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# TYPES OF STRATEGIES IN POINT-PICKING GAMES

by

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# TYPES OF STRATEGIES IN POINT-PICKING GAMES[1]

**Andrew J. Berner**

## 1. Introduction

The following ordinal game is defined in [B-J, Definition 1.1]:

*Definition* 1.1. If X is a topological space, and $\alpha$ is an ordinal, the game $\underline{G^D_\alpha(X)}$ is played in the following manner:

Two players take turns playing. A round consists of Player I choosing a non-empty open set $U \subset X$ and Player II choosing a point $x \in U$. A round is played for each ordinal less than $\alpha$. Player I wins the game if the set of points Player II played is dense. Otherwise, Player II wins.

The formal definitions of strategies can be found in [B-J, Definitions 1.2, 1.3, 1.6 and Lemma 1.7]. Informally, a strategy for a player is a function from partial plays of the game that tells a player what to play on her next turn; a winning strategy is, of course, one that guarantees a win if followed.

*Definition* 1.2 [B-J, Def. 1.4]. We write $\underline{\mathrm{I} \uparrow G^D_\alpha(X)}$ (read Player I wins $G^D_\alpha(X)$) if there is a winning strategy for Player I in $G^D_\alpha(X)$. $\underline{\mathrm{II} \uparrow G^D_\alpha(X)}$ is defined similarly. Also, we write $\underline{\mathrm{I} \not\uparrow G^D_\alpha(X)}$ (resp. $\underline{\mathrm{II} \not\uparrow G^D_\alpha(X)}$) if there is

[1]This work supported by funding from the O'Hara Chemical Sciences Institute of the University of Dallas.

no winning strategy for Player I (resp. Player II) in $G^D_\alpha(X)$. If I $\not\uparrow$ $G^D_\alpha(X)$ and II $\not\uparrow$ $G^D_\alpha(X)$, we say $G^D_\alpha(X)$ is neutral.

The main results concerning these games proved in [B-J] are:

(a) If no non-empty open subset of X has a countable $\pi$-base then I $\not\uparrow$ $G^D_\omega(X)$.

(b) If X is an HFD, then I $\uparrow$ $G^D_{\omega\cdot\omega}(X)$.

(c) ($\diamond \Rightarrow$) There is an HFD X such that II $\not\uparrow$ $G^D_\omega(X)$, and thus (a) shows that $G^D_\omega(X)$ is neutral.

(d) (CH $\Rightarrow$) There is an HFD X such that II $\uparrow$ $G^D_\omega(X)$. The construction of this example forms the basis for Section 3 of this paper.

*Definition* 1.3. If X is a topological space, then <u>ow(X)</u> = min($\{\alpha$: I $\uparrow$ $G^D_\alpha(X)\}$).

If Player I plays the elements of a $\pi$-base for X, Player II is forced to play a dense set. Thus ow(X) $\le$ $\pi$(X).

We will be interested in how much of the history of the game Player II needs to remember. Following the terminology of [G-T], a stationary strategy for a player is a strategy that depends only on the opponent's preceding move, and a Markov strategy for a player is one which depends only on the preceding move and the ordinal number of the round. More formally:

*Definition* 1.4. A *winning stationary strategy for Player II in* <u>$G^D_\alpha(X)$</u> is a function s: $\tau(X) \to X$ such that s(U) $\in$ U for every U $\in$ $\tau(X)$ (where $\tau(X)$ is the set of non-empty open subsets of X) and whenever $((U_\beta, s(U_\beta)): \beta < \alpha)$ is a play of the game, $\{s(U_\beta): \beta < \alpha\}$ is not dense. If Player II has a winning stationary strategy for $G_\alpha^D(X)$, we will write $\underline{II \uparrow_S G_\alpha^D(X)}$.

*Definition* 1.5. A *winning Markov strategy for Player II in* $\underline{G_\alpha^D(X)}$ is a function $s: \tau(X) \times \alpha \to X$ such that $s(U,\beta) \in U$ for every $(U,\beta)$ in $\tau(X) \times \alpha$, and whenever $(U_\beta, s(U_\beta,\beta): \beta < \alpha)$ is a play of the game, $\{s(U_\beta,\beta): \beta < \alpha\}$ is not dense. If Player II has a winning Markov strategy for $G_\alpha^D(X)$ we will write $\underline{II \uparrow_M G_\alpha^D(X)}$.

*Definition* 1.6. A *uniform strategy for Player II in* $\underline{G^D(X)}$ is a function $s: \tau(X)^{<ow(X)} \times \tau(X) \to X$ with $s((S,U)) \in U$ for all $(S,U) \in \tau(X)^{<ow(X)} \times \tau(X)$ (where $A^{<\alpha}$ is the set of all well ordered sequences of elements of A with order type less than $\alpha$, including the null sequence). A uniform strategy is winning if whenever $\alpha < ow(X)$ and $((U_\beta, x_\beta): \beta < \alpha)$ is a play for $G_\alpha^D(X)$ with $x_\beta = s(((U_\gamma: \gamma < \beta), U_\beta))$ then $\{x_\beta: \beta < \alpha\}$ is not dense (thus for each $\alpha < ow(X)$, $s|\tau(X)^{<\alpha} \times \tau(X)$ is a winning strategy for Player II in $G_\alpha^D(X)$). If Player II has a winning uniform strategy for $G^D(X)$, we will write $\underline{II \uparrow_U G^D(X)}$.

It should be noted that if $\alpha < ow(X)$, it does not follow that $II \uparrow G_\alpha^D(X)$; $G_\alpha^D(X)$ may be neutral.

Since Player II can elect to "forget" parts of the history of a game, $II \uparrow_S G_\alpha^D(X) \Rightarrow II \uparrow_M G_\alpha^D(X) \Rightarrow II \uparrow G_\alpha^D(X)$. In Section 2, we will show that the converses of these implications need not hold, and show that in some circumstances, the existence of a uniform strategy is equivalent to the existence of a stationary strategy.

In Section 3, CH will be used to construct a space X for which II $\uparrow G_\alpha^D(X)$ for every $\alpha < ow(X) = \pi(X) = \omega_1$, but for which II $\not\uparrow_M G_\alpha^D(X)$ for every countable $\alpha$.

If A is a set and $\alpha$ is an ordinal, $|A|$ will denote the cardinality of A, $[A]^{\leq\alpha}$ (resp. $[A]^{<\alpha}$, $[A]^\alpha$) will denote the collection of all subsets of A of cardinality at most $|\alpha|$ (resp. less than $|\alpha|$, equal to $|\alpha|$), and H(A) will denote the set of finite partial functions from A to 2, i.e., if $h \in H(A)$, then h maps a finite subset of A into $\{0,1\}$.

## 2. Relations Among Strategies

*Theorem* 2.1. II $\uparrow_S G_\alpha^D(X)$ *if and only if there is a dense set* $D \subset X$ *such that for every* $S \in [D]^{\leq\alpha}$, S *is not dense.*

*Proof.* Suppose $t: \tau(X) \to X$ is a winning stationary strategy for $G_\alpha^D(X)$. Let D be the image of t. Since $t(U) \in U$ for every $U \in \tau(X)$, D is dense. Suppose $S = \{x_\beta: \beta < \alpha\}$ is a subset of D with $|S| \leq |\alpha|$. For each $\beta < \alpha$, choose $U_\beta \in \tau(X)$ such that $t(U_\beta) = x_\beta$. Then $((U_\beta, x_\beta): \beta < \alpha)$ is a play of the game with Player II following t; since t is a winning strategy, S is not dense.

Conversely, suppose D is a dense subset of X such that no element of $[D]^{\leq\alpha}$ is dense. Choose $t: \tau(X) \to X$ such that $t(U) \in U \cap D$ for each $U \in \tau(X)$. On any play of $G_\alpha^D(X)$ where Player II follows t, Player II will play an element of $[D]^{\leq\alpha}$. Thus t is a winning stationary strategy.

Note: It is not assumed t is one-to-one, nor that Player II must play a "new" point on each round.

*Corollary* 2.2. *If* $|\alpha| = |\beta|$, *then* II $\uparrow_S G_\alpha^D(X)$ *if and only if* II $\uparrow_S G_\beta^D(X)$.

*Example* 2.3. A space with a stationary strategy.

Let $X = 2^{\omega_1}$. Let $D = \Sigma(2^{\omega_1}) = \{f \in X: \exists \alpha < \omega_1$ s.t. $f(\beta) = 0$ for all $\beta > \alpha\}$. D is dense in X and every countable subset of D is nowhere dense, so Theorem 2.1 shows that II $\uparrow_S G_\omega^D(X)$. In fact, II $\uparrow_U G^D(X)$ since if Player II always plays an element of D and follows the rules for $G_\alpha^D(X)$, $\alpha < \omega_1$, Player II can't lose (see Theorem 2.4 below; also see [B-J], Example 2.6).

*Theorem* 2.4. *If* $ow(X)$ *is a successor cardinal* $\kappa^+$, *then* II $\uparrow_S G_\kappa^D(X)$ *if and only if* II $\uparrow_U G^D(X)$.

*Proof.* Suppose $t: \tau(X) \to X$ is a winning stationary strategy for $G_\kappa^D(X)$. Let $t': \tau(X)^{<ow(X)} \times \tau(X) \to X$ be defined by $t'(((U_\beta: \beta < \alpha), U)) = t(U)$. Then t' is a winning uniform strategy.

Suppose, conversely, we have a winning uniform strategy $t': \tau(X)^{<ow(X)} \times \tau(X) \to X$ for Player II. We can think of t' as a strategy for Player II in $G_{ow(X)}^D(X)$, although it is not a winning strategy for that game. Since I $\uparrow G_{ow(X)}^D(X)$, there is a winning strategy $s: X^{<ow(X)} \to \tau(X)$ for Player I. Imagine the play $((U_\beta, x_\beta): \beta < ow(X))$ of $G_{ow(X)}^D(X)$ where Player I follows s and Player II follows t'. Let $D = \{x_\beta: \beta < ow(X)\}$. Since s is a winning strategy for Player I, D is dense in X. Suppose $S \in [D]^{\leq\kappa}$. Then there is $\alpha < ow(X) = \kappa^+$ such that $S \subset \{x_\beta: \beta < \alpha\}$. Since t' is a winning uniform strategy for Player II, $\{x_\beta: \beta < \alpha\}$ and, therefore, S are not dense. Theorem 2.1, then, shows $II \uparrow_S G^D_\kappa(X)$.

*Theorem* 2.5. *If* $II \uparrow_M G^D_\alpha(X)$ *and* $|\beta| = |\alpha|$ *then* $II \uparrow_M G^D_\beta(X)$.

*Proof*. Let $f: \beta \to \alpha$ be a bijection. Let $s: \tau(X) \times \alpha \to X$ be a winning Markov strategy for Player II in $G^D_\alpha(X)$. Define $s': \tau(X) \times \beta \to X$ by $s'((U,\gamma)) = s((U,f(\gamma)))$. Suppose $((U_\gamma,x_\gamma): \gamma < \beta)$ is a play for $G^D_\beta(X)$ with $x_\gamma = s'((U_\gamma,\gamma))$ for each $\gamma < \beta$. Then $((U_{f^\leftarrow(\delta)},x_{f^\leftarrow(\delta)}): \delta < \alpha)$ is a play for $G^D_\alpha(X)$ and $x_{f^\leftarrow(\delta)} = s'((U_{f^\leftarrow(\delta)},f^\leftarrow(\delta))) = s((U_{f^\leftarrow(\delta)},\delta))$. Thus since s is a winning Markov strategy, $\{x_{f^\leftarrow(\delta)}: \delta < \alpha\} = \{x_\gamma: \gamma < \beta\}$ is not dense, showing s' is a winning Markov strategy in $G^D_\beta(X)$.

*Theorem* 2.6. $II \uparrow_M G^D_\alpha(X)$ *if and only if there is a collection* $\{D_\beta: \beta < \alpha\}$ *of dense subsets of* X *such that if* $\{x_\beta: \beta < \alpha\}$ *is a set with* $x_\beta \in D_\beta$ *for all* $\beta < \alpha$, *then* $\{x_\beta: \beta < \alpha\}$ *is not dense in* X.

*Sketch of proof*. If $s: \tau(X) \times \alpha \to X$ is a winning Markov strategy, let $D_\beta = s(\tau(X) \times \{\beta\})$. Conversely, given the collection $\{D_\beta: \beta < \alpha\}$, define $s: \tau(X) \times \alpha \to X$ such that $s((U,\beta)) \in U \cap D_\beta$.

*Example* 2.7. A space with a Markov strategy, but no stationary or uniform strategy.

Let X be the countable dense subset of $2^{\mathbb{R}}$ constructed in [E, Theorem 2.3.7]. A point of X is specified by a finite collection of disjoint intervals with rational endpoints; the point is the function which is 0 on the union of the intervals and 1 off the union. For $x \in X$, define $m(x)$ to be the measure of $\{r \in \mathbb{R}: x(r) = 0\}$. Define $s: \tau(X) \times \omega \to X$ as follows: if $(U,i) \in \tau(X) \times \omega$, choose $x \in U$ such that $m(x) \leq 2^{-i}$ and let $s(U,i) = x$. Suppose $((U_i,x_i): i \in \omega)$ is a play for $G_{\omega}^{D}(X)$ with $w_i = s(U_i,i)$. Then $\Sigma_{i\in\omega} m(x_i) \leq 2$, thus there is $r \in \mathbb{R}$ such that $x_i(r) = 1$ for all $i$. Therefore $\{x_i: i \in \omega\}$ is not dense, showing that s is a winning Markov strategy in $G_{\omega}^{D}(X)$. Since II $\uparrow_M G_{\omega}^{D}(X)$, Theorem 2.5 shows that II $\uparrow_M G_{\alpha}^{D}(X)$ for all $\alpha < \omega_1$. By [B-J, Cor. 2.3a], I $\uparrow G_{\omega_1}^{D}(X)$ since X is countable, and so $ow(X) = \omega_1$ (note that $\pi(X) = c$, by the way). Suppose $D \subset X$ is dense. D itself is countable, so Theorem 2.1 shows II $\not\uparrow_S G_{\omega}^{D}(X)$ and thus Theorem 2.4 shows II $\not\uparrow_U G^{D}(X)$.

*Example* 2.8. A space with a uniform strategy but no Markov or stationary strategy.

Consider the HFD X constructed in [B-J, Theorem 3.1] under CH for which II $\uparrow G_{\omega}^{D}(X)$. By [B-J, Theorem 2.7], I $\uparrow G_{\omega\cdot\omega}^{D}(X)$. Therefore Theorem 2.5 shows II $\not\uparrow_M G_{\omega}^{D}(X)$, and thus II $\not\uparrow_S G_{\omega}^{D}(X)$. The strategy given in [B-J] for $G_{\omega}^{D}(X)$ had the stronger property that any set Player II played following the strategy in $G_{\omega}^{D}(X)$ was nowhere dense (discrete, even!). Since the finite union of nowhere dense sets is nowehre dense, Player II can repeat this strategy on rounds $\{\omega\cdot n+i: i \in \omega\}$ for fixed $n \in \omega$. Thus $ow(X) = \omega\cdot\omega$ and II $\uparrow_U G^{D}(X)$. Thus the hypothesis on $ow(X)$ in Theorem 2.4 cannot be eliminated.

**3. A Space With No Winning Markov or Uniform Strategies**

*Example* 3.1 (CH) A space X with $ow(X) = \pi(X) = \omega_1$ such that II $\uparrow G^D_\alpha(X)$ for every $\alpha < \omega_1$, but II $\not\uparrow_M G^D_\omega(X)$.

We will construct $X \subset 2^{\omega_1}$ in a manner similar to the construction in [B-J, Section 3]. The new idea in this paper is that we will define a collection of infinite subsets to be called anti-strategic sets, each of which will be made dense in a tail. Note, though, that X cannot be an HFD since I $\not\uparrow G^D_{\omega\cdot\omega}(X)$ ([B-J, Theorem 2.7]).

As in the standard inductive construction of an HFD, at stage $\alpha < \omega_1$ we will define functions $f_{\beta\alpha}: \alpha + 1 \to 2$ for each $\beta < \omega_1$ that extend those defined at earlier stages. X will then be $\{f_\beta = \cup\{f_{\beta\alpha}: \alpha < \omega_1\}: \beta < \omega_1\}$ (actually, for notational convenience, we will define X to be homeomorphic to this). To do this, we will have, at stage $\alpha$, a countable collection $Z(\alpha)$ of countably infinite subsets of $\omega_1$. We find a set $B(\alpha) \subset \omega_1$ such that for each $A \in Z(\alpha)$, both $A \cap B(\alpha)$ and $A - B(\alpha)$ are infinite. We will say $B(\alpha)$ splits $Z(\alpha)$.

We will pre-define some values of the $f_\beta$'s by defining functions $\{p_\beta: \beta < \omega_1\}$ with $\mathrm{dom}(p_\beta) \subset \omega_1$ and range $(p_\beta) \subset 2$; we will assure that $p_\beta \subset f_\beta$ for each $\beta < \omega_1$.

To begin, let $\mathcal{S} = [\omega_1]^{\underline{<\omega}}$. Let $\{C_S: S \in \mathcal{S}\}$ be a partition of $\omega_1$ into uncountable, pairwise disjoint subsets such that if $\alpha \in C_S$ then $\alpha > \sup(S)$ (let $0 \in C_\emptyset$). Further, let $i: C_\emptyset \to \omega_1$ be a function such that $i^{\leftarrow}(\alpha)$ is uncountable for each $\alpha < \omega_1$. For $S \subset \omega_1$, let $ot(S)$ be the order type of S. Let $\pi: \omega_1 \to \mathcal{S}$ be defined by $\pi(\alpha) = S$ if $\alpha \in C_S$.

We say a subset $S \subset \omega_1$ has the *strategy property* if for every $\alpha \in S$, $\pi(\alpha) = \alpha \cap S$. Note that initial segments of $S$ also will have the strategy property. We say $S \in \mathcal{S}$ is a *strategic set* if $S$ is infinite, $S$ has the strategy property and $i(\min(S)) = ot(S)$ (note that since $S$ has the strategy property, $\min(S) \in C_\emptyset$). A set $S \in \mathcal{S}$ is called *anti-strategic* if $|S| = \omega$ and $S \cap S'$ is finite for every strategic set $S'$.

Index the anti-strategic sets as $\{A_\alpha : \alpha \in I\}$ for some index set $I \subset \omega_1$ such that $A_\alpha \subset \alpha$ for each $\alpha \in I$. Index the strategic sets as $\{S_\lambda : \lambda \in L\}$ for some set of limit ordinals $L \subset \omega_1$, with $S_\lambda \subset \lambda$ for each $\lambda \in L$. For each $\lambda \in L$ and $\beta \in S_\lambda$ define a function $h^\lambda_\beta \in 2^{\lambda+\omega-\lambda}$ as follows: reindex $S_\lambda$ as $\{\beta_i : i \in \omega\}$. Then let $h^\lambda_{\beta_i}(\lambda + j) = \begin{cases} 0 \text{ if } j = i \\ 1 \text{ if } j \neq i \end{cases}$.

For each $\beta < \omega_1$ choose a function $g_\beta \in 2^\beta$ such that for each $h \in H(\omega_1)$ and $S \in \mathcal{S}$, there is $\beta \in C_S$ such that $h \subset g_\beta$ and also for each $h \in H(\omega_1)$ and each $\alpha < \omega_1$ there is $\beta \in C_\emptyset$ such that $h \subset g_\beta$ and $i(\beta) = \alpha$. Note that if $\beta \in S_\lambda \cap S_{\lambda'}$, then $\mathrm{dom}(h^\lambda_\beta) \cap \mathrm{dom}(g_\beta) = \emptyset$ and $\mathrm{dom}(h^\lambda_\beta) \cap \mathrm{dom}(h^{\lambda'}_\beta) = \emptyset$.

We can now define $p_\beta$ for $\beta < \omega_1$:

$$p_\beta = \cup\{h^\lambda_\beta : \beta \in S_\lambda\} \cup g_\beta$$

This will guarantee that in the space $X$ we construct, $\{f_\beta : \beta \in C_S\}$ is dense for each $S \in \mathcal{S}$ and $\{f_\beta : \beta \in C_\emptyset$ and $i(\beta) = \alpha\}$ is dense for each $\alpha < \omega_1$. Also, it will guarantee that if $S$ is a strategic set, $\{f_\beta : \beta \in S\}$ is discrete and hence nowhere dense.

At long last, we are ready for the induction! Suppose we are at stage $\alpha$. We need to define functions $\{f_{\beta\alpha} : \beta < \omega_1\}$

and a countable collection $Z(\alpha)$ of anti-strategic sets such that if $A \in Z(\alpha)$ then $A \subset \alpha$, and we assume we have done this for all $\gamma < \alpha$. First define:

$$Z_1(\alpha) = \begin{cases} \cup\{Z(\gamma): \gamma < \alpha\} & \text{if } \alpha \notin I \\ \cup\{Z(\gamma): \gamma < \alpha\} \cup \{A_\alpha\} & \text{if } \alpha \in I \end{cases}$$

If $\alpha \neq \lambda + i$ for any $\lambda \in L$ and $i < \omega$, let $Z_2(\alpha) = Z_1(\alpha)$. If $\alpha = \lambda + i$ for some $\lambda \in L$ and $i < \omega$, let

$$Z_2(\alpha) = \{A - S_\lambda: A \in Z_1(\alpha)\}.$$

Note that the definition of anti-strategic set guarantees that elements of $Z_2(\alpha)$ are infinite (and anti-strategic). Also, if $\beta \in A \in Z_2(\alpha)$, then $\alpha \notin \mathrm{dom}(p_\beta)$.

Let $B(\alpha) \subset \omega_1$ be a set that splits $Z_2(\alpha)$, i.e. for each $A \in Z_2(\alpha)$, $A - B(\alpha)$ and $A \cap B(\alpha)$ are both infinite.

For all $\beta < \omega_1$, define $f_{\beta\alpha}: \alpha + 1 \to 2$ to extend $p_\beta|(\alpha + 1)$ and $f_{\beta\gamma}$ for all $\gamma < \alpha$ such that if $\beta \in A \in Z_2(\alpha)$ then

$$f_{\beta\alpha}(\alpha) = \begin{cases} 1 \text{ if } \beta \in B(\alpha) \\ 0 \text{ if } \beta \notin B(\alpha) \end{cases}$$

Finally, let

$$Z(\alpha) = Z_1(\alpha) \cup \{A \cap B(\alpha): A \in Z_2(\alpha)\} \cup \{A - B(\alpha): A \in Z_2(\alpha)\}.$$

This completes stage $\alpha$ of the induction.

Let $f_\beta = \cup\{f_{\beta\alpha}: \alpha < \omega_1\}$ for each $\beta < \omega_1$. It will be convenient to identify $f_\beta$ with its index $\beta$. More formally, we can define a topology $\tau$ on $\omega_1$ such that the function $f: \omega_1 \to \{f_\beta: \beta < \omega_1\} \subset 2^{\omega_1}$ which takes $\beta$ to $f_\beta$ is a homeomorphism; we then let $X = (\omega_1, \tau)$.

To see that $\mathrm{II} \uparrow G^D_\alpha(X)$ for $\alpha < \omega_1$, recall that since $f_\beta$ extends $g_\beta$ for each $\beta < \omega_1$, $C_S$ is dense in $X$ for each

$S \in \mathcal{S}$ and $i^{\leftarrow}(\alpha) \subset C_{\emptyset}$ is dense for each $\alpha < \omega_1$. Therefore, given $\alpha < \omega_1$, we can define a function $s_\alpha$: $[X]^{<\alpha} \times \tau(X) \to X$ such that $s_\alpha(((S,U))) \in U \cap C_S$ for all $(S,U) \in [X]^{<\alpha} \times \tau(X)$ and $i(s_\alpha(\emptyset,U)) = \alpha$ for all $U \in \tau(X)$. If $\{(U_\beta,\gamma_\beta): \beta < \alpha\}$ is a play in $G_\alpha^D(X)$ with $\gamma_\beta = s((\{\gamma_\delta: \delta < \beta\},U_\beta))$, then $\{\gamma_\beta: \beta < \alpha\}$ is a strategic set, thus not dense in X. Therefore, $s_\alpha$ is a winning strategy for Player II.

To show II $\not\uparrow_M G_\omega^D(X)$ we will need a lemma, which will be proved later.

*Lemma* 3.2. *If* O *is a non-empty open subset of* X *and* $\{D_i: i < \omega\}$ *is a countable collection of dense subsets of* O*, then there is an infinite subset* $J \subset \omega$ *and an anti-strategic set* $\{\beta_i: i \in J\}$ *such that* $\beta_i \in D_i$ *for each* $i \in J$.

We will use this lemma in conjunction with Theorem 2.6. Suppose we have a countable collection of dense subsets of X which we can index as $\{D_{j,k,i}: j,k,i < \omega\}$. We can construct a dense set $\{\beta_{j,k,i}: j,k,i < \omega\}$ with $\beta_{j,k,i} \in D_{j,k,i}$ for all $j,k,i < \omega$ as follows. If $h \in H(\omega_1)$, let $\langle h \rangle = \{\beta \in X$: $f_\beta$ extends $h\}$. Thus $\{\langle h \rangle: h \in H(\omega_1)\}$ is a basis for X. We will define a sequence of countable ordinals $(\alpha_j: j \in \omega)$ and the points $\{\beta_{j,k,i}: j,k,i < \omega\}$ by induction on j. First, let $\alpha_0 = \omega$. Continuing inductively, suppose we have defined $\alpha_j$. Index $H(\alpha_j)$ as $\{h_{j,k}: k < \omega\}$. For each $k < \omega$, apply Lemma 3.2 to $\{D_{j,k,i} \cap \langle h_{j,k} \rangle: i < \omega\}$. We get a set $J_{j,k} \subset \omega$ and an anti-strategic set $\{\beta_{j,k,i}: i \in J_{j,k}\}$ with $\beta_{j,k,i} \in D_{j,k,i} \cap \langle h_{j,k} \rangle$. For $i \notin J_{j,k}$, choose $\beta_{j,k,i} \in D_{j,k,i}$. When we constructed X, we indexed the anti-strategic sets, so $\{\beta_{j,k,i}: i \in J_{j,k}\} = A_{\alpha(j,k)}$ for some $\alpha(j,k) < \omega_1$. The construction of X guaranteed that if $h \in H(\omega_1 - \alpha(j,k))$ then $A_{\alpha(j,k)} \cap \langle h \rangle \neq \emptyset$. Let $\alpha_{j+1} = \sup(\{\alpha(j,k): k < \omega\})$. Note that if $h \in H(\alpha_j \cup (\omega_1 - \alpha_{j+1}))$ then $\{\beta_{j,k,i}: k,i < \omega\} \cap \langle h \rangle \neq \emptyset$.

Let $\alpha = \sup(\{\alpha_j: j < \omega\})$. Suppose $h \in H(\omega_1)$. Then $h = h_1 \cup h_2$, where $h_1 \in H(\alpha)$ and $h_2 \in H(\omega_1 - \alpha)$. There is $j < \omega$ such that $h_1 \in H(\alpha_j)$. Then $h_2 \in H(\omega_1 - \alpha_{j+1})$ so $h \in H(\alpha_j \cup (\omega_1 - \alpha_{j+1}))$. Thus there is $\beta_{j,k,i} \in \langle h \rangle$ for some $k,i < \omega$. This shows that $\{\beta_{j,k,i}: j,k,i < \omega\}$ is dense. Theorem 2.6, then, tells us II $\not\uparrow_M G_\omega^D(X)$.

Before we can prove Lemma 3.2, we need to further examine the strategic sets. We call a set $S \in \mathcal{S}$ *pre-strategic* if there is a strategic set S' such that $S \subset S'$. Note that pre-strategic sets are nowhere dense in X. We call an infinite set $S \in \mathcal{S}$ an *initial strategic segment* if S has the strategy property and $ot(S) \leq i(\min(S))$. If S is an initial strategic segment, then S can be extended to a strategic set. For $S \subset \omega_1$ let $\Pi(S) = \cup_{\alpha \in S}(\pi(\alpha) \cup \{\alpha\})$. Then, if S is an initial strategic segment, $\Pi(S) = S$, and, for infinite S, S is pre-strategic if and only if $\Pi(S)$ is an initial strategic segment.

*Lemma* 3.3. *Suppose* $\{S_\alpha: \alpha \in J\}$ *is a chain of pre-strategic sets. Then* $\cup\{S_\alpha: \alpha \in J\}$ *is pre-strategic.*

*Proof*. Let $S = \Pi(\cup\{S_\alpha: \alpha \in J\})$. Suppose $\beta \in S$. Then for some $\alpha \in J$ and $\beta' \in S_\alpha$, $\beta \in \pi(\beta') \cup \{\beta'\}$. There is a strategic set S' containing $S_\alpha$. Since $\beta' \in S'$, $S' \cap \beta' = \pi(\beta')$. Thus $\beta \in S'$, so $S' \cap \beta = \pi(\beta)$. Since $\beta \leq \beta'$,

$S' \cap \beta \subset S' \cap \beta'$, thus $\pi(\beta) \subset \pi(\beta') \subset S$. Now suppose further that $\gamma \in S \cap \beta$. There is $\alpha' \in J$ and $\beta'' \in S_{\alpha'}$ such that $\gamma \in \pi(\beta'') \cup \{\beta''\}$. But then for some $\delta \in J$, $\{\beta',\beta''\} \subset S_\delta$ and there is a strategic set $S''$ containing $S_\delta$. Since $\gamma \in \pi(\beta'') \cup \{\beta''\}$ and $S''$ is strategic, $\gamma \in S''$. Likewise, since $\beta \in \pi(\beta') \cup \{\beta'\}$, $\beta \in S''$. Thus $\gamma \in S'' \cap \beta$ so $\gamma \in \pi(\beta)$. Thus $S \cap \beta = \pi(\beta)$, i.e. $S$ has the strategy property. Suppose $ot(S) > i(\min(S))$. Let $\beta$ be the element of $S$ such that $ot(\pi(\beta)) = i(\min(S))$. For some $\alpha \in J$, $\beta \in \Pi(S_\alpha)$, so for some strategic $S'$, $\beta \in S' \supset S_\alpha$. But then, since $S' \cap \beta = S \cap \beta = \pi(\beta)$, $\min(S') = \min(S)$. $\pi(\beta) \cup \{\beta\} \subset S'$ and $ot(\pi(\beta) \cup \{\beta\}) = ot(\pi(\beta)) + 1 = i(\min(S)) + 1 > i(\min(S'))$, contradicting the fact that $S'$ is strategic. (Note: This is the only place where the condition on the order type of strategic sets matters!) Therefore, $ot(S) \leq i(\min(S))$, so $S$ is an initial strategic segment. This shows that $\cup\{S_\alpha: \alpha \in J\}$ is pre-strategic.

*Proof of Lemma* 3.2. Suppose $O \in \tau(X)$ and $\{D_i: i < \omega\}$ is a collection of dense subsets of $O$. For each $i < \omega$ we will inductively define $J_i \subset \omega$ with $|J_i| = \omega$ and $J_{i+1} \subset J_i$, pre-strategic sets $M_i'$ and $M_i$ such that $M_i \subset \cup\{D_j: j \in J_i\}$ and $M_i' \subset M_i \cap \cup\{D_j: j \in J_i - J_{i+1}\}$ and a strategic set $S_i \supset M_i$. Let $J_0 = \omega$. Suppose we have defined $J_i$ for $i \leq k$ and $M_i$, $M_i'$ and $S_i$ for $i < k$. Since each $S_i$ is strategic and thus nowhere dense, $D_j - \cup\{S_i: i < k\}$ is dense in $O$ (thus non-empty!) for each $j$. Lemma 3.3 and Zorn's Lemma let us choose a maximal pre-strategic set $M_k \subset \cup\{D_j - \cup\{S_i: i < k\}: j \in J_k\}$. Let $S_k$ be a strategic set containing

$M_k$. Choose a cofinal set $M''_k \subset M_k$ with $ot(M''_k) \leq \omega$. If there is $j \in J_k$ such that $M''_k \cap D_j$ is cofinal in $M''_k$, let $M'_k = M''_k \cap D_j$ and $J_{k+1} = J_k - \{j\}$. Otherwise choose $M'''_k = \{\alpha_i: i < \omega\}$ to be a cofinal subset of $M''_k$ and an increasing sequence $(j(i): i < \omega)$ from $J_k$ such that $\alpha_i \in D_{j(i)}$. Let $M'_k = \{\alpha_{2i}: i < \omega\}$ and $J_{k+1} = J_k - \{j(2i): i < \omega\}$. This completes the inductive definitions. Note that if $M_k$ is finite at any stage $k < \omega$, we can let $J = J_k$ and choose $\beta_j \in D_j - \cup\{S_i: i < k\}$ for each $j \in J$ to satisfy the lemma. So assume $M_k$ is infinite for each $k < \omega$.

For each $i < \omega$, let $m_i = \min(M'_i)$. If $\{m_i: i < \omega\}$ is anti-strategic, then for each i we can choose $j(i) < \omega$ such that $m_i \in D_{j(i)}$ and $j(i) \in J_i - J_{i+1}$. Then $\{m_i: i < \omega\}$ and $J = \{j(i): i < \omega\}$ satisfy the conclusion of Lemma 3.2.

So suppose there is a strategic set S such that $|S \cap \{m_i: i < \omega\}| = \omega$. Choose a subset $\{m_{i(j)}: j < \omega\} \subset S \cap \{m_i: i < \omega\}$ such that if $j < k$ then $i(j) < i(k)$ and (the ordinal!) $m_{i(j)} < m_{i(k)}$. Since S has the strategy property, (*) $m_{i(j)} \in \pi(m_{i(k)})$ for $j < k$. Suppose $j < \omega$. If $m_{i(j+1)} < \sup(M'_{i(j)})$, let $q_j = \min(\{q \in M'_{i(j)}: q > m_{i(j+1)}\})$. Since $m_{i(j+1)} \notin S_{i(j)}$ but $q_j \in S_{i(j)}$ and $S_{i(j)}$ has the strategy property, $m_{i(j+1)} \notin \pi(q_j)$. If, on the other hand, $m_{i(j+1)} \geq \sup(M'_{i(j)})$, it cannot be the case that $M'_{i(j)} \subset \pi(m_{i(j+1)})$ for if that were the case then $M'_{i(j)} \cup \{m_{i(j+1)}\} \subset S_{i(j+1)}$. But $M_{i(j)} \subset \Pi(M'_{i(j)})$ since $M'_{i(j)}$ is cofinal in the pre-strategic set $M_{i(j)}$. This would imply that $M_{i(j)} \cup \{m_{i(j+1)}\} \subset S_{i(j+1)}$, contradicting the maximality of $M_{i(j)}$. So there must be $q_j \in M'_{i(j)} - \pi(m_{i(j+1)})$. Either way, we have found $q_j \in M'_{i(j)}$ such that

$\{q_j, m_{i(j+1)}\}$ is not pre-strategic. Pick $k(j) \in J_{i(j)} - J_{i(j)+1}$ such that $q_j \in D_{k(j)}$. If S' is a strategic set, then S' contains at most one element of $\{q_j: j < \omega\}$, for suppose, on the contrary, $\{q_j, q_{j'}\} \subset S'$, with $j < j'$. Since $\{m_{i(j')}, q_{j'}\} \subset S_{i(j')}$ and $m_{i(j')} \leq q_{j'}$, $m_{i(j')} \in \pi(q_{j'}) \cup \{q_{j'}\}$, thus $m_{i(j')} \in S'$. But since $j + 1 \leq j'$, (*) implies that $m_{i(j+1)} \in \pi(m_{i(j')}) \cup \{m_{i(j')}\}$. Thus $m_{i(j+1)} \in S'$, which contradicts the choice of $q_j$. Therefore $\{q_j: j < \omega\}$ is anti-strategic, and $\{q_j: j < \omega\}$ and $J = \{k(j): j < \omega\}$ satisfy the conclusion of Lemma 3.2.

*Remark* 3.4. We noted in the proof of Lemma 3.3 the only place where the condition on the order types of strategic sets plays a crucial role. Since we are aiming for a space without winning Markov strategies, we know from Theorem 2.4 that we must ensure that II $\not\uparrow_U$ $G^D(X)$. Eric van Douwen pointed out that the condition on order types of strategic sets is the only reason why $\alpha$ must be mentioned in a strategy for $G_\alpha^D(X)$. Indeed, there are dense subsets D of X with the strategy property, but not all initial segments of D are pre-strategic! While the condition on order types was not necessary for the inductive construction of X, had it been omitted, Player II would have had a uniform strategy for the resulting space.

## 4. Open Problems

(a) Is there a neutral game in ZFC?

(b) Can CH be eliminated from Example 3.1?

(c) Is there a space X such that $\omega \cdot \omega < ow(X) < \omega_1$?

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