onto a* T_1 *θ-space.*

It is easily seen from the previous definition that the folloiwng is also true.

Theorem 19. *A space* X *is a strong pre-*θ*-space if and only if there is a perfect mapping from* X *onto a* T_1 θ*-space.*

We next discuss the class of wθ-spaces which is larger than the class of θ-spaces.

Definition 20. A space (X,τ) is a wθ-*space* if and only if there is a mapping $h\colon N \times X \to \tau$ such that for all $x \in X$ and $n \in N$, $x \in h(n+1,x) \subseteq h(n,x)$ and if $g(n,x) = \cup\{h(n,z)\colon z \in h(n,x) \text{ and } x \in h(n,z)\}$ then if $\langle y_n\colon n \in N\rangle$ is a sequence in X such that $y_n \in g(n,x)$ for each $n \in N$ then $\langle y_n\colon n \in N\rangle$ clusters ([H], [W]).

We will also need the following definition.

Definition 21. A space X is a *primitive* σ*-space* if and only if there is a sequence $\mathcal{U} = \langle \mathcal{U}_n\colon n \in N\rangle$ of well-ordered open covers of X such that if $F(x,\mathcal{U}_n) = F(x_n,\mathcal{U}_n)$ for each $n \in N$ implies that $\langle x_n\colon n \in N\rangle$ clusters to x [R].

It is clear that if h and g are mappings on a space X as described in Definition 20 which also satisfies (02) then X is a wθ-space. Thus a pre-θ-space is a wθ-space and a wθ-space which is also a primitive σ-space is a primitive q-space [R]. Hence a pre-θ-space which is also a primitive σ-space is a primitive q-space.

It is clear that every perfect pre-image of a D_0-space [FL] (compact sets have countable character) is a D_0-space.

Also it is not hard to see that quasi-perfect pre-images of first countable spaces are q-spaces in the sense of Michael [M] and perfect pre-images of first countable spaces are spaces of pointwise countable type (for each $x \in X$ there is a compact set K such that $x \in K$ and K has countable character [A_1]).

Notice that a condition like (01) will always be true for continuous pre-images of T_1-spaces with a given base property because we are considering inverse images of points and points are closed in a T_1-space. A property like (02) will always be true for closed continuous quasi-perfect pre-images because of the countable compactness of inverse images of points, the base property of the image space and the closedness of the mapping as Lemma 14 shows.

It should be noted that the work presented here could have been limited to spaces having a base of countable order but since the underlying techniques of proof yield results for other important uniformized versions of first countability we adopt a more general point of view by proving the basic result for θ-spaces. In further papers we will show that by adding the appropriate conditions we are able to get analogous results for more specialized spaces. If desired, one could simplify the method and restrict consideration to spaces having a base of countable order or spaces having a primitive base. The theorems and methods of base of countable order theory are fundamental in the work.

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