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1. Introduction

The class of *p-spaces* was introduced by A. V. Arhangel'skii in 1963 [1]. Metrizable and locally compact Hausdorff spaces are examples of p-spaces. Although in the original definition every p-space was supposed to be completely regular and T_1 (in fact, that definition depends on the existence of the Stone-Cech compactification of the space), D. Burke [2] gave a characterization of p-spaces which allows to generalize the concept to T_3-spaces. Under this new definition, Moore spaces are p-spaces (see [2; Thm. 2.1]). R. Hodel introduced the concept of *pluming degree* plX of a T_3-space X [5]. According to his definition, a T_3-space is a p-space if and only if $plX \leq \aleph_0$. A remarkable property of p-spaces is *countable typeness* [1], i.e., every compact subset of a p-space is contained in a compact subset which has a countable local basis for its neighborhood system. Chaber, Coban and Nagami [3] introduced the class of *monotonic p-spaces* which lies between the class of p-spaces and the class of T_3-spaces of countable type. A common feature of these generalizations is their intrinsic character, i.e., the definitions concrete to the space in question.

In this paper we give an extrinsic characterization of p-spaces using the Wallman instead of the Stone-Cech compactification. In a natural way we give an equivalent formulation of the inequality $plX \leq \lambda$, where λ is any infinite cardinal number. If we replace *countable typeness* by *$\leq\lambda$-typeness* (a space X has *$\leq\lambda$-type* if every compact subset lies in a compact set which has a local basis for its neighborhood system consisting of at most λ elements), it is natural to ask if $plX \leq \lambda$ implies $\leq\lambda$-typeness. This seems to be a difficult problem and a possible solution may depend on a characterization of T_3-spaces of $\leq\lambda$-type using their Wallman compactification. We content ourselves with a characterization of T_3-spaces of countable type which makes trivial the statement that a p-space has countable type.

2. Definitions and Preliminary Results

If X is a T_1-space, we denote by $\mathcal{F}_X$ the family of all closed subsets of X. The collection wX of all ultrafilters in $\mathcal{F}_X$ may be topologized as follows: for each $A \subset X$, A^* denotes $\{\xi \in wX | F \subset A \text{ for some } F \in \xi\}$. The collection $\mathcal{B}^* = \{U^* | U \subset X \text{ open}\}$ is closed under finite unions and finite intersections and it is obvious that $\Phi^* = \Phi$ and $X^* = wX$. Hence there is a topology τ of wX having $\mathcal{B}^*$ as a basis and the space (wX,τ) turns out to be compact and T_1. If we identify each $x \in X$ with its fixed ultrafilter $\xi_x = \{F \in \mathcal{F}_X | x \in F\}$, we get an embedding of X into wX and hence wX is a T_1-compactification of X, called the *Wallman compactification of* X *associated* to $\mathcal{F}_X$. In case X is a T_3-space, wX is a *nearly* T_2*-extension* of X, that is, if a,b are different points of wX and at least one of them belongs to X, then a and b have disjoint neighborhoods. We need the following properties of wX (the reader may easily provide the proofs):

2.1. *Let* $F_1, \cdots, F_n \in \mathcal{F}_X$, *where* X *is a* T_1*-space. Then* $Cl_{wX}(F_1 \cap \cdots \cap F_n) = \cap_{i=1}^n Cl_{wX}F_i$.

2.2. *Let* X *be a* T_3*-space. If* A *and* B *are disjoint closed subsets of* wX *and* $A \subset X$, *then* A *and* B *have disjoint neighborhoods.*

Let A be a subset of a space X. A family $\mathcal{G}$ of subsets of X containing A is a *local net* of A in X if for every open $U \supset A$, there exists an element $G \in \mathcal{G}$ contained in U. A local net of A in X consisting of open sets is called a *local basis* of A in X.

A space X is of *point countable type* if each point of X lies in a compact set having a countable local basis in X.

X is of *countable type* if each compact set in X lies in a compact set having a countable local basis in X.

A family $\{\mathcal{G}_\alpha | \alpha \in J\}$ of open covers of a space X is a *pluming* of X if the following conditions are fulfilled:

1) If $G_\alpha \in \mathcal{G}_\alpha$, $\alpha \in J$ and $\cap_{\alpha \in J} G_\alpha \neq \Phi$, then $\cap_{\alpha \in J} \overline{G}_\alpha$ is compact;

2) If $G_\alpha \in \mathcal{G}_\alpha$, $\alpha \in J$ and $\cap_{\alpha \in J} G_\alpha \neq \Phi$, then the family of finite intersections $\cap_{i=1}^n \overline{G}_{\alpha_i}$, $n \in \mathbb{N}$, $\alpha_1, \cdots, \alpha_n \in J$ is a local net of $\cap_{\alpha \in J} \overline{G}_\alpha$ in X.

Hodel proves in [5] that every T_3-space has a pluming. The least infinite cardinal number λ such that X has a pluming $\{\mathcal{G}_\alpha | \alpha \in J\}$ with $|J| \leq \lambda$ is called the *pluming degree* of X. A T_3-space X is a p-*space* if $\mathrm{pl}X = \aleph_0$.

3. Main Results

For each space X, $\Delta(X)$ denotes its diagonal $\{(x,x) | x \in X\}$.

3.1. *Let* X *be a* T_3*-space and let* λ *be an infinite cardinal number. Then* $\mathrm{pl}X \leq \lambda$ *iff there exists a set* $A \subset X \times wX$ *such that* $\Delta(X) \subset A \subset X \times X$ *and such that* A *is the intersection of at most* λ *open subsets of* $X \times wX$.

Proof (Necessity). Let $\{\mathcal{G}_i | i < \lambda\}$ be a pluming of X. For each open G in X, let $G' = wX - \mathrm{Cl}_{wX}(X - G)$. Define $U_i = \cup\{G \times G' | G \in \mathcal{G}_i\}$ and $A = \cap\{U_i | i < \lambda\}$. We have only to prove that $A \subset X \times X$. Assume, on the contrary, that there exists a point $(x,z) \in A \cap (X \times (wX - X))$. For each $i < \lambda$, obtain a set $G_i \in \mathcal{G}_i$ such that $(x,z) \in G_i \times G_i'$. By the definition of pluming, $L = \cap\{G_i^- | i < \lambda\}$ is compact and

$$\{G_{i_1}^- \cap \cdots \cap G_{i_k}^- | k \in \mathbb{N},\ i_1,\cdots,i_k < \lambda\}$$

is a local net of L in X. By 2.2, there exists an open set T in wX such that

$$L \subset T \subset \mathrm{Cl}_{wX}T \subset wX - \{z\}.$$

Hence there exist $i_1,\cdots,i_k < \lambda$ such that

$$G_{i_1}^- \cap \cdots \cap G_{i_k}^- \subset T.$$

But by 2.1,

$$\cap_{j=1}^k \mathrm{Cl}_{wX}G_{i_j} = \cap_{j=1}^k \mathrm{Cl}_{wX}G_{i_j}^- = \mathrm{Cl}_{wX} \cap_{j=1}^k G_{i_j}^- \subset \mathrm{Cl}_{wX}T$$

Therefore, for some j, $z \notin \mathrm{Cl}_{wX}G_{i_j}$. Since X is dense in wX, we have $G_{i_j}' \subset \mathrm{Cl}_{wX}G_{i_j}$. Hence, $z \notin G_{i_j}'$, a contradiction.

(Sufficiency). By hypothesis, there exist open sets $\{U_i \mid i < \lambda\}$ in $X \times wX$ such that $A = \cap\{U_i \mid i < \lambda\}$ lies between $\Delta(X)$ and $X \times X$. For each $i < \lambda$, let

$$\mathcal{G}_i = \{G \mid G \text{ open in } X,\ G^- \times Cl_{wX}G \subset U_i\}$$

Clearly each $\mathcal{G}_i$ is an open cover of X. To prove $\{\mathcal{G}_i \mid i < \lambda\}$ is a pluming of X, take $G_i \in \mathcal{G}_i\, (i < \lambda)$ and assume there is a point $x \in \cap\{G_i \mid i < \lambda\}$. We must prove $L = \cap\{G_i^- \mid i < \lambda\}$ is compact and

$$\{G_{i_1}^- \cap \cdots \cap G_{i_k}^- \mid k \in \mathbb{N},\ i_1, \cdots, i_k < \lambda\}$$

is a local net of L in X. Observe

$$\cap\{Cl_{wX}G_i \mid i < \lambda\} \subset X:$$

if $z \in \cap\{Cl_{wX}G_i \mid i < \lambda\}$ and $z \notin X$, then $(x,z) \notin A$ and hence $(x,z) \notin U_i$ for some $i < \lambda$. But $(x,z) \in G_i^- \times Cl_{wX}G_i \subset U_i$, a contradiction. Therefore, $L = \cap\{Cl_{wX}G_i \mid i < \lambda\}$ is a compact subset of X. Let $T \subset X$ be open and assume $L \subset T$. Let T' be any open set in wX such that $T = X \cap T'$. According to [4; 2.27], there exist $i_1, \cdots, i_k < \lambda$ such that $\cap_{j=1}^{k} Cl_{wX}G_{i_j} \subset T'$. Therefore,

$$\cap_{j=1}^{k} G_{i_j}^- \subset T' \cap X = T$$

and the proof is complete.

3.2. *Lemma. Let* Q *be a compact subset of a* T_3*-space* X *and let* $V_1, V_2, \cdots$ *be open sets in* $X \times wX$ *such that* $\Delta(Q) \subset A = \cap_{i=1}^{\infty} V_i \subset X \times X$. *Then there exists a compact set* $K \subset X$ *such that* K *has a countable local basis in* X *and* $Q \subset K \subset X$.

Proof. For each $i = 1, 2, \cdots$, let $\mathcal{G}_i$ be the family of open non-empty subsets of $X \times wX$ which may be written in the form $(V \cap X) \times V$, with V open in wX, and such that $\mathrm{Cl}_{wX}[(V \cap X) \times V] \subset V_i$. 2.2 implies that each $\mathcal{G}_i$ is a cover of $\Delta(X)$. Let $\mathcal{U}_1 \subset \mathcal{G}_1$ be a finite subfamily such that $\mathcal{U}_1$ covers $\Delta(Q)$ irreducibly. Proceeding by induction, assume we have finite families $\mathcal{U}_1,\cdots,\mathcal{U}_n$ such that $\mathcal{U}_i \subset \mathcal{G}_i$, $\mathcal{U}_i$ covers $\Delta(Q)$ irreducibly $(i = 1,\cdots,n)$ and such that $\{\mathrm{Cl}_{X\times wX}L \mid L \in \mathcal{U}_{i+1}\}$ refines $\mathcal{U}_i$ for $i < n$. We find then a finite subfamily $\mathcal{U}_{n+1}$ of $\mathcal{G}_{n+1}$ such that $\mathcal{U}_{n+1}$ covers $\Delta(Q)$ irreducibly and $\{\mathrm{Cl}_{X\times wX}L \mid L \in \mathcal{U}_{n+1}\}$ refines $\mathcal{U}_n$. Let $q \in Q$ be arbitrary. Let $(T \cap X) \times T$ be the intersection of members of $\mathcal{U}_n$ containing (q,q) and let W_q be an open set in wX such that

$$(q,q) \in (W_q \cap X) \times W_q \subset \mathrm{Cl}_{X\times wX}[(W_q \cap X) \times W_q] \subset V_{n+1} \cap [(T \cap X) \times T].$$

By definition, $(W_q \cap X) \times W_q \subset \mathcal{G}_{n+1}$. Since $\Delta(Q)$ is compact, a finite subcollection $\mathcal{U}_{n+1}$ of the family $\{(W_q \cap X) \times W_q \mid q \in Q\}$ covers $\Delta(Q)$ irreducibly and $\mathcal{U}_{n+1}$ fulfills the required properties.

Define now $K = \bigcap_{n=1}^{\infty} S_n$, where

$$S_n = \bigcup\{V \mid (V \cap X) \times V \in \mathcal{U}_n\}.$$

By construction, $\mathrm{Cl}_{wX}S_{n+1} \subset S_n$ for each $n = 1,2,\cdots$. Hence, K is a compact G_δ in wX. To complete the proof, it will be enough to show that $K \subset X$. Assume, on the contrary, that there exists a point $z \in K - X$. For each $n = 1,2,\cdots$, pick a sequence $W_1^{(n)},W_2^{(n)},\cdots,W_n^{(n)}$ of open sets in wX such that $(W_i^{(n)} \cap X) \times W_i^{(n)} \in \mathcal{U}_i$, $z \in W_i^{(n)}$ for each $i = 1,\cdots,n$ and such that $\mathrm{Cl}_{wX}W_{i+1}^{(n)} \subset W_i^{(n)}$ for $i = 1,\cdots,n-1$. Since the families $\mathcal{U}_1,\mathcal{U}_2,\cdots$ are finite, there exists indices

$\lambda_1, \lambda_2, \cdots$ such that $1 \leq \lambda_1 < \lambda_2 < \cdots$ and $Cl_{wX} W_{n+1}^{(\lambda_{n+1})} \subset W_n^{(\lambda_n)}$ for $n = 1,2,\cdots$. For brevity, put $W_n = W_n^{(\lambda_n)}$. Therefore $Cl_{wX} W_{n+1} \subset W_n$ for $n = 1,2,\cdots$. Select a point $q^* \in Q \cap \bigcap_{n=1}^{\infty} (W_n \cap X)^-$. Then $(q^*,z) \in \bigcap_{n=1}^{\infty} [(W_n \cap X) \times W_n] \subset A \subset X \times X$, a contradiction.

3.3. *Theorem. Let* X *be a* T_3*-space and let* $\mathcal{G}$ *be the family of* G_δ *subsets of* $X \times wX$ *which are contained in* $X \times X$. *Then:*

a) X *is of point countable type iff every point of* $\Delta(X)$ *lies in an element of* $\mathcal{G}$.

b) X *is of countable type iff every compact subset of* $\Delta(X)$ *lies in an element of* $\mathcal{G}$.

c) X *is a* p*-space iff* $\Delta(X)$ *lies in an element of* $\mathcal{G}$.

Proof. Lemma 3.2 takes care of the sufficiency conditions. c) is a direct consequence of 3.1. The proof will be complete if we show that whenever K is a compact subset of X having a countable local basis in X, then $X \times K \in \mathcal{G}$. Let $V_1 \supset V_2 \supset \cdots$ be open sets in wX such that $V_1 \cap X$, $V_2 \cap X, \cdots$ is a local basis of K in X. Since X is dense in wX and wX is compact, $V_1 \supset V_2 \supset \cdots$ is a local basis of K in wX. Hence, $K = \bigcap_{n=1}^{\infty} V_n$ and K is a G_δ is wX. Clearly $X \times K$ is a G_δ in $X \times wX$ and $X \times K \subset X \times X$, that is $X \times K \in \mathcal{G}$.

3.3.1. *Corollary. Every* p*-space is of countable type.*

Bibliography

[1] A. V. Arhangel'skii, *On a class of spaces containing all metric and all locally bicompact spaces*, Soviet Math. Dokl. 4 (1963), 1051-1055. Amer. Math. Soc. Transl. 2 (1970), 1-39.

[2] D. K. Burke, *On* p-*spaces and* wΔ-*spaces*, Pacific J. Math. 35 (1970), 285-296.

[3] J. Chaber, M. M. Coban and K. Nagami, *On monotonic generalizations of Moore spaces, Cech complete spaces and* p-*spaces*, Fund. Math. 84 (1974), 107-119.

[4] A. García-Máynez, *Introducción a la topología de conjuntos*, México, Trillas, 1971.

[5] R. E. Hodel, *On the weight of a topological space*, Proc. of the Amer. Math. Soc. 43 (1974), 470-474.

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