TOPOLOGY PROCEEDINGS

Volume 10, 1985

Pages 33–46

http://topology.auburn.edu/tp/

PERFECT RIMCOMPACT IMAGES OF ALMOST RIMCOMPACT SPACES

by

Beverly Diamond

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

PERFECT RIMCOMPACT IMAGES OF ALMOST RIMCOMPACT SPACES

Beverly Diamond

1. Introduction and Known Results

A 0-*space* is a completely regular Hausdorff space possessing a compactification with zero-dimensional remainder. Recall that a Hausdorff space is *rimcompact* if it possesses a base of open sets with compact boundaries. A rimcompact space X has a compactification FX in which each point of FX has a base of open sets whose boundaries lie in X (see, for example [Sk]). Thus a rimcompact space is a 0-space; the converse is not true ([Sk]). In the process of internally characterizing the class of 0-spaces ([Di3]), a theory was developed for an intermediate class of spaces. A space X is *almost rimcompact* if and only if X possesses a compactification KX in which each point of $KX\setminus X$ has a base of open sets in KX whose boundaries lie in X (see [Di1], [Di2] for the internal characterization and description of properties of almost rimcompact spaces).

An open set U of X is π-*open* in X if $bd_X U$ is compact. The sets A,B of X are π-*separated* in X if there is a π-open set U of X such that $A \subseteq U \subseteq c\ell_X U \subseteq X\setminus B$. One of the defining conditions of almost rimcompactness is the following: for each $x \in X$, there is a compact connected set K_x of X such that if F is closed in X and $F \cap K_x = \phi$, then $\{x\}$ and F are π-separated.

In fact, if X is almost rimcompact, and for $x \in X$ we define $G_x = \cap\{c\ell_{\beta X}U\colon U \text{ is } \pi\text{-open in } X,\ x \in U\}$, then G_x has the properties of K_x listed above ([Di1]). In addition, G_x is precisely the set of points of X from which x cannot be π-separated. (Notice that $G_x = \{x\}$ for each $x \in X$ if and only if X is rimcompact.) The compactness of G_x, combined with this last property, makes the following a natural question: for an almost rimcompact space X, is there a perfect map from X onto a rimcompact space Y? In particular, can we collapse G_x to a point, for each $x \in X$, and obtain a rimcompact quotient space?

We mention two related facts. First, if Z is any zero-dimensional, non-strongly zero-dimensional space, then there are spaces X, Y such that X is almost rimcompact, nonrimcompact, Y is rimcompact, $\beta X\setminus X \approx Z \approx \beta Y\setminus Y$, and X can be mapped onto Y by a perfect monotone map ([Di1]). (Recall that a map $f\colon X \to Y$ is monotone if $f^{\leftarrow}(y)$ is connected for each $y \in Y$.) Secondly, although the perfect preimage of a rimcompact space need not be a 0-space, if the perfect preimage is a 0-space then that preimage is almost rimcompact (4.1 and 4.3 of [Di2]).

In this paper we show that an almost rimcompact space need not be the perfect preimage of a rimcompact space (3.1) and investigate the existence of perfect maps onto rimcompact spaces. In particular, the collection $\mathcal{G} = \{G_x\colon x \in X\}$ (where G_x is as defined above) need not be a partition of X. If $\mathcal{G}$ is a partition of X, then $X/\mathcal{G}$ is rimcompact (2.8). However, even if $\mathcal{G}$ is not a partition of X, X may be the perfect preimage of a rimcompact space (3.2). If an almost rimcompact space X is the perfect preimage of a rimcompact space then 1) X is the perfect monotone preimage of a rimcompact space (2.3) and 2) there is a maximal rimcompact perfect image of X, where the perfect map will of necessity be monotone (2.5).

In the remainder of this section, we present some terminology and known results. All spaces, unless constructed, are assumed to be completely regular and Hausdorff. A function $f: X \to Y$ is *closed* if whenever F is a closed subset of X, $f[F]$ is closed in Y. The symbol ω_α is used to denote the αth cardinal.

In the following, Rim(X) will denote the set of points of X possessing a neighborhood base of π-open sets, L(X) will denote the locally compact part of X, and R(X) will denote the residue of X, that is, the non-locally compact part of X.

The following is 1.2 of [HI].

1.1 *Proposition. Suppose that* $f: X \to Y$ *is perfect. Then* $f^{\leftarrow}[K]$ *is compact for each compact subset* K *of* Y.

It follows immediately from 1.1 that if X is the perfect preimage of a rimcompact space Y via the map f, then $\{f^{\leftarrow}(y): y \in Y\}$ is an upper semicontinuous decomposition of X into compact sets, each having a neighborhood base of π-open sets.

2. The Main Results

We begin this section by presenting some consequences of an almost rimcompact space X being the perfect preimage of a rimcompact space X (2.1-2.5) and continue with some results on specific types of decompositions in an attempt to build rimcompact images (2.6-2.8).

The first result states that when collapsing subsets of X in an attempt to obtain a rimcompact quotient space, we need not collapse any sets in the interior of the rimcompact part of X. It is well known that L(X) is open in X; an almost rimcompact space X can be constructed so that Rim(X) is not open in X.

2.1 *Theorem. Suppose that* $f: X \to Y$ *is perfect, where* Y *is rimcompact, and that* U *is any open subset of* $\mathrm{int}_X \mathrm{Rim}(X)$. *Then there is a rimcompact space* Z, *and perfect maps* $g: X \to Z$, $h: Z \to Y$ *such that* $h \circ g = f$, $g^{\leftarrow}[g[U]] = U$, *and* $g|_U$ *is a homeomorphism.*

Proof. Suppose that $f: X \to Y$ is perfect. Let $\mathcal{D}$ denote the decomposition of X consisting of $\{\{x\}: x \in U\} \cup \{f^{\leftarrow}[f(p)]\setminus U: p \in X\setminus U\}$, and g the quotient map from X onto $Z = X/\mathcal{D}$. Notice that $g^{\leftarrow}(z)$ is compact for each $z \in Z$, and that $g^{\leftarrow}[g[U]] = U$. It is straightforward to verify that $\mathcal{D}$ is an upper semicontinuous decomposition of X, hence that g is perfect. If we define $h: Z \to Y$ as follows: $h(z) = f[g^{\leftarrow}(z)]$; then h is well-defined, $f = h \circ g$ and h is perfect.

We wish to show that if Y is rimcompact, then Z is rimcompact. Suppose $z \in Z$ and that $z \in V$ open in Z. If $z = \{x\}$ for $x \in U$, then x has a base of π-open sets of X which are contained in U, hence z has a base of π-open sets in Z. Suppose $z = f^{\leftarrow}[f(p)]\backslash U$ for $p \in X\backslash U$. Then $h^{\leftarrow}[h(z)] = \{z\} \cup \{\{x'\}: x' \in U, f(x') = f(p)\}$. Let $K = h^{\leftarrow}[h(z)]\backslash V$. The set K is compact and contained in the open set $g[U]$. Since each element of K has a base of π-open sets of Z contained in $g[U]$, using compactness we can choose W_1 to be a π-open set of Z such that $K \subseteq W_1 \subseteq g[U]$. Then $h^{\leftarrow}[h(z)] \subseteq V \cup W_1$. Since h is closed, there is a π-open set A of Y such that $h^{\leftarrow}[h(z)] \subseteq h^{\leftarrow}[A] \subseteq V \cup W_1$. It follows from 1.1 that $h^{\leftarrow}[A]$ is π-open in Z. Then $h^{\leftarrow}[A]\backslash c\ell_Z W_1$ is π-open in Z, and $z \in h^{\leftarrow}[A]\backslash c\ell_Z W \subseteq V$. Therefore Z is rimcompact.

As previously mentioned, the perfect preimage of a rimcompact space need not be rimcompact, thus in the preceding proof some verification that Z is rimcompact is required.

As a special case of 2.1 we have the following: if $f: X \to Y$ is perfect, where Y is rimcompact, then there is a rimcompact space Z and a perfect map $g: X \to Z$ such that $g|_{L(X)}$ is a homeomorphism.

The details of the following construction appear in [Po]. Let $f: X \to Y$ be perfect. For each $y \in Y$, $f^{\leftarrow}(y)$ is compact, thus each connected component of $f^{\leftarrow}(y)$ is a compact quasi-component of $f^{\leftarrow}(y)$. Let $\mathcal{D}$ denote the decomposition of X whose elements are the connected components of the sets $f^{\leftarrow}(y)$, $y \in Y$. If $g: X \to X/\mathcal{D}$ is the quotient map, then g is perfect and monotone. If $h: X/\mathcal{D} \to Y$ is defined as follows: $h(z) = f[q^{\leftarrow}(z)]$, then h is perfect and

$h^{\leftarrow}(y)$ is zero-dimensional for each $y \in Y$. (Although the hypotheses in [Po] include compactness of X, the perfectness of f is the only aspect of compactness used.)

This construction, together with the next result, allows us to build monotone images from arbitrary images.

2.2 *Lemma. Suppose* $f: X \to Y$ *is perfect,* Y *is rimcompact, and* $f^{\leftarrow}(y)$ *is* 0*-dimensional for each* $y \in Y$. *Then* X *is rimcompact.*

Proof. Suppose that $x \in W$, where W is open in X. Let $S = f^{\leftarrow}[f(x)]\backslash W$. Then S is a compact subset of $f^{\leftarrow}[f(x)]$, which is compact and 0-dimensional. Since $x \notin S$, there is a set U clopen in $f^{\leftarrow}[f(x)]$ such that $x \in U$ while $S \cap U = \phi$. Note that $U \subseteq W$. Since U and $f^{\leftarrow}[f(x)]\backslash U$ are disjoint compact subsets of X, we can choose W_1 and W_2 to be disjoint open subsets of X such that $U \subseteq W_1 \subseteq W$ and $f^{\leftarrow}[f(x)]\backslash U \subseteq W_2$. Then $f^{\leftarrow}[f(x)] \subseteq W_1 \cup W_2$, and f is closed, so there is a π-open subset A of Y such that $f^{\leftarrow}[f(x)] \subseteq f^{\leftarrow}[A] \subseteq f^{\leftarrow}[c\ell_Y A] \subseteq W_1 \cup W_2$. It follows from 1.1 that $f^{\leftarrow}[A]$ is π-open in X. It is easy to verify that since $W_1 \cap W_2 = \phi$, $f^{\leftarrow}[A] \cap W_1$ is π-open in X. Since $x \in f^{\leftarrow}[A] \cap W_1 \subseteq W$, x has a base of π-open sets.

2.3 *Corollary. Let* $f: X \to Y$ *be perfect, where* Y *is rimcompact. Then there is a rimcompact space* Z *and perfect maps* $g: X \to Z$, $h: Z \to Y$ *such that* $h \circ g = f$ *and* g *is monotone.*

Thus if X has a decomposition $\mathcal{D}$ consisting of compact sets so that $X/\mathcal{D}$ is rimcompact, then X has a decomposition

$\mathcal{D}'$ consisting of compact connected sets so that $X/\mathcal{D}'$ is rimcompact.

In addition, the next results indicate that given any rimcompact perfect image, there is a maximal rimcompact perfect image.

2.4 *Theorem. Suppose that for* $\alpha \in A$, $f_\alpha: X \to X_\alpha$ *is perfect, where* X_α *is rimcompact. Let* $\mathcal{D} = \{\cap_{\alpha \in A} f_\alpha^{\leftarrow}[f_\alpha(x)]:$ $x \in X\}$. *Then the quotient map* $g: X \to X/\mathcal{D}$ *is perfect and* $X/\mathcal{D}$ *is rimcompact.*

Proof. If g is the quotient map from X onto $X/\mathcal{D}$, then g is continuous and $g^{\leftarrow}(p)$ is compact for each $p \in X/\mathcal{D}$. We define, for each $\beta \in A$, a map $g_\beta: X/\mathcal{D} \to X_\beta$ in the obvious way. For $p \in X/D$, $g^{\leftarrow}(p) = \cap_{\alpha \in A} f_\alpha^{\leftarrow}[f_\alpha(x)]$ for some $x \in X$. Define $g_\beta(p) = f_\beta(x)$. The map g_β is well-defined, for if $g^{\leftarrow}(p) = \cap_{\alpha \in A} f_\alpha^{\leftarrow}[f_\alpha(y)]$ for $y \in X$, then $f_\beta(x) = f_\beta(y)$. It is clear that g_β is continuous, since $g_\beta \circ g = f_\beta$. It is easy to verify that g_β is closed and perfect. Also, for each $p \in X/\mathcal{D}$, $p = \cap_{\alpha \in A} g_\alpha^{\leftarrow}[g_\alpha(p)]$. For suppose $q \in X/\mathcal{D}$ and $q \neq p$. Then $g^{\leftarrow}(p) = \cap_{\alpha \in A} f_\alpha^{\leftarrow}[f_\alpha(x)]$ for $x \in X$, and $g^{\leftarrow}(q) = \cap_{\alpha \in A} f_\alpha^{\leftarrow}[f_\alpha(y)]$ for $y \in Y$, where $x \neq y$ and $f_\beta(x) \neq f_\beta(y)$ for some $\beta \in A$. Then $g_\beta(q) \neq g_\beta(p)$ and $q \notin g_\beta^{\leftarrow}[g_\beta(p)]$.

We show that g is perfect and that $X/\mathcal{D}$ is rimcompact. To show g is perfect, it suffices to show that $\mathcal{D}$ is an upper semicontinuous decomposition of X. Suppose that $D \subseteq V$, where V is open in X and $D = \cap_{\alpha \in A} f_\alpha^{\leftarrow}[f_\alpha(x)]$ for some $x \in X$. Since $\{f_\alpha^{\leftarrow}[f_\alpha(x)]: \alpha \in A\}$ is a collection of compact sets, there is a finite set $F \subseteq A$ such that $\cap_{\gamma \in F} f_\gamma^{\leftarrow}[f_\gamma(x)] \subseteq V$. For each $\gamma \in F$, $[f_\gamma^{\leftarrow}[f_\gamma(x)]]\backslash V$ is compact.

Choose W_γ open in X such that $[f_\gamma^{\leftarrow}[f_\gamma(x)]]\backslash V \subseteq W_\gamma$ and $c\ell_X W_\gamma \cap c\ell_X W_\beta = \phi$ if $\gamma \neq \beta$. Then $f_\gamma^{\leftarrow}[f_\gamma(x)] \subseteq V \cup W_\gamma$, so there is a π-open set U_γ of X_γ such that $f_\gamma^{\leftarrow}[f_\gamma(x)] \subseteq f_\gamma^{\leftarrow}[U_\gamma] \subseteq f_\gamma^{\leftarrow}[c\ell U_\gamma] \subseteq V \cup W_\gamma$.

Then $D \subseteq \cap_{\gamma\in F} f_\gamma^{\leftarrow}[f_\gamma(x)] \subseteq \cap_{\gamma\in F} f_\gamma^{\leftarrow}[U_\gamma] \subseteq \cap_{\gamma\in F}(V \cap W_\gamma) \subseteq V$. Since $f_\gamma^{\leftarrow}[U_\gamma]$ is saturated with respect to $\mathcal{D}$, $\cap_{\gamma\in F} f_\gamma^{\leftarrow}[U_\gamma]$ is saturated with respect to $\mathcal{D}$. It follows that $\mathcal{D}$ is upper semicontinuous. Also, $g[\cap_{\gamma\in F} f_\gamma^{\leftarrow}[U_\gamma]] = \cap_{\gamma\in F} g_\gamma^{\leftarrow}[U_\gamma] \subseteq q[V]$. Since g_γ is perfect, and U_γ is π-open in X_γ, it follows from 1.1 that $\cap_{\gamma\in F} g_\gamma^{\leftarrow}[U_\gamma]$ is π-open in X/D. To complete the proof that $X/\mathcal{D}$ is rimcompact, it is sufficient to note that for $p \in W$ open in $X/\mathcal{D}$, $g^{\leftarrow}(p) = D \subseteq g^{\leftarrow}[W]$. The above construction, with $V = g^{\leftarrow}[W]$ yields the desired π-open neighborhood of p in X/D.

2.5 *Corollary. Suppose that a space* X *is the perfect preimage of a rimcompact space. Then there is a rimcompact space* Z *and a perfect monotone map* $g: X \to Z$ *such that*

a) $g^{\leftarrow}[g[L(X)]] = L(X)$, *and* $g|_{L(X)}$ *is a homeomorphism.*

b) *If* Y *is any rimcompact space, and* $f: X \to Y$ *is perfect, then there is a perfect map* $h: Z \to Y$ *such that* $h \circ g = f$.

Proof. This follows from 2.3 and 2.4.

The preceding results indicate that in attempting to determine when X can be mapped onto a rimcompact space Y by a perfect map, we can restrict ourselves to looking at decompositions of X consisting of compact connected sets. The next results show the utility of this restriction.

2.6 *Theorem. Let* $\mathcal{D}$ *be a decomposition of* X *into closed connected sets, each having a base of* π*-open sets. The following are equivalent.*

a) $X/\mathcal{D}$ *is* T_2.

b) *The quotient map* $g\colon X \to X/\mathcal{D}$ *is closed.*

If either a) *or* b) *holds, then* $X/\mathcal{D}$ *is rimcompact.*

Proof. a $\Rightarrow$ b. We show that $\mathcal{D}$ is upper semicontinuous and hence that g is closed. In the process, we shall also prove that $X/\mathcal{D}$ is rimcompact. Suppose $D \subseteq V$, where V is open in X. There is a π-open set U such that $D \subseteq U \subseteq C\ell_X U \subseteq V$. According to [Ku], if $f\colon X \to Y$ is a monotone quotient map, where X,Y are Hausdorff, and U is π-open in X, then $bd_Y f[U] \subseteq f[bd_X U]$. Thus if $Y = X/\mathcal{D}$ is T_2, then $bd_Y g(X\setminus c\ell_X U) \subseteq g(bd_X(X\setminus c\ell_X U) \subseteq g(bd_X U)$, hence $bd_Y g(X\setminus c\ell_X U)$ is compact. Let $W = Y\setminus c\ell_Y g(X\setminus c\ell_X U)$. Then W is open in Y, and $bd_Y W \subseteq bd_Y g(X\setminus c\ell_X U)$ hence $bd_Y W$ is compact. Since $D \cap bd_X U = \phi$, $g(D) \in W$. Then $g^{\leftarrow}(W)$ is a saturated open neighborhood of D contained in V, while W is an open neighborhood of g(D) having compact boundary. To complete the proof that Y is rimcompact, it is sufficient to note that if $g(D) \in W'$ open in Y, then $D \subseteq g^{\leftarrow}(W')$ which is open in X, and the process of choosing W can be completed with $V = g^{\leftarrow}(W')$.

b $\Rightarrow$ a. Suppose $D_1 \neq D_2$ and $g\colon X \to X/D$ is closed. Choose V to be open in X such that $D_1 \subseteq V \subseteq c\ell_X V \subseteq X\setminus D_2$. The set $g(bd_X V)$ is closed in X/D, so $g^{\leftarrow}[g(bd_X V)]$ is closed in X. For $D \in D$, if $D \cap bd_X V = \phi$, then $D \cap g^{\leftarrow}[g(bd_X V)] = \phi$ and since D is connected, either $D \subseteq W_1 = V\setminus g^{\leftarrow}[g(bd_X V)]$ or

$D \subseteq W_2 = (X \backslash c\ell_X V) \backslash g^{\leftarrow}[g(bd_X V)]$. Then the two disjoint open sets W_1 and W_2 are complete preimages of sets in X/D. That is, there are disjoint sets V_1 and V_2 in X/D such that $g^{\leftarrow}(V_1) = W_1$ and $g^{\leftarrow}(V_2) = W_2$. Then V_1 and V_2 are open, since g is a quotient map, hence D_1 and D_2 have disjoint open nieghborhoods in X/D.

Notice that the prccf that $b \Rightarrow a$ required only that $D \in \mathcal{D}$ has a base of open sets. A base of π-open sets is required to complete the proof that $X/\mathcal{D}$ is rimcompact.

We point out that the hypotheses do not imply that $X/\mathcal{D}$ is T_2. For example, let $X = I \times \{0,1,1/2,1/3,\cdots\}$ and let $\mathcal{D}$ consist of $\{\{x\}: x \in I \times \{0\}\} \cup \{\{I \times 1/n\}: n \in N\}$. Also, if "connected" is deleted from the hypotheses, $X/\mathcal{D}$ need not be rimcompact, even if $X/\mathcal{D}$ is T_2 and the quotient map is perfect: for any space X, there is an extremely disconnected (hence 0-dimensional) space $E(X)$ which can be mapped onto X by a perfect irreducible map.

We now return to a discussion of the sets which motivated the original question. Recall that $G_x = \cap\{c\ell_{\beta X} U: U$ is π-open in X, $x \in U\}$.

2.7 *Lemma. Suppose* X *is almost rimcompact. The following are equivalent.*

a) *For each* $x \in X$, G_x *has a base of* π*-open sets.*

b) *For* $x, y \in X$, $G_x \cap G_y \neq \phi \Rightarrow G_x = G_y$.

c) *If* U *is* π*-open in* X, *and* $G_x \cap U \neq \phi$, *then* $G_x \subseteq C\ell_X U$.

Proof. a $\Rightarrow$ b. Suppose $G_x = G_y$, for $x, y \in X$. Without loss of generality, there is $p \in G_x \backslash G_y$. Since $p \notin G_y$, there are π-open sets U_1 and U_2 of X such that $G_y \subseteq U_2 \subseteq C\ell_X U_2 \subseteq U_1$ and $p \notin C\ell_X U_1$. Since $p \in G_x$, $x \notin U_1$, so $G_x \subseteq X \backslash U_2$. Then $G_x \cap G_y = \phi$.

c $\Rightarrow$ a. Suppose $x \in X$ and $G_x \cap F = \phi$, where F is closed in X. There is a π-open set U of X such that $x \in U$ and $C\ell_X U \cap F = \phi$. Then $G_x \subseteq C\ell_X U$. If $K = G_x \cap bd_X U \neq \phi$, then for $y \in K$, $x \in G_y$. For suppose $x \notin G_y$. Then there is a π-open set V of X such that $y \in V$ and $x \notin C\ell_X V$. It follows that $G_x \subseteq X \backslash V$, contradicting the fact that $y \in G_x$. Hence $G_y \subseteq C\ell_X U$ for $y \in K$.

In particular, $G_y \cap F = \phi$, so there is a π-open set U_y of X such that $y \in U_y$ and $C\ell_X U_y \cap F = \phi$. Then $K \subseteq \cup\{U_y: y \in K\}$ hence there is a finite set $\{y_1, y_2, \cdots, y_n\}$ such that $K \subseteq \cup_{i=1}^n U_{y_i}$.

It follows that $G_x \subseteq U \cup \cup_{i=1}^n U_{y_i}$, which is π-open in X, while $F \cap C\ell[U \cup (\cup_{i=1}^n U_{y_i})] = \phi$.

b $\Rightarrow$ c. Suppose $U \cap G_z \neq \phi$. If $y \in U \cap G_z$, then $G_y \subseteq C\ell_X U_1$, while $G_y = G_z$.

Thus if $\mathcal{G} = \{G_x: x \in X\}$ is a partition of X, then $\mathcal{G}$ is a decomposition of X into compact, connected sets having a base of π-open sets. Even though in general such a decomposition need not yield a rimcompact quotient space, this particular decomposition does.

2.8 *Theorem. Suppose* X *is almost rimcompact, and that* $\mathcal{G} = \{G_x: x \in X\}$ *is a partition of* X. *Then* $X/\mathcal{G}$ *is rimcompact.*

Proof. According to 2.6 and 2.7, it suffices to show that $\mathcal{G}$ is an upper semicontinuous decomposition. Suppose

$G_x \subseteq V$, where $x \in X$ and V is open in X. Let $V_s = \{G_y: G_y \subseteq V\}$. Clearly V_s is saturated with respect to $\mathcal{G}$. We show V_s is open in X. Suppose $z \in V_s$. Then $G_z \subseteq V$, since $z \in G_y$ for some y so that $G_y \subseteq V$ and $G_z = G_y$. Since $G_z \subseteq V$, there is a π-open set U of X such that $G_z \subseteq U \subseteq C\ell_X U \subseteq V$. We claim that $U \subseteq V_s$. If $p \in U$ then $G_p \cap U \neq \phi$. According to 2.7, $G_p \subseteq C\ell_X U \subseteq V_s$, hence $p \in V_s$. It follows that V_s is open, as z was an arbitrary element of V_s.

3. Examples

In 3.1 we construct an almost rimcompact space X which is not the perfect preimage of any rimcompact space, and for which $\mathcal{G} = \{G_x: x \in X\}$ is not a partition of X. In 3.2 we modify 3.1 slightly to obtain an almost rimcompact space X_2 which is the perfect preimage of a rimcompact space but for which $\mathcal{G}$ is not a partition of X_2.

Let $\mathcal{R}$ denote a maximal almost disjoint collection of subsets of the natural numbers $\mathbb{N}$. The space $\mathbb{N} \cup \mathcal{R}$ will have the following topology: each point of $\mathbb{N}$ is isolated and $\lambda \in \mathcal{R}$ has as an open base $\{\{\lambda\} \cup (\lambda \setminus F): F$ is a finite subset of $\mathbb{N}\}$.

According to 2.1 of [Te], we can choose a family $\mathcal{R}$ so that $\beta(\mathbb{N} \cup \mathcal{R}) \setminus (\mathbb{N} \cup \mathcal{R})$ is homeomorphic to the unit interval I. It follows from 3.1 of [Di1] that if $W = [\beta(\mathbb{N} \cup \mathcal{R}) \times (\omega_1 + 1)] \setminus [(\mathbb{N} \cup \mathcal{R}) \times \{\omega_1\}]$ then W is almost rimcompact but not rimcompact. For each $w \in I \times \{\omega_1\}$, w cannot be π-separated from any element w_1 of $I \times \{\omega_1\}$, hence $G_w = I \times \{\omega_1\}$.

3.1 *Example*. For each $n \in N$, let $(N \cup \mathcal{R})_n$ be a copy of the space $N \cup \mathcal{R}$ above. Form the disjoint union of $\{\beta[(N \cup \mathcal{R})_n]: n \in N\}$ and let Y be the quotient space obtained by collapsing 1 of $\beta[(N \cup \mathcal{R})_n]$ and 0 of $\beta[(N \cup \mathcal{R})_{n+1}]$ to a single point, for each $n \in N$. For simplicity of notation, we shall think of Y as a copy of the real line R with a copy of $N \cup \mathcal{R}$ attached to each interval $[n,n+1]$. The space Y is locally compact; let $KY = Y \cup \{p\}$ denote the one-point compactification of Y. Note that $Z = [\cup_{n \in N}(N \cup \mathcal{R})_n] \cup \{p\}$ is zero-dimensional.

Define $X = [KY \times (\omega_1 + 1)]\backslash[Z \times \{\omega_1\}]$. Then $\beta X = KY \times (\omega_1 + 1)$ and $\beta X \backslash X = Z \times \{\omega_1\}$, hence X is a 0-space. It is easy to show directly that each element of $\beta X \backslash X$ has a base of open sets of βX whose boundaries are contained in X, hence X is almost rimcompact. Using the ideas contained in the proof of 3.1 of [Di1], one can see that if $x \in (n,n+1) \times \{\omega_1\}$, then $G_x = [n,n+1] \times \{\omega_1\}$, while for $x = n \times \{\omega_1\}$, $G_x = [n-1,n+1] \times \{\omega_1\}$. Thus $\mathcal{G} = \{G_x: x \in X\}$ is not a partition of X. Since any rimcompact quotient space of X would have to collapse $R \times \{\omega_1\}$, no such space could be a perfect image of X.

3.2 *Example*. Construct X as in 3.1, and let $X_2 = X \cup \{(p,\omega_1)\}$. Then $R(X_2)$ is compact. If X_3 is the quotient space of X_2 obtained by collapsing $R(X_2)$ to a point, then X_3 is rimcompact. Notice that X_3 is the maximal perfect rimcompact image of X_2. The collection $\mathcal{G} = \{G_x: x \in X_2\}$ is not a partition of X_2, and $R(X_2) \nsubseteq G_x$ for any $x \in X_2$.

We leave the reader with the following open question: Suppose that a (almost rimcompact) space X has a decomposition $\mathcal{D}$ into (connected) compact sets, each having a base of π-open sets. Does this imply that X has a decomposition $\mathcal{D}'$ with the same properties so that $X/\mathcal{D}'$ is rimcompact?

References

[Di1] B. Diamond, *Almost rimcompact spaces* (submitted).

[Di2] ________, *Some properties of almost rimcompact spaces*, Pac. J. Math. 118 (1985), 63-77.

[Di3] ________, *A characterization of those spaces having zero-dimensional remainders*, Rocky Mountain J. 15 (1985), 47-60.

[HI] M. Henriksen and J. R. Isbell, *Some properties of compactifications*, Duke Math. J. 25 (1958), 83-105.

[KU] T. A. Kuznetsova, *Continuous mappings and bicompact extensions of topological spaces*, Vestnik Moskovskogo Universiteta. Matematika 28 (1973), 48-53.

[Po] V. I. Ponomarev, *On continuous decompositions of bicompacta*, Amer. Math. Soc. Trans. 30 (1963), 235-240.

[Sk] E. G. Sklyarenko, *Some questions in the theory of bicompactifications*, Amer. Math. Soc. Trans. 58 (1966), 216-244.

[Te] J. Terasawa, *Spaces* N ∪ $\mathcal{R}$ *and their dimensions*, Top. and Its Appl. 11 (1980), 93-102.

College of Charleston

Charleston, South Carolina 29424