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CONCERNING THE EXTENSION OF CONNECTIVITY FUNCTIONS

Richard G. Gibson and Fred Roush

In his classic paper, Stallings [7] asked if a connectivity function $I \to I$ could always be extended to a connectivity function $I^2 \to I$ when I is considered embedded in I^2 as $I \times 0$. Several authors answered this negatively by giving examples of connectivity functions $I \to I$ which are not almost continuous, [1], [6]. In [7] Stallings proved that an almost continuous function $I \to I$ is a connectivity function and, curiously enough, a connectivity function $I^2 \to I$ is an almost continuous function. Later it was shown by Kellum [4] that an almost continuous function $I \to I$ can be extended to an almost continuous function $I^2 \to I$. This naturally leaves the question "can an almost continuous function $I \to I$ be extended to a connectivity function $I^2 \to I$?" Theorem 2 of this paper together with the first example of [2] shows that this is not the case.

For simplicity no distinction will be made between points of $I \times 0$ and I. Also, $B(y,r)$ denotes an open ball about y with radius r where d is the usual distance function.

Definition 1. A function $f: X \to Y$ between spaces X and Y is said to be almost continuous if each open set containing the graph of f also contains the graph of a continuous function with the same domain. The function f is said to be a connectivity function if for each connected subset

C of X the graph of f restricted to C, denoted by $f|C$, is a connected subset of $X \times Y$. The function f is said to be a Darboux function if f(C) is connected whenever C is a connected subset of X.

Definition 2. A function $f: I \to I$ has the Cantor Intermediate Value Property (CIVP) if for any Cantor set K in the interval $(f(x),f(y))$ the interval (x,y) or (y,x) contains a Cantor set C such that $f(C) \subset K$ where $x,y \in I = [0,1]$. The function f has the Weak Cantor Intermediate Value Property (WCIVP) if there exists a Cantor set C between x and y such that $f(C) \subset (f(x),f(y))$.

Theorem 1. *If* $f: I \to I$ *has the* CIVP, *then* f *has the* WCIVP.

Proof. Obvious.

Example 1. There exists a function $f: I \to I$ that has the WCIVP but does not have the CIVP. Let S_y, $y \in I$, be the collection of Cantor dense subsets of I constructed in [2]. Let $r \in I$ be fixed. Let $g: I \to \cup_{y \neq r} S_y$ where $y \in I$ be 1-1 and onto. Define $f(x) = g(y)$ where $x \in S_y$ and $y \neq r$. If $x \in S_r$, let $f(x) = 0$. If x is not in any S_y, let $f(x) = 0$. Let $a,b \in I$ and assume that $f(a) < f(b)$. Let K be a Cantor set in $(f(a),f(b))$ such that $K \subset S_y$ for some $y \neq r$. Choose $z \in K$ such that $r \neq g^{-1}(z) = w$. Consider S_w. If $x \in S_w$, then $f(x) = g(w) = z$ and $f(S_w) \subset K$. By Cantor density there exists a Cantor set $C \subset S_w$ such that $C \subset (a,b)$ or $C \subset (b,a)$. Therefore $f(C) \subset (f(a),f(b))$ and hence f has the WCIVP.

Let K be a Cantor set in (f(a),f(b)) such that $K \subset S_r$. Since K contains no points of the range of f, there exists no Cantor set $C \subset I$ such that $f(C) \subset K$. Therefore f does not have the CIVP.

Theorem 2. *If* $f\colon I^2 \to I$ *is a connectivity function, then* $f|I \times 0$ *has the* WCIVP. *Moreover, the Cantor set can be selected such that* f *restricted to it is continuous.*

Proof. It follows that a function $I^2 \to I$ is a connectivity function if and only if it is peripherally continuous [3]. The function $f\colon I^2 \to I$ is peripherally continuous if and only if U is an open subset of I^2 containing a point x and V is an open subset of I containing f(x), then there is an open subset W of U containing x such that f(bd(W)) is a subset of V, where bd(W) is the boundary of W.

Assume $a,b \in I$ such that $f(a) < f(b)$. Choose $y \in I$ between a and b such that $f(y) \in (f(a),f(b))$. Let $\varepsilon = \min\{d(f(a),f(y)),d(f(y),f(b))\}$. Let U be a connected open subset of I^2 with connected boundary C such that $y \in U \subset \overline{U} \subset B(y,\eta/5)$ where $\eta = \min\{d(y,a),d(y,b)\}$, and $f(C) \subset B(f(y),\varepsilon/5)$. Then there exists $y_0,y_1 \in I$ which are in C such that $y_0 < y < y_1$.

$$y_0 \in B(y,\eta/5), \quad f(y_0) \in B(f(y),\varepsilon/5),$$
$$y_1 \in B(y,\eta/5), \quad f(y_1) \in B(f(y),\varepsilon/5).$$

Clearly $d(y_0,y) < \eta/5$ and $d(y_1,y) < \eta/5$. Also $d(f(y_0),f(y)) < \varepsilon/5$ and $d(f(y_1),f(y)) < \varepsilon/5$.

Now there exist connected open subsets U_0 and U_1 of I^2 with connected boundaries C_0 and C_1 such that

$$y_0 \in U_0,\ \overline{U}_0 \subset B(y_0,\eta_0/5),\ f(C_0) \subset B(f(y_0),\varepsilon/5^2)$$

and

$$y_1 \in U_1,\ \overline{U}_1 \subset B(y_1,\eta_1/5),\ f(C_1) \subset B(f(y_1),\varepsilon/5^2)$$

where $\eta_0 = d(y_0,y)$ and $\eta_1 = d(y_1,y)$. So $\eta_0 < \eta/5$ and $\eta_1 < \eta/5$.

Now C_0 has points $y_{00},y_{01} \in I$ and C_1 has points $y_{10},y_{11} \in I$ such that

$$a < y_{00} < y_0 < y_{01} < y < y_{10} < y_1 < y_{11} < b,$$
$$y_{00},y_{01} \in B(y_0,\eta_0/5), f(y_{00}),f(y_{01}) \in B(f(y_0),\varepsilon/5^2),$$
$$y_{10},y_{11} \in B(y_1,\eta_1/5), f(y_{10}),f(y_{11}) \in B(f(y_1),\varepsilon/5^2).$$

There exists connected open subsets $U_{00},U_{01},U_{10},U_{11}$ of I^2 with connected boundaries $C_{00},C_{01},C_{10},C_{11}$ such that

$$y_{00} \in U_{00},\ \overline{U}_{00} \subset B(y_{00},\eta_{00}/5),\ f(C_{00}) \subset B(f(y_{00}),\varepsilon/5^3),$$
$$y_{01} \in U_{01},\ \overline{U}_{01} \subset B(y_{01},\eta_{01}/5),\ f(C_{01}) \subset B(f(y_{01}),\varepsilon/5^3),$$
$$y_{10} \in U_{10},\ \overline{U}_{10} \subset B(y_{10},\eta_{10}/5),\ f(C_{10}) \subset B(f(y_{10}),\varepsilon/5^3),$$
$$y_{11} \in U_{11},\ \overline{U}_{11} \subset B(y_{11},\eta_{11}/5),\ f(C_{11}) \subset B(f(y_{11}),\varepsilon/5^3),$$

where $\eta_{00} = d(y_{00},y_0),\eta_{01} = d(y_{01},y_0),\eta_{10} = d(y_{10},y_1)$, and $\eta_{11} = d(y_{11},y_1)$.

Now C_{00} has points $y_{000},y_{001} \in I$, C_{01} has points $y_{010},y_{011} \in I$, C_{10} has points $y_{100},y_{101} \in I$, and C_{11} has points $y_{110},y_{111} \in I$ such that $a < y_{000} < y_{00} < y_{001} < y_0 < y_{010} < y_{01} < y_{011} < y < y_{100} < y_{10} < y_{101} < y_1 < y_{110} < y_{11} < y_{111} < b$.

$$y_{000},y_{001} \in B(y_{00},\eta_{00}/5),\ f(y_{000}),f(y_{001}) \in B(f(y_{00}),\varepsilon/5^3),$$
$$y_{010},y_{011} \in B(y_{01},\eta_{01}/5),\ f(y_{010}),f(y_{011}) \in B(f(y_{01}),\varepsilon/5^3),$$
$$y_{100},y_{101} \in B(y_{10},\eta_{10}/5),\ f(y_{100}),f(y_{101}) \in B(f(y_{10}),\varepsilon/5^3),$$

and

$$y_{110},y_{111} \in B(y_{11},\eta_{11}/5),\ f(y_{110}),f(y_{111}) \in B(f(y_{11}),\varepsilon/5^3).$$

Continuing this process let α be a finite sequence of 0's and 1's of length k. Thus for y_α we obtain

$$y_{\alpha 0} < y_\alpha < y_{\alpha 1},$$
$$y_{\alpha 0} \in B(y_\alpha, \eta_\alpha/5),$$
$$y_{\alpha 1} \in B(y_\alpha, \eta_\alpha/5),$$
$$\eta_{\alpha 0} = d(y_{\alpha 0}, y_\alpha),$$
$$\overline{U}_{\alpha 0} \subset B(y_{\alpha 0}, \eta_{\alpha 0}/5),$$
$$f(C_{\alpha 0}) \subset B(f(y_{\alpha 0}), \varepsilon/5^{k+2}),$$
$$\eta_{\alpha 1} = d(y_{\alpha 1}, y_\alpha),$$
$$\overline{U}_{\alpha 1} \subset B(y_{\alpha 1}, \eta_{\alpha 1}/5), \text{ and}$$
$$f(C_{\alpha 1}) \subset B(f(y_{\alpha 1}), \varepsilon/5^{k+2})$$

where $\eta_{\alpha 0} = d(y_{\alpha 0}, y_\alpha)$ and $\eta_{\alpha 1} = d(y_{\alpha 1}, y_\alpha)$. Now $C_{\alpha 0}$ has points $y_{\alpha 00}, y_{\alpha 01} \in I$ and $C_{\alpha 1}$ has points $y_{\alpha 10}, y_{\alpha 11} \in I$ such that

$$y_{\alpha 00} < y_{\alpha 0} < y_{\alpha 01} < y_\alpha < y_{\alpha 10} < y_{\alpha 1} < y_{\alpha 11},$$
$$y_{\alpha 00}, y_{\alpha 01} \in B(y_{\alpha 0}, \eta_{\alpha 0}/5),\ f(y_{\alpha 00}), f(y_{\alpha 01}) \in B(f(y_{\alpha 0}), \varepsilon/5^{k+2}),$$
$$y_{\alpha 10}, y_{\alpha 11} \in B(y_{\alpha 1}, \eta_{\alpha 1}/5),\ f(y_{\alpha 10}), f(y_{\alpha 11}) \in B(f(y_{\alpha 1}), \varepsilon/5^{k+2}).$$

We now claim that if α and β are finite binary sequences of equal length n of the form $\alpha = \gamma 0 \mu$ and $\beta = \gamma 1 \nu$ where γ is of length $k \leq n-1$, then

(1) $y_\alpha < y_\beta$,

(2) $3/4(\eta_{\gamma 0} + \eta_{\gamma 1}) \leq |y_\alpha - y_\beta| \leq 5/4(\eta_{\gamma 0} + \eta_{\gamma 1})$, and

(3) $|f(y_\alpha) - f(y_\beta)| < \varepsilon/2(5^k)$.

By construction $y_\alpha < y_\beta$ and $y_{\gamma 0} < y_{\gamma 1}$. Thus $y_{\gamma 1} - y_{\gamma 0} = y_{\gamma 1} - y_\gamma + y_\gamma - y_{\gamma 0} = \eta_{\gamma 0} + \eta_{\gamma 1}$. Also

$$d(y_\alpha, y_{\gamma 0}) < \eta_{\gamma 0}((1/5)+(1/5^2)+\cdots+(1/5^{n-k})) < \tfrac{1}{4}\eta_{\gamma 0} \text{ and}$$
$$d(y_\beta, y_{\gamma 1}) < \eta_{\gamma 1}((1/5)+(1/5^2)+\cdots+(1/5^{n-k})) < \tfrac{1}{4}\eta_{\gamma 1}.$$

From this it follows that (2) is true.

Now $|f(y_{\gamma 0})-f(y_\gamma)| < \varepsilon/5^{k+1}$ and $|f(y_{\gamma 1})-f(y_\gamma)| < \varepsilon/5^{k+1}$,
$|f(y_\alpha)-f(y_\gamma)| < \varepsilon((1/5^k)+(1/5^{k+1})+\cdots+(1/5^n))$, and
$|f(y_\beta)-f(y_\gamma)| < \varepsilon((1/5^k)+(1/5^{k+1})+\cdots+(1/5^n))$.

So

$$\begin{aligned}|f(y_\alpha)-f(y_\beta)| &< 2\varepsilon((1/5^k)+(1/5^{k+1})+\cdots+(1/5^n))\\ &< 2\varepsilon(1/5^k)(1/4)\\ &= \varepsilon/2(5^k).\end{aligned}$$

Let $\alpha(n)$ denote a binary sequence with n terms such that the first n-1 terms of $\alpha(n)$ is $\alpha(n-1)$. Define

$$y_\alpha = \lim_{n\to\infty} y_{\alpha(n)}.$$

Then the previous claim holds true for infinite sequences α and β. We now prove that $f(y_\alpha) = \lim_{n\to\infty} f(y_{\alpha(n)})$. Each $C_{\alpha(k+1)}$ intersects $C_{\alpha(k)}$ since one point of $C_{\alpha(k+1)}$ is inside the interval formed by $C_{\alpha(k)}$ and one point is outside. Thus for any γ of length k the union of all sets $C_{\gamma\nu}$ is a connected set and its image points differ from $f(y_\gamma)$ by at most $(\varepsilon/5^k)+(\varepsilon/5^{k+1})+\cdots = \varepsilon/4(5^{k-1})$. Since f is a Darboux function (the image of connected sets is connected),

$$f(\overline{\cup C_{\gamma\nu}}) \subset \overline{f(\cup C_{\gamma\nu})} \subset \overline{B(f(y_\gamma),\varepsilon/4(5^{k-1}))}.$$

Thus $d(f(y_\alpha),f(y_{\alpha(n)})) < \varepsilon/4(5^{n-1})$ where $\alpha(n) = \gamma$ and it follows that $f(y_{\alpha(n)})$ converges to $f(y_\alpha)$.

Now it follows that the function defined by the assignment $\alpha \to y_\alpha$ is a homeomorphism from a Cantor set to $S = \{y_\alpha\}$. Thus S is a Cantor set and $f(S) \subset (f(a),f(b))$. So $f|I \times 0$ has the WCIVP and $f|S$ is continuous.

Example 2. The first example of [2] is an example of an almost continuous function $I \to I$ which does not have the WCIVP. For completeness that example will be described here. There exists a subset $G \subset I$ which intersects every

Cantor set in every interval (a,b) but contains no Cantor set. Thus $G \cap (a,b)$ contains c points. Let $F_1 = \{(x,0): x \notin G\}$. Consider the collection $\{K\}$ of closed subsets of I^2 such that the x-projection of K has cardinality c. The x-projection of any set in the collection is closed and contains a Cantor set. Hence it contains a point of G. Select a subset $F_2 \subset I^2$ by transfinite induction such that

(1) F_2 intersects each member of the collection $\{K\}$ and

(2) if p and q are distinct points of F_2, then their x-projections are distinct points of G.

Let $F_3 = \{(t,1): t \in I$ but t is neither in the x-projection of F_1 nor in the x-projection of $F_2\}$.

Let $f = F_1 \cup F_2 \cup F_3$. Then the x-projection of f is I and f is the graph of a function $f: I \to I$.

Remarks. The second example of [2] is an example of a function $I \to I$ which has the WCIVP but is not a Darboux function. Also, it follows that if $f: I \to I$ is continuous then f has th P.

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