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CONCERNING THE EXTENSION OF CONNECTIVITY FUNCTIONS

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In his classic paper, Stallings [7] asked if a connectivity function $I \rightarrow I$ could always be extended to a connectivity function $I^2 \rightarrow I$ when I is considered embedded in I^2 as $I \times 0$. Several authors answered this negatively by giving examples of connectivity functions $I \rightarrow I$ which are not almost continuous, [1], [6]. In [7] Stallings proved that an almost continuous function $I \rightarrow I$ is a connectivity function and, curiously enough, a connectivity function $I^2 \rightarrow I$ is an almost continuous function. Later it was shown by Kellum [4] that an almost continuous function $I \rightarrow I$ can be extended to an almost continuous function $I^2 \rightarrow I$. This naturally leaves the question "can an almost continuous function $I \rightarrow I$ be extended to a connectivity function $I^2 \rightarrow I$?" Theorem 2 of this paper together with the first example of [2] shows that this is not the case.

For simplicity no distinction will be made between points of $I \times 0$ and I . Also, $B(y,r)$ denotes an open ball about y with radius r where d is the usual distance function.

Definition 1. A function $f: X \rightarrow Y$ between spaces X and Y is said to be almost continuous if each open set containing the graph of f also contains the graph of a continuous function with the same domain. The function f is said to be a connectivity function if for each connected subset

C of X the graph of f restricted to C , denoted by $f|_C$, is a connected subset of $X \times Y$. The function f is said to be a Darboux function if $f(C)$ is connected whenever C is a connected subset of X .

Definition 2. A function $f: I \rightarrow I$ has the Cantor Intermediate Value Property (CIVP) if for any Cantor set K in the interval $(f(x), f(y))$ the interval (x, y) or (y, x) contains a Cantor set C such that $f(C) \subset K$ where $x, y \in I = [0, 1]$. The function f has the Weak Cantor Intermediate Value Property (WCIVP) if there exists a Cantor set C between x and y such that $f(C) \subset (f(x), f(y))$.

Theorem 1. If $f: I \rightarrow I$ has the CIVP, then f has the WCIVP.

Proof. Obvious.

Example 1. There exists a function $f: I \rightarrow I$ that has the WCIVP but does not have the CIVP. Let $S_y, y \in I$, be the collection of Cantor dense subsets of I constructed in [2]. Let $r \in I$ be fixed. Let $g: I \rightarrow \bigcup_{y \neq r} S_y$ where $y \in I$ be 1-1 and onto. Define $f(x) = g(y)$ where $x \in S_y$ and $y \neq r$. If $x \in S_r$, let $f(x) = 0$. If x is not in any S_y , let $f(x) = 0$. Let $a, b \in I$ and assume that $f(a) < f(b)$. Let K be a Cantor set in $(f(a), f(b))$ such that $K \subset S_y$ for some $y \neq r$. Choose $z \in K$ such that $r \neq g^{-1}(z) = w$. Consider S_w . If $x \in S_w$, then $f(x) = g(w) = z$ and $f(S_w) \subset K$. By Cantor density there exists a Cantor set $C \subset S_w$ such that $C \subset (a, b)$ or $C \subset (b, a)$. Therefore $f(C) \subset (f(a), f(b))$ and hence f has the WCIVP.

Let K be a Cantor set in $(f(a), f(b))$ such that $K \subset S_f$. Since K contains no points of the range of f , there exists no Cantor set $C \subset I$ such that $f(C) \subset K$. Therefore f does not have the CIVP.

Theorem 2. If $f: I^2 \rightarrow I$ is a connectivity function, then $f|I \times 0$ has the WCIVP. Moreover, the Cantor set can be selected such that f restricted to it is continuous.

Proof. It follows that a function $I^2 \rightarrow I$ is a connectivity function if and only if it is peripherally continuous [3]. The function $f: I^2 \rightarrow I$ is peripherally continuous if and only if U is an open subset of I^2 containing a point x and V is an open subset of I containing $f(x)$, then there is an open subset W of U containing x such that $f(\text{bd}(W))$ is a subset of V , where $\text{bd}(W)$ is the boundary of W .

Assume $a, b \in I$ such that $f(a) < f(b)$. Choose $y \in I$ between a and b such that $f(y) \in (f(a), f(b))$. Let $\epsilon = \min\{d(f(a), f(y)), d(f(y), f(b))\}$. Let U be a connected open subset of I^2 with connected boundary C such that $y \in U \subset \bar{U} \subset B(y, \eta/5)$ where $\eta = \min\{d(y, a), d(y, b)\}$, and $f(C) \subset B(f(y), \epsilon/5)$. Then there exists $y_0, y_1 \in I$ which are in C such that $y_0 < y < y_1$.

$$y_0 \in B(y, \eta/5), \quad f(y_0) \in B(f(y), \epsilon/5),$$

$$y_1 \in B(y, \eta/5), \quad f(y_1) \in B(f(y), \epsilon/5).$$

Clearly $d(y_0, y) < \eta/5$ and $d(y_1, y) < \eta/5$. Also $d(f(y_0), f(y)) < \epsilon/5$ and $d(f(y_1), f(y)) < \epsilon/5$.

Now there exist connected open subsets U_0 and U_1 of I^2 with connected boundaries C_0 and C_1 such that

$$y_0 \in U_0, \bar{U}_0 \subset B(y_0, \eta_0/5), f(C_0) \subset B(f(y_0), \epsilon/5^2)$$

and

$$y_1 \in U_1, \bar{U}_1 \subset B(y_1, \eta_1/5), f(C_1) \subset B(f(y_1), \epsilon/5^2)$$

where $\eta_0 = d(y_0, y)$ and $\eta_1 = d(y_1, y)$. So $\eta_0 < \eta/5$ and $\eta_1 < \eta/5$.

Now C_0 has points $y_{00}, y_{01} \in I$ and C_1 has points $y_{10}, y_{11} \in I$ such that

$$\begin{aligned} a < y_{00} < y_0 < y_{01} < y < y_{10} < y_1 < y_{11} < b, \\ y_{00}, y_{01} &\in B(y_0, \eta_0/5), f(y_{00}), f(y_{01}) \in B(f(y_0), \epsilon/5^2), \\ y_{10}, y_{11} &\in B(y_1, \eta_1/5), f(y_{10}), f(y_{11}) \in B(f(y_1), \epsilon/5^2). \end{aligned}$$

There exists connected open subsets $U_{00}, U_{01}, U_{10}, U_{11}$ of I^2 with connected boundaries $C_{00}, C_{01}, C_{10}, C_{11}$ such that

$$\begin{aligned} y_{00} &\in U_{00}, \bar{U}_{00} \subset B(y_{00}, \eta_{00}/5), f(C_{00}) \subset B(f(y_{00}), \epsilon/5^3), \\ y_{01} &\in U_{01}, \bar{U}_{01} \subset B(y_{01}, \eta_{01}/5), f(C_{01}) \subset B(f(y_{01}), \epsilon/5^3), \\ y_{10} &\in U_{10}, \bar{U}_{10} \subset B(y_{10}, \eta_{10}/5), f(C_{10}) \subset B(f(y_{10}), \epsilon/5^3), \\ y_{11} &\in U_{11}, \bar{U}_{11} \subset B(y_{11}, \eta_{11}/5), f(C_{11}) \subset B(f(y_{11}), \epsilon/5^3), \end{aligned}$$

where $\eta_{00} = d(y_{00}, y_0), \eta_{01} = d(y_{01}, y_0), \eta_{10} = d(y_{10}, y_1)$, and $\eta_{11} = d(y_{11}, y_1)$.

Now C_{00} has points $y_{000}, y_{001} \in I$, C_{01} has points $y_{010}, y_{011} \in I$, C_{10} has points $y_{100}, y_{101} \in I$, and C_{11} has points $y_{110}, y_{111} \in I$ such that $a < y_{000} < y_{00} < y_{001} < y_0 < y_{010} < y_{01} < y_{011} < y < y_{100} < y_{10} < y_{101} < y_1 < y_{110} < y_{11} < y_{111} < b$.

$$\begin{aligned} y_{000}, y_{001} &\in B(y_{00}, \eta_{00}/5), f(y_{000}), f(y_{001}) \in B(f(y_{00}), \epsilon/5^3), \\ y_{010}, y_{011} &\in B(y_{01}, \eta_{01}/5), f(y_{010}), f(y_{011}) \in B(f(y_{01}), \epsilon/5^3), \\ y_{100}, y_{101} &\in B(y_{10}, \eta_{10}/5), f(y_{100}), f(y_{101}) \in B(f(y_{10}), \epsilon/5^3), \end{aligned}$$

and

$$y_{110}, y_{111} \in B(y_{11}, \eta_{11}/5), f(y_{110}), f(y_{111}) \in B(f(y_{11}), \epsilon/5^3).$$

Continuing this process let α be a finite sequence of 0's and 1's of length k . Thus for y_α we obtain

$$\begin{aligned} y_{\alpha 0} &< y_\alpha < y_{\alpha 1}, \\ y_{\alpha 0} &\in B(y_\alpha, \eta_\alpha/5), \\ y_{\alpha 1} &\in B(y_\alpha, \eta_\alpha/5), \\ \eta_{\alpha 0} &= d(y_{\alpha 0}, y_\alpha), \\ \bar{U}_{\alpha 0} &\subset B(y_{\alpha 0}, \eta_{\alpha 0}/5), \\ f(C_{\alpha 0}) &\subset B(f(y_{\alpha 0}), \epsilon/5^{k+2}), \\ \eta_{\alpha 1} &= d(y_{\alpha 1}, y_\alpha), \\ \bar{U}_{\alpha 1} &\subset B(y_{\alpha 1}, \eta_{\alpha 1}/5), \text{ and} \\ f(C_{\alpha 1}) &\subset B(f(y_{\alpha 1}), \epsilon/5^{k+2}) \end{aligned}$$

where $\eta_{\alpha 0} = d(y_{\alpha 0}, y_\alpha)$ and $\eta_{\alpha 1} = d(y_{\alpha 1}, y_\alpha)$. Now $C_{\alpha 0}$ has points $y_{\alpha 00}, y_{\alpha 01} \in I$ and $C_{\alpha 1}$ has points $y_{\alpha 10}, y_{\alpha 11} \in I$ such that

$$\begin{aligned} y_{\alpha 00} &< y_{\alpha 0} < y_{\alpha 01} < y_\alpha < y_{\alpha 10} < y_{\alpha 1} < y_{\alpha 11}, \\ y_{\alpha 00}, y_{\alpha 01} &\in B(y_{\alpha 0}, \eta_{\alpha 0}/5), \quad f(y_{\alpha 00}), f(y_{\alpha 01}) \in B(f(y_{\alpha 0}), \epsilon/5^{k+2}), \\ y_{\alpha 10}, y_{\alpha 11} &\in B(y_{\alpha 1}, \eta_{\alpha 1}/5), \quad f(y_{\alpha 10}), f(y_{\alpha 11}) \in B(f(y_{\alpha 1}), \epsilon/5^{k+2}). \end{aligned}$$

We now claim that if α and β are finite binary sequences of equal length n of the form $\alpha = \gamma 0\mu$ and $\beta = \gamma 1\nu$ where γ is of length $k \leq n-1$, then

- (1) $y_\alpha < y_\beta$,
- (2) $3/4(\eta_{\gamma 0} + \eta_{\gamma 1}) \leq |y_\alpha - y_\beta| \leq 5/4(\eta_{\gamma 0} + \eta_{\gamma 1})$, and
- (3) $|f(y_\alpha) - f(y_\beta)| < \epsilon/2(5^k)$.

By construction $y_\alpha < y_\beta$ and $y_{\gamma 0} < y_{\gamma 1}$. Thus $y_{\gamma 1} - y_{\gamma 0} = y_{\gamma 1} - y_\gamma + y_\gamma - y_{\gamma 0} = \eta_{\gamma 0} + \eta_{\gamma 1}$. Also

$$\begin{aligned} d(y_\alpha, y_{\gamma 0}) &< \eta_{\gamma 0}((1/5) + (1/5^2) + \dots + (1/5^{n-k})) < \frac{1}{4} \eta_{\gamma 0} \text{ and} \\ d(y_\beta, y_{\gamma 1}) &< \eta_{\gamma 1}((1/5) + (1/5^2) + \dots + (1/5^{n-k})) < \frac{1}{4} \eta_{\gamma 1}. \end{aligned}$$

From this it follows that (2) is true.

$$\begin{aligned}
 \text{Now } |f(y_{\gamma_0}) - f(y_{\gamma})| &< \varepsilon/5^{k+1} \text{ and } |f(y_{\gamma_1}) - f(y_{\gamma})| < \varepsilon/5^{k+1}, \\
 |f(y_{\alpha}) - f(y_{\gamma})| &< \varepsilon((1/5^k) + (1/5^{k+1}) + \dots + (1/5^n)), \text{ and} \\
 |f(y_{\beta}) - f(y_{\gamma})| &< \varepsilon((1/5^k) + (1/5^{k+1}) + \dots + (1/5^n)). \\
 \text{So } |f(y_{\alpha}) - f(y_{\beta})| &< 2\varepsilon((1/5^k) + (1/5^{k+1}) + \dots + (1/5^n)) \\
 &< 2\varepsilon(1/5^k)(1/4) \\
 &= \varepsilon/2(5^k).
 \end{aligned}$$

Let $\alpha(n)$ denote a binary sequence with n terms such that the first $n-1$ terms of $\alpha(n)$ is $\alpha(n-1)$. Define

$$y_{\alpha} = \lim_{n \rightarrow \infty} y_{\alpha(n)}.$$

Then the previous claim holds true for infinite sequences α and β . We now prove that $f(y_{\alpha}) = \lim_{n \rightarrow \infty} f(y_{\alpha(n)})$. Each $C_{\alpha(k+1)}$ intersects $C_{\alpha(k)}$ since one point of $C_{\alpha(k+1)}$ is inside the interval formed by $C_{\alpha(k)}$ and one point is outside. Thus for any γ of length k the union of all sets $C_{\gamma\nu}$ is a connected set and its image points differ from $f(y_{\gamma})$ by at most $(\varepsilon/5^k) + (\varepsilon/5^{k+1}) + \dots = \varepsilon/4(5^{k-1})$. Since f is a Darboux function (the image of connected sets is connected),

$$f(\overline{UC_{\gamma\nu}}) \subset \overline{f(UC_{\gamma\nu})} \subset \overline{B(f(y_{\gamma}), \varepsilon/4(5^{k-1}))}.$$

Thus $d(f(y_{\alpha}), f(y_{\alpha(n)})) < \varepsilon/4(5^{n-1})$ where $\alpha(n) = \gamma$ and it follows that $f(y_{\alpha(n)})$ converges to $f(y_{\alpha})$.

Now it follows that the function defined by the assignment $\alpha \rightarrow y_{\alpha}$ is a homeomorphism from a Cantor set to $S = \{y_{\alpha}\}$. Thus S is a Cantor set and $f(S) \subset (f(a), f(b))$. So $f|I \times 0$ has the WCIVP and $f|S$ is continuous.

Example 2. The first example of [2] is an example of an almost continuous function $I \rightarrow I$ which does not have the WCIVP. For completeness that example will be described here. There exists a subset $G \subset I$ which intersects every

Cantor set in every interval (a,b) but contains no Cantor set. Thus $G \cap (a,b)$ contains c points. Let $F_1 = \{(x,0) : x \notin G\}$. Consider the collection $\{K\}$ of closed subsets of I^2 such that the x -projection of K has cardinality c . The x -projection of any set in the collection is closed and contains a Cantor set. Hence it contains a point of G . Select a subset $F_2 \subset I^2$ by transfinite induction such that

- (1) F_2 intersects each member of the collection $\{K\}$ and
- (2) if p and q are distinct points of F_2 , then their x -projections are distinct points of G .

Let $F_3 = \{(t,1) : t \in I \text{ but } t \text{ is neither in the } x\text{-projection of } F_1 \text{ nor in the } x\text{-projection of } F_2\}$.

Let $f = F_1 \cup F_2 \cup F_3$. Then the x -projection of f is I and f is the graph of a function $f: I \rightarrow I$.

Remarks. The second example of [2] is an example of a function $I \rightarrow I$ which has the WCIVP but is not a Darboux function. Also, it follows that if $f: I \rightarrow I$ is continuous then f has the P.

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