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NON-EXISTENCE OF UNIVERSAL SPACES FOR SOME STRATIFIABLE SPACES

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1. Introduction

One of the remarkable properties of metric spaces is that there are universal spaces with a given weight (for example, the Baire's 0-dimensional spaces and generalized Hilbert spaces). The purpose of this paper is to show that there are none for some stratifiable spaces.

Theorem 1. There are no universal spaces for the following subclasses of stratifiable spaces with a given network weight $\alpha \geq \omega$;

- (1) σ -discrete stratifiable spaces,
- (2) M_0 -spaces in the sense of [6],
- (3) stratifiable μ -spaces [11],
- (4) stratifiable spaces with encircling nets [15],
- (5) (strongly zero-dimensional) stratifiable spaces,
- (6) (strongly zero-dimensional) L-spaces in the sense of [13],
- (7) (strongly zero-dimensional) free L-spaces [14].

When we use a given weight instead of a given network weight, we have a similar result:

Corollary 1. For any given infinite cardinal number α there exists a cardinal number $\beta > \alpha$ such that there exist no universal spaces for the above classes (1) ~ (7) with a given weight β .

When we consider more restricted subclasses, contained in Lasnev spaces or Nagata spaces, we also have a similar result:

Theorem 2. There are no universal spaces for either countable Lasnev spaces or first-countable, separable, strongly zero-dimensional stratifiable spaces.

Remark 1. We have universal spaces for several subclasses of *metric* spaces. In particular, we have:

Theorem 3 [12]. There exists a universal space for σ -discrete metric spaces with a given weight.

In this paper all spaces are assumed to be regular T_1 . $w(X)$ and $nw(X)$ denote the weight of X and the network weight of X , respectively. A cardinal number is an initial ordinal number, and an ordinal number is the set of its predecessors.

2. Proofs of Theorems

We start with the following proposition.

Proposition 1. For every infinite cardinal number α there exists a family S of cardinality 2^{2^α} , none of whose members are homeomorphic to each other, and which consists of σ -discrete stratifiable spaces S with $|S| = nw(S) = \alpha$ and $w(S) \leq 2^\alpha$.

Proof. Let $\{\alpha_i\}$ be countable disjoint copies of α . Then, for each ultrafilter p on α , let S_p be the space $\alpha_0 \cup \{p\}$, where $\alpha_0 = \bigcup \{\alpha_i\}$, with the following topology τ_p .

(i) $U \in \tau_p$ if and only if $p \notin U$, or $p \in U$ and there exist an $n \in \omega$ and an $F \in p$ such that $U \cap \alpha_i \supset F$ for every $i \geq n$.

Then, since each α_i is a clopen discrete subspace of S_p , it is readily seen that S_p is a σ -discrete stratifiable space and that $|S_p| = nw(S_p) = \alpha$ and $w(S_p) \leq 2^\alpha$. Next, we shall show that

(ii) the family of all S_q 's which are homeomorphic to a fixed S_p is of the cardinality at most 2^α .

Let S_q and S_r be homeomorphic to a fixed S_p . Then, take any homeomorphisms $h_q: S_p \rightarrow S_q$ and $h_r: S_p \rightarrow S_r$. To show (ii) it suffices to show that

(iii) $h_q|_{\alpha_0} \neq h_r|_{\alpha_0}$ if $q \neq r$.

On the contrary, assume that there exists a permutation $h: \alpha_0 \rightarrow \alpha_0$ such that $h = h_q|_{\alpha_0} = h_r|_{\alpha_0}$. Since $q \neq r$, we can assume without loss of generality that there exists an $F \in q \setminus r$. Put

$$F_0 = \bigcup_{i=1}^{\infty} F_i, \text{ where each } F_i \text{ is a copy of } F \text{ in } \alpha_i,$$

and put

$$U_0 = \{q\} \cup F_0.$$

Then, U_0 is a neighborhood of q by the definition (i) of τ_q . Hence, $h_r h_q^{-1}(U_0)$ is a neighborhood of r , and by the definition (i) there exist an $n \in \omega$ and a $G \in r$ such that $G \subset \alpha_i \cap h_r h_q^{-1}(U_0)$ for $i \geq n$. Hence, $F_n \supset G$, since $\alpha_n \cap h_r h_q^{-1}(U_0) = \alpha_n \cap h h^{-1}(U_0) = F_n$. Since F_n is a copy of F and r is an ultrafilter, we have $F \in r$. That contradiction shows that (iii) holds. Since (ii) holds and there exist 2^{2^α} many ultrafilters on α [3, Theorem 3.6.11], there

exists a family $\mathcal{S} = \{S_p : p \in \Lambda\}$, with $|\Lambda| = 2^{2^\alpha}$, none of whose members are homeomorphic to each other. Thus, our family \mathcal{S} is the required one.

Remark 2. For the case $\alpha = \omega$ we can show the above proposition much easier, using subspaces $\omega \cup \{p\}$ of the Stone-Čech compactification $\beta\omega$ (cf. [13 and 16]).

Proofs of Theorem 1 and Corollary 1. The proofs for all the cases of Theorem 1 and Corollary 1 are derived simultaneously from the following observation, since every class (i+1) contains the preceding class (i) except $i = 5$, the class (3) contains the class (7), and the collection \mathcal{S} in Proposition 1 consists of σ -discrete L-spaces (cf. [4, 5, 7, 8, 9, 11, 13, 14, and 15]).

Let us denote by $[A]^\kappa$ the family consisting of all subsets of a set A having cardinality κ . If κ and λ are infinite and $\kappa \leq \lambda$, then $|[\lambda]^\kappa| = \lambda^\kappa$. It follows that if X is a stratifiable space (more generally, a paracompact σ -space) with $\text{nw}(X) \leq 2^\kappa$, where κ is infinite, then $|X| \leq |[2^\kappa]^\omega| = |(2^\kappa)^\omega| = 2^\kappa$; consequently, $|[X]^\kappa| \leq 2^\kappa$, and we see that X cannot contain copies of all the spaces in Proposition 1, with $\alpha = \kappa$.

Proof of Theorem 2. It suffices to show that there exists a family \mathcal{J} (respectively, \mathcal{U}) of cardinality 2^c , where $c = 2^\omega$, none of whose members are homeomorphic to each other, and which consists of countable Lasnev spaces (respectively, σ -compact, first-countable, separable, strongly zero-dimensional stratifiable spaces).

Let C be the Cantor set in the real line with usual Euclidean topology, and let $0 \in C$. Take a countable dense subset D of C , and let $\{x_n\} \subset C$ be a sequence converging to 0 . At first, we show the existence of \mathcal{J} . Put

$$F = \{0\} \times C, \text{ and } A = \{(x_n, d_k) : k \leq n, \text{ and } n \in \omega\},$$

where $D = \{d_k : k \in \omega\}$. Then, A is a countable discrete subset in C^2 with $\text{cl}(A) \supset F$. For every subset $T \subset F$ let

$$Y_T = T \cup A.$$

Take a point $p_T \notin A$, and let $X_T = \{p_T\} \cup A$. Define a function $\phi_T : Y_T \rightarrow X_T$ as $\phi_T(T) = p_T$ and $\phi_T(a) = a$ for every $a \in A$. We topologize the set X_T as

$$U \text{ is open in } X_T \text{ if } \phi_T^{-1}(U) \text{ is open in } Y_T.$$

By the definition of ϕ_T , ϕ_T is a closed map between Y_T and X_T . Thus, each X_T is a countable Lasnev space. By the proof of Proposition 1, one can show that there exists a family $\mathcal{J} = \{X_T : T \in \mathcal{J}\}$, with $|\mathcal{J}| = 2^C$, none of whose members are homeomorphic to each other. Next, we show the existence of \mathcal{U} . We shall modify the examples constructed in [2]. Let \mathcal{J} be a free ultrafilter on ω . Then, enumerate the family of all elements of \mathcal{J} as $\{F_s : s \in C\}$, and for each $s \in C$ and $m \in \omega$ choose a $q_s(m) \in D$ such that $0 < |s - q_s(m)| < 1/m$, and put

$$D_s = \{q_s(m) : m \in \omega\}, \text{ and } E_s = \{x_n : n \in F_s\}.$$

We topologize the set

$$\Delta_{\mathcal{J}} = (C \times \{0\}) \cup (D \times \{x_n : n \in \omega\})$$

as follows. Points of $D \times \{x_n : n \in \omega\}$ are isolated, and basic neighborhoods of a point $(s, 0) \in C \times \{0\}$ have the form

$$B_m(s) = \{(x, y) \in \Delta_{\mathcal{J}} : |s - x| < 1/m\} \setminus (D_s \times E_s \cup \{s\} \times \{x_n : n \in \omega\})$$

for $m \in \omega$. Then, it is known [2] that each Δ_J is first-countable σ -compact, strongly zero-dimensional, cosmic. We can show without difficulty that each Δ_J is stratifiable. Here, we show moreover that it admits a free L-structure.

Let $V = \{V_i : i \in \omega\}$ be a countable clopen base of C . Put

$$L_0 = C \times \{0\}, L_{2i} = V_i \times C, \text{ and } L_{2i+1} = \{u_i\},$$

where $\Delta_J \setminus L_0 = \{u_i : i \in \omega\}$. Put

$$\mathcal{U}_{L_0} = \{\{u\} : u \notin L_0\}, \text{ and } \mathcal{U}_{L_i} = \{\Delta_J \setminus L_i\}$$

for each $L_i \neq L_0$. Then, one can easily check that

$(\mathcal{L} = \{L_i : i \in \omega\}, \mathcal{U}_L : L \in \mathcal{L})$ is a free L-structure of Δ_J .

Again, by the proof of Proposition 1, we can show that there exists a family $\mathcal{U} = \{\Delta_J : J \in \Lambda\}$, with $|\Lambda| = 2^C$, none of whose members are homeomorphic to each other. That completes the proof of Theorem 2.

Professor Junnila [10] has kindly communicated to the author the following problem as well as the reference [12] of Theorem 3.

Problem. Does there exist a universal space for closed images of *countable metrizable* spaces?

Added in Proof. By the proof of Theorem 2 it follows that there are no universal spaces for separable Lasnev spaces.

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