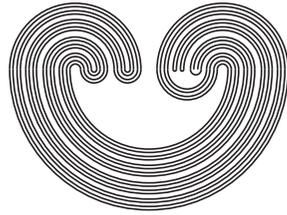


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## EXTENSIONS OF COMPACT GROUP ACTIONS

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## EXTENSIONS OF COMPACT GROUP ACTIONS

Beverly L. Brechner

### 1. Introduction

Compact actions on  $S^2$  were characterized by Kerekjarto in [K], where it is shown that the only regular homeomorphisms of  $S^2$  are the (topological) rotations and reflections. Using this result, we observe (Theorem 2.3) that if  $H$  is an orientation preserving compact group action on a nondegenerate subcontinuum of  $S^2$  such that  $H$  is extendable to a compact action  $G$  on  $S^2$ , then  $G$  and  $H$  are isomorphic. (Note that this result does not necessarily hold when  $H$  is not orientation preserving. For example, take  $H$  to be the group generated by the reflection through the equator.) While Theorem 2.3 is an easy consequence of [K], it is remarkable in light of our Example 2.4, which explains the necessity of the hypothesis that the extension  $G$  be compact. (Or perhaps the theorem is natural, while the example is remarkable.)

Recall that a homeomorphism is *regular* iff its full family of iterates is equicontinuous.

### 2. Results

2.1 *Theorem.* Let  $X$  be a nondegenerate continuum in  $S^2$  (or  $B^2$ ), and let  $K$  be an orientation preserving, compact group action on  $S^2$  ( $B^2$ ) such that each element of  $K$  is the identity on  $X$ . Then  $K$  is the identity.

*Proof.* Kerekjarto [K] (see also [E]) has shown that the only orientation preserving, regular homeomorphisms of  $S^2$  are the rotations. (See also [B,2] and [R].) Now every element of a compact group action is regular. Thus, since each non-identity rotation on  $S^2$  has exactly a pair of points as its fixed point set, we see that  $K$  must be the identity.

If  $K$  acts on  $B^2$ , then it can be extended to an isomorphic action on  $S^2$  (simply reflect the action), and by the above, it must again be the identity.

2.2 *Remark.* This theorem does not hold for  $S^3$ : Let  $S^3 = T_1 \cup T_2$ , where each  $T_i$  is a torus, and where  $T_1$  and  $T_2$  are identified by the homeomorphism  $g$  along their boundaries, where  $g$  identifies meridians on  $T_1$  with longitudes on  $T_2$ . Then let  $h: S^3 \rightarrow S^3$  be the period  $n$  homeomorphism that naturally extends a period  $n$  meridional rotation on  $T_1$ . Then the fixed point set of  $h$  is the center circle,  $S$ , of  $T_1$ . Let  $K = \{h, h^2, \dots, h^n\}$ , and let  $X = S$ . Then  $K$  is an orientation preserving extension of the identity on  $X$ , but  $K$  is not the identity.

2.3 *Theorem.* Let  $X$  be a nondegenerate continuum in  $S^2$  ( $B^2$ ), and let  $H$  be a compact group action on  $X$ . Suppose that  $H$  can be extended to an orientation preserving compact group action  $G$  on  $S^2$  ( $B^2$ ). Then  $G$  and  $H$  are isomorphic.

*Proof.* Let  $f: G \rightarrow H$  be the natural continuous homomorphism; that is,  $f(g) = g|X$ . We shall show that  $f$  is 1-1. Let  $K = f^{-1}(e)$ . Then  $K$  is a compact subgroup of  $G$ . By

Theorem 2.1,  $K$  must be the identity. It follows that  $f$  is 1-1, and therefore an isomorphism.

2.4 *Example.* The following example shows that Theorem 2.3 does not hold in general; that is, restricted compact groups may not be isomorphic to *any* group extension.

Let  $X$  be the outer boundary simple closed curve of the Sierpinski curve  $Y \subseteq E^2 \subseteq S^2$ , and let  $H$  be a circle group of homeomorphisms on  $X$ . The proof of the main theorem (3) of [W] shows that each element of  $H$  can be extended to a homeomorphism of  $Y$  onto itself. Choose such an extension for each  $h \in H$ , and let  $G$  be the group generated by these extensions. Then

(1)  $G$  is totally disconnected (see Theorem 1.2 of R. D. Anderson of [B,1]), but the orbit of a point in  $X$  is a simple closed curve.

(2) By (1),  $G'$  and  $H$  cannot be isomorphic, for *any* group extension  $G'$  of  $H$ .

(3)  $G$  can be extended, in a natural way, to a group action  $\tilde{G}$  on  $S^2$ , with  $\tilde{G}$  isomorphic to  $G$ , since each complementary domain of  $X$  in  $S^2$  is a disk.

(4)  $G'$  cannot be compact, for any group extension  $G'$  of  $H$ . For if it were, by (3) and Theorem 2.3,  $G$  would be a circle group. But this contradicts (1).

(5) It follows from (4) and Theorem 2.3, that every compact subgroup of  $G$  is finite.

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