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1. Introduction

Let X be a metric continuum. Denoted by 2^X and C(X) the hyperspaces of nonempty closed subsets and subcontinua of X respectively and endow with the Hausdorff metric H.

In 1938 Wojkyslowski proved that 2^X is contractible if X is locally connected [11]. In 1942 Kelly [2] proved that the contractibility of 2^X is equivalent to the contractibility of C(X). Furthermore, he introduced a sufficient condition, namely property (3.2), for the contractibility of the hyperspace of metric continua. In 1978 Nadler [3] called the Kelley's condition property K and raised a question. Find a necessary and/or sufficient condition in terms of X in order that 2^X is contractible. In [6] a necessary condition, call it admissible condition, was given and introduced a notion of property C and proved that a space X with property C has a contractible hyperspace C(X) if and only if there is a continuous fiber map α such that $\alpha(x) \subset a(x)$ for each $x \in X$, where $a(x)$ is the admissible fiber at x. Subsequently Curtis [1] proved that C(X) is contractible if and only if there exists a lower semicontinuous set-valued map $\Phi: X \to C^2(X)$ such that for each $x \in X$, each element of $\Phi(x)$ is an ordered arc in C(X) between $\{x\}$ and X. The last two results do not fully provide the topological characterization of the space X having contractible hyperspaces. The obstruction lies on certain subsets of X, call it the $\mathcal{M}$-set of X, which is our object to investigate and to prove a theorem characterizing the contractibility of C(X) and a theorem on the hyperspace contraction of the image of confluent maps.

Let $\mu: C(X) \to I = [0,1]$ be a Whitney map [10] such that $\mu(x) = 0$ for each $x \in X$, and $\mu(X) = 1$. For each $x \in X$, we define a total fiber map $F: X \to 2^{C(X)}$ (not necessarily continuous) by $F(x) = \{A \in C(X) | x \in A\}$. An element $A \in F(x)$ is admissible at x if, for each $\varepsilon > 0$, there is $\delta > 0$ such that each y in the δ-neighborhood of x has an element $B \in F(y)$ such that $H(A,B) < \varepsilon$. For each $x \in X$, the collection $\mathcal{a}(x) = \{A \in F(x) | A$ is admissible at $x\}$ is called the admissible fiber at x. We say that the space X is admissible if $\mathcal{a}_t(x) = \mathcal{a}(x) \cap \mu^{-1}(t)$ is nonempty for each $(x,t) \in X \times I$. We define the $\mathcal{M}$-set of X to be the set $M = \{x \in X | F(x) \neq \mathcal{a}(x)\}$ and the points of $X \backslash M$ as K-points of X. We state here some known results in [7] and [9].

Theorem 1.0. *Let* X *be a metric continuum.*

1. *For each* $x \in X$, $\mathcal{a}(x)$ *is closed in* C(X), $\{x\} \in \mathcal{a}(x)$, *and* $X \in \mathcal{a}(x)$.

2. *If* $A \in \mathcal{a}(\xi)$ *and* $B \in A(x)$ *and* $\xi \in A \cap B$ *then* $A \cup B \in \mathcal{a}(x)$.

3. *For each* $B \in F(x)$, $C = \cup\{A \in \mathcal{a}(x) | A \subset B\} \in \mathcal{a}(x)$.

Theorem 1.1. *If* $h: X \times I \to C(X)$ *is a continuous icnreasing map such that* $\{x\} \in h(x,0)$ *then* $h(x,t) \in \mathcal{a}(x)$ *for* $(x,t) \in X \times I$. *Thus, if* C(X) *is contractible then* X *is an admissible space.*

Theorem 1.2. *For any metric continuum* X, *the following statements are equivalent:*

1. $F(x) = \mathcal{a}(x)$,
2. X *has property* K *at* x,
3. F *is continuous at* x.

Theorem 1.3. *Let* X *be any metric continuum. If* X *is locally connected then* X *has property* K *at* x.

2. $\mathcal{M}$-set

In this section, we investigate $\mathcal{M}$-sets of admissible spaces.

Proposition 2.1. *Let* X *be an admissible space. Then the components of its* $\mathcal{M}$-*set are nondegenerate.*

Proof. This proposition follows easily from the next proposition since $\mu(A) > 0$ if and only if A is nondegenerate.

Proposition 2.2. *Let* X *be an admissible space and* M *be its* $\mathcal{M}$-*set. For each* $x \in M$, *let* $\mathcal{M}_x = \{A \in \mathcal{a}(x) | A \subset M\}$. *Then there is a positive number* $t(x) \in I$ *such that* $\mathcal{a}_s(x) = \mathcal{a}(x) \cap \mu^{-1}(s) \subset \mathcal{M}_x$ *for* $0 \leq s < t(x)$.

Proof. Let $x \in M$. Since $F(x) \neq \mathcal{a}(x)$ there is $A_0 \in F(x)\backslash\mathcal{a}(x)$. We show the nonexistence of t(x) implies $A_0 \in \mathcal{a}(x)$. Suppose no such t(x) exists. Let $\varepsilon > 0$. There is $t_0 > 0$ such that the diameter of A is less than $\varepsilon/2$ for all $A \in F(x) \cap \mu^{-1}(t)$ with $0 \leq t \leq t_0$. There is t with $0 < t < t_0$ and $B \in \mathcal{a}_t(x)$ such that $B\backslash M \neq \emptyset$. One easily shows $H(A_0, A_0 \cup B) < \varepsilon/2$. Let $x_1 \in B\backslash M$. Then $F(x_1) = \mathcal{a}(x_1)$. Since $A_0 \cup B$ is a continuum, $A_0 \cup B \in \mathcal{a}(x_1)$. There is $\delta_1 > 0$ such that for each z with $d(z,x_1) < \delta_1$ there is

$C \in F(z)$ such that $H(A_0 \cup B, C) < \varepsilon/2$. Also, since $B \in \mathcal{a}(x)$, there is $\delta > 0$ such that for each y with $d(x,y) < \delta$ there is $D \in F(y)$ such that $H(B,D) < \min\{\delta_1, \varepsilon/2\}$. Consequently, let y be such that $d(x,y) < \delta$ and $D \in F(y)$ such that $H(B,D) < \min\{\delta_1, \varepsilon/2\}$. Then there is $z \in D$ such that $d(z,x_1) < \delta_1$. Let $C \in F(z)$ be such that $H(A_0 \cup B, C) < \varepsilon/2$. Since $z \in C \cap D$, we have $C \cup D \in F(y)$. By Lemma 1.4 [7] $H(A_0 \cup B, C \cup D) = H((A_0 \cup B) \cup B, C \cup D) \leq \max\{H(A_0 \cup B, C), H(B,D)\} < \varepsilon/2$. Therefore $H(A_0, C \cup D) \leq H(A_0, A_0 \cup B) + H(A_0 \cup B, C \cup D) < \varepsilon$. We conclude that $A_0 \in \mathcal{a}(x)$, a contradiction. Hence a positive number exists and the proposition is proved.

Corollary 2.3. *For each* $x \in M$, *let* $\tilde{\mathcal{M}}_x = \{A \in \mathcal{a}(x) \mid A \subset \overline{M}\}$. *Then there is a positive number* $\overline{t}(x)$ *such that* $\mathcal{a}_s(x) \subset \tilde{\mathcal{M}}_x$ *for* $0 \leq s \leq \overline{t}(x)$.

Proof. Choose $\overline{t}(x)$ with $0 < \overline{t}(x) < t(x)$.

We remark that since $F(x) \neq \mathcal{a}(x)$ for $x \in M$ any increasing contraction h of X in $C(X)$, if it exists, must take admissible elements as its values, the above propositions and corollary provide some insight into the behavior of such map h.

3. T-admissibility

We introduce another condition on admissible fiber of X to give a characterization of contractibility of $C(X)$ and a theorem on the contractibility of $C(\hat{X})$ when $\hat{X}$ is a confluent image of a T-admissible space X.

Definition 3.1. A metric continuum X is said to be top-admissible (abbreviated T-admissible) if, for each $(x,s) \in X \times I$ the following condition is true:

For each $A \in \mathcal{a}_s(x)$ and $t \in [s,1]$, there is an element $B \in \mathcal{a}_t(x)$ such that $A \subset B$.

Since $\mathcal{a}_0(x) = \{x\}$ for each $x \in X$, we have that T-admissibility implies admissibility of a space X. We make a further remark that the contractibility of C(X) implies T-admissibility of X.

Proposition 3.2. *Let* X *be* T-*admissible. Suppose* M_α *is a component of the* $\mathcal{M}$-*set* M *of* X. *Then, for each* $x \in M_\alpha$ *and each* $t \in [0,\mu(\overline{M}_\alpha)]$, *there is an element* $A \in \mathcal{a}_t(x)$ *such that* $A \subset \overline{M}_\alpha$.

Proof. Let $S = \{t \in [0,\mu(\overline{M}_\alpha)] \mid \exists A \in \mathcal{a}_t(x) \ni A \subset \overline{M}_\alpha\}$. Obviously $0 \in S$. Since $\mathcal{a}(x)$ is a compact set in C(X) by Theorem 1.0 and μ is continuous we have that S is closed. Suppose $[0,\mu(\overline{M}_\alpha)]\backslash S \neq \emptyset$. Then there are t_0,t_1 such that $t_0 \in S$, $0 < t_0 < t_1 \leq \mu(\overline{M}_\alpha)$ and $(t_0,t_1) \cap S = \emptyset$. Let $t \in (t_0,t_1)$ and $A_0 \in \mathcal{a}_{t_0}(x)$ with $A_0 \subset \overline{M}_\alpha$. Then there is a $B \in F(x)$ such that $A_0 \subset B \subset \overline{M}_\alpha$ and $\mu(B) = t$. Since $t \in [0,\mu(\overline{M}_\alpha)]\backslash S$, we conclude that $B \notin \mathcal{a}_t(x)$. By T-admissibility, there is for each positive integer n an element $A_n \in \mathcal{a}(x)$ such that $A_0 \subset A_n$, $t_0 < \mu(A_n) < t_1$ and $\lim_{n\to\infty} \mu(A_n) = t_0$. Then, the sequence A_n converges to A_0 in C(X). Since $\mu(A_n) \in (t_0,t_1)$ we have $A_n\backslash\overline{M}_\alpha \neq \emptyset$. Consequently $A_n\backslash M \neq \emptyset$ because M_α is a component of M. Let $x_n \in A_n\backslash M$. Then $F(x_n) = \mathcal{a}(x_n)$, $A_n \cup B \in \mathcal{a}(x_n)$ and

$x_n \in A_n \in \mathcal{a}(x)$. By Theorem 1.0 $A_n \cup B = (A_n \cup B) \cup A_n \in \mathcal{a}(x)$. Since $A_n \cup B$ converges to $A_0 \cup B = B$ in $C(X)$, we have by the compactness of $\mathcal{a}(x)$ that $B \in \mathcal{a}(x)$. Since $\mu(B) = t$ and $B \subset \overline{M}_\alpha$, we have $t \in S$, a contradiction. We conclude that $S = [0, \mu(\overline{M}_\alpha)]$ and the proposition is proved.

We remark that there is an example of a T-admissible space X having contractible hyperspace C(X) and connected $\mathcal{M}$-set M in which there is an element $A \in \mathcal{a}(x)$ for some $x \in M$ such that $0 < \mu(A) < \mu(\overline{M})$ and $A \backslash \overline{M} \neq \emptyset$.

Proposition 3.3. *Let* X *be* T-*admissible. Suppose* M_α *is a component of the* $\mathcal{M}$-*set* M *of* X. *Then for each* $x \in M_\alpha$ *and* $B \in F(x)$ *such that* $M_\alpha \subset B$ *we have* $B \in \mathcal{a}(x)$.

Proof. The proof is similar to that of Proposition 3.2. Let $S = \{t \in [\mu(\overline{M}_\alpha), 1] \mid B \in F_t(x) \text{ and } B \supset M \Rightarrow B \in \mathcal{a}(x)\}$. Since $B \in C(X)$, $B \supset M_\alpha$, $\mu(B) = \mu(\overline{M}_\alpha)$ imply $B = \overline{M}_\alpha$, Proposition 3.2 yields $\mu(\overline{M}_\alpha) \in S$. Moreover, $1 = \mu(X)$ implies $1 \in S$. Once S is proved to be closed, the connectedness of S is proved with an argument similar to that found in Proposition 3.2. We prove the closedness of S and leave the connectedness of S to the reader.

Let t be a limit point of S and let $t_n \in S$ such that $t_n \to t$ as $n \to \infty$. We may suppose $t > \mu(\overline{M}_\alpha)$. Let $B \in F(x)$, $\mu(B) = t$ and $B \supset M_\alpha$. If $\mu(\overline{M}_\alpha) \leq t_n < t$, there is $A_n \in C(X)$ such that $\overline{M}_\alpha \subset A_n \subset B$ and $\mu(A_n) = t_n$. Since $t_n \in S$ and $M_\alpha \subset A_n$ we have $A_n \in \mathcal{a}(x)$. If $t < t_n$, there is $A_n \in C(X)$ such that $B \subset A_n$ and $\mu(A_n) = t_n$. Since $t_n \in S$ and $M_\alpha \subset A_n$ we have $A_n \in \mathcal{a}(x)$. Because $\mu(A_n) \to \mu(B)$ as $n \to \infty$ and either $B \subset A_n$ or $A_n \subset B$, we have A_n converging to B in C(X).

Since $a(x)$ is compact in $C(X)$ we have $B \in a(x)$. S is now proved to be closed.

Definition 3.4. Let N and Z be subcontinua of X such that $N \subset Z$.

A set-valued function α: $N \to C(Z)$ is a fiber function if, for each $x \in N$, (1) $\alpha(x) \subset a(x)$, (2) $\{\{x\},Z\} \subset \alpha(x)$, and (3) $\alpha(x)$ is path-connected. α is monotone-connected (4) if there is a path in $\alpha(x) \cap C(A)$ between $\{x\}$ and A for each $A \in \alpha(x)$. A monotone-connected, lower semicontinuous fiber function α: $X \to C(X)$ is called a c-function for X.

We rephrase Curtis' result [1] here in terms of c-function to prove the next theorem. $C(X)$ is contractible if and only if there is a c-function α: $X \to C(X)$.

Theorem 3.5. *Let* X *be a* T-*admissible space with its* $\mathcal{M}$-*set* M. *Then* $C(X)$ *is contractible if and only if there exists a subcontinuum* Z *of* X *containing* M *and a monotone-connected lower semicontinuous fiber function* α': $\overline{M} \to C(Z)$.

Proof. Suppose $C(X)$ is contractible. Let h: $X \times I \to C(X)$ be an increasing contraction map [7]. Then $h(x,t) \in a(x)$ for each $x \in X$ and the set-valued function α defined by $\alpha(x) = \{h(x,t) \mid t \in I\}$ is a c-function for X. The restriction of α on $\overline{M}$ is a monotone-connected continuous fiber map on $\overline{M}$ into $C(X)$. For the converse, we let $\mathcal{S}$ be a monotone segment from Z to X which is provided by [2]. Since X is T-admissible and $M \subset Z$, by Proposition 3.3, each element of $\mathcal{S}$ is admissible at each point of M. If $x \in \overline{M} \setminus M$, then such a point is a K-point, thus, element of

$\mathcal{S}$ containing x is admissible at x. Define a set-valued function $\alpha: X \to C(X)$ by

$$\alpha(x) = \begin{cases} a(x), & x \in X \setminus \overline{M} \\ \alpha'(x) \cup \mathcal{S}, & x \in \overline{M}. \end{cases}$$

Let $x \in X \setminus \overline{M}$. Then x is a K-point and hence $a(x) = F(x)$. The total fiber $F(x)$ is always path-connected and monotone-connected by [2]. If $x \in \overline{M}$ then $\alpha'(x)$ is monotone-connected and $A \subset Z$ for all $A \in \alpha'(x)$ and $\mathcal{S}$ is a monotone segment from Z to X. Thus $\alpha'(x) \cup \mathcal{S}$ is monotone-connected.

To prove the lower semicontinuity of α, let $x \in X \setminus \overline{M}$. Then x is a K-point and $\alpha(x) = a(x) = F(x)$. Therefore α is continuous at x by [9].

Suppose $x \in \overline{M}$, $A_0 \in \alpha'(x) \cup \mathcal{S}$, and $\varepsilon > 0$. Suppose $A_0 \in \alpha'(x)$. Since α' is lower semicontinuous at x in $\overline{M}$, there exists $\delta_1 > 0$ such that each point y in the δ_1-neighborhood of x in $\overline{M}$ has an element $B \in \alpha'(y)$ such that $H(A_0,B) < \varepsilon$. Also, since $A_0 \in a(x)$, there is $\delta_2 > 0$ such that each point y in the δ_2-neighborhood of x in X has an element $B \in F(y)$, $F(y) = a(x)$ if $y \in X \setminus M$, such that $H(A_0,B) < \varepsilon$. Combining the above two statements for $\delta = \min\{\delta_1, \delta_2\}$, each point y in the δ-neighborhood of x in X has an element $B \in \alpha(y)$ such that $H(A_0,B) < \varepsilon$. Thus we conclude that α is a c-function for X. Hence by [1], $C(X)$ is contractible.

Since it is rather easier to obtain a monotone-connected fiber function $\alpha: \overline{M} \to C(\overline{M})$ and in view of Proposition 2.2 and Corollary 2.3, we state the following corollaries.

Corollary 3.6. *Suppose* X *is a* T-*admissible space with a locally connected and connected subspace* M *as its* $\mathcal{M}$-*set such that each element* $A \in F(x) \cap C(\overline{M})$ *is admissible at* x *in* X *for* $x \in \overline{M}$. *Then* C(X) *is contractible.*

Proof. Let $A \in F(x) \cap C(\overline{M})$, $x \in M$, and $\varepsilon > 0$. Then there is an arbitrarily small connected neighborhood N of x in M such that $H(A, A \cup \overline{N}) < \varepsilon$ and the element $A \cup \overline{N} \in F(x) \cap C(\overline{M})$.

Define $\alpha: \overline{M} \to C(\overline{M})$ by $\alpha(x) = F(x) \cap C(\overline{M})$. Then α is a monotone-connected fiber function.

Corollary 3.7. *Suppose the* $\mathcal{M}$-*set* M *of a* T-*admissible space* X *is the union of two components* M_1 *and* M_2 *with* $\overline{M}_1 \cap \overline{M}_2 = \emptyset$. *If there is a lower semi-continuous monotone-connected fiber function* $\alpha_i^!: \overline{M}_i \to C(\overline{M}_i)$, $i = 1,2$. *Then* C(X) *is contractible.*

In [4], we introduced a notion of a space X being contractible im kleinen at a closed set K and proved that if C(X) is contractible and X is contractible im kleinen at K then the hyperspace C(X/K) is contractible. In this line, we use the T-admissiblity condition on admissible fiber of X to investigate certain confluent maps associated with the $\mathcal{M}$-set of X and the contractibility of the hyperspace of the quotient space $X/\overline{M}$.

We recall the definition of a confluent map. Let X and $\hat{X}$ be continua. A map $f: X \to \hat{X}$ is called confluent if f is a continuous surjection such that for each component B of $f^{-1}(\hat{B})$ of each subcontinuum $\hat{B}$ of $\hat{X}$ it is true that $f(B) = \hat{B}$. Clearly, continuous monotone surjections are confluent.

Lemma 3.8. *Let* $f\colon X \to \hat{X}$ *be a confluent map and* M *be the* $\mathcal{M}$*-set of* X. *Suppose* $M \cap f^{-1}(\hat{x}) = \emptyset$. *Then* $\hat{x}$ *is a* K*-point of* $\hat{X}$.

Proof. Let $\hat{H}$ denote the Hausdorff metric on $C(\hat{X})$ and $f^*\colon C(X) \to C(\hat{X})$ be the map induced by f. Then f* is uniformly continuous. Let $\varepsilon > 0$. There is $\delta_1 > 0$ such that $H(A,B) < \delta_1$, $A,B \in C(X)$ imply $\hat{H}(f(A), f(B)) < \varepsilon$. Let $x \in f^{-1}(\hat{x})$. Suppose $\hat{A} \in C(\hat{X})$ such that $\hat{x} \in \hat{A}$ and denote by A the component of $f^{-1}(\hat{A})$ containing x. Since x is a K-point of X there is $\eta_x > 0$ such that for each y in the η_x-neighborhood of x there is $B \in F(y)$ such that $H(A,B) < \delta_1$. By the confluency of f we have $\hat{H}(\hat{A}, f(B)) = \hat{H}(f(A), f(B)) < \varepsilon$. The compactness of $f^{-1}(\hat{x})$ implies there is $\eta > 0$ such that for each y in the η-neighborhood V of $f^{-1}(\hat{x})$ there is $B \in F(y)$ such that $\hat{H}(\hat{A}, f(B)) < \varepsilon$. There is $\delta > 0$ such that the δ-neighborhood W of $\hat{x}$ in $\hat{X}$ has $f^{-1}(W) \subset V$. For each $\hat{y}$ in the δ-neighborhood of $\hat{x}$ we have $f^{-1}(\hat{y}) \subset V$. Let $y \in f^{-1}(\hat{y})$. Then there is $B \in F(y)$ such that $\hat{H}(\hat{A}, f(B)) < \varepsilon$. Since $\hat{y} = f(y) \in f(B)$, we have $\hat{x}$ is a K-point of $\hat{X}$.

The above lemma includes a result of [9] where the $\mathcal{M}$-set is assumed to be empty.

Lemma 3.9. *Let* $f\colon X \to \hat{X}$ *be a confluent map and* M *be the* $\mathcal{M}$*-set of* X. *Suppose* X *is* T*-admissible and* $\hat{x} \in \hat{X}$ *is such that, for each component* M_α *of* M*, either* $M_\alpha \cap f^{-1}(\hat{x}) = \emptyset$ *or* $f^{-1}(\hat{x}) \supset M_\alpha$. *Then* $\hat{x}$ *is a* K*-point of* $\hat{X}$.

Proof. The proof is similar to that of the previous lemma. Let $x \in M \cap f^{-1}(\hat{x})$ and denote by M_α the component of M containing x. Then $M_\alpha \subset f^{-1}(\hat{x})$. As before, let $\hat{A} \in C(\hat{X})$ such that $\hat{x} \in \hat{A}$ and denote by A the component of $f^{-1}(\hat{A})$ containing x. Then $M_\alpha \subset A$ and hence $A \in \mathcal{a}(x)$ by Proposition 3.3. Consequently, there is $n_x > 0$ such that each y in the n_x-neighborhood of x has an element $B \in F(y)$ such that $H(A,B) < \delta_1$. The proof is completed just as in Lemma 3.8.

Immediate consequences are following.

Theorem 3.10. *Let* X *be* T-*admissible and* $f: X \to \hat{X}$ *be confluent. If, for each component* M_α *of the* $\mathcal{M}$-*set of* X *and each* $\hat{x} \in \hat{X}$, *either* $M_\alpha \cap f^{-1}(\hat{x}) = \emptyset$ *or* $M_\alpha \subset f^{-1}(\hat{x})$, *then* $\hat{X}$ *has property* K *and hence* $C(\hat{X})$ *is contractible.*

Corollary 3.11. *Let* X *be* T-*admissible. If the* $\mathcal{M}$-*set* M *of* X *is connected then the quotient space* $X/\overline{M}$ *has property* K *and hence* $C(X/\overline{M})$ *is contractible.*

4. Obstructing Sets

In [5], we introduced the notion of S-point and proved that any space having an S-point does not have contractible hyperspaces. In this section, we generalize this notion. Let X be a nonvoid metric continuum. By Theorem 1.0, each admissible fiber $\mathcal{a}(x)$ is nonempty. However, if X is not an admissible space, there is an element $(x,t) \in X \times I$ such that $\mathcal{a}_t(x) = \mathcal{a}(x) \cap \mu^{-1}(t) = \emptyset$. This occurs at some point x of the $\mathcal{M}$-set of X.

Proposition 4.1. *Suppose* $\mathcal{a}_t(x) = \emptyset$ *for some* $(x,t) \in X \times I$. *Let* $S = \{t \in I | \mathcal{a}_t(x) = \emptyset\}$. *Then* S *is nonempty open* *subset of the reals* R *contained in* I. *Moreover, if* $t_0 \in I\backslash S$ *such that* $t_0 = \text{glb } S'$, *for some nonempty subset* $S' \subset S$, *then* $\mathcal{a}_{t_0}(x) \subset \mathcal{M}_x = \{A \in \mathcal{a}_{t_0}(x) | A \subset M\}$. *In particular, if* $s_0 = \text{glb } S$, *then* $\mathcal{a}_s(x) \subset \mathcal{M}_x$ *for all* $0 \leq s \leq s_0$.

Proof. Let s be a limit point of S. For each positive integer n there is $s_n \in I\backslash S$ such that $|s_n - s| < \frac{1}{n}$. Let $A_n \in \mathcal{a}(x) \cap \mu^{-1}(s_n)$. Since $\mathcal{a}(x)$ is compact in C(X), we may assume that the sequence A_n converges to $A_0 \in \mathcal{a}(x)$. Because μ is continuous we have $A_0 \in \mathcal{a}(x) \cap \mu^{-1}(s)$. Hence $s \in I\backslash S$ and S is open in I. Since $0 \notin S$, $1 \notin S$ we have S is open in R.

The proof of the second assertion is similar to that of the last assertion. Let $0 \leq s \leq s_0$, $s_0 = \text{glb } S$. We suppose there is $B \in \mathcal{a}_s(x)$ such that $B\backslash M \neq \emptyset$. Let $\xi \in B\backslash M$. Then $F(\xi) = \mathcal{a}(\xi)$. There is a monotone segment $\mathcal{S}$ from B to X in C(X) by [3]. Since $\mu(\mathcal{S}) = [s,1]$, there is an element $A \in \mathcal{S}$ such that $\mu(A) \in S$ and $A \supset B$. This means that A is not admissible at x. On the other hand, we have $A \in \mathcal{a}(\xi)$, $B \in \mathcal{a}(x)$ and $\xi \in A \cap B$. So by Theorem 1.0, $A = A \cup B \in \mathcal{a}(x)$, a contradiction. Hence $\mathcal{a}_s(x) \subset \mathcal{M}_x$ for $0 \leq s \leq s_0$. The proposition is now proved.

If $s_0 = 0$ then $\mathcal{a}_{s_0}(x) = \{x\}$. In this case x is an S-point as defined in [5]. It is clear that the concept of S-point is independent of the choice of the Whitney function μ. By Theorem 1.1, an increasing continuous map $h: X \times I \to C(X)$ with $h(x,0) = \{x\}$ must have $h(x,t) \in \mathcal{a}(x)$ for all $(x,t) \in X \times I$. We have by Proposition 4.1 that such an h must stabilize in a subcontinuum (element) of $\mathcal{a}_{t_0}(x)$.

Let us call element of $\mathcal{a}_{t_0}(x)$ S-set.

Proposition 4.2. *If a metric continuum* X *contains an* S-*set then* C(X) *is not contractible.*

5. Examples

We will give three examples to illustrate Theorem 3.5 and Corollary 3.6 and an example of a space which contains an S-set.

Example 5.1. In the plane, let P_n and q_n, be points defined by $P_n = (0,\frac{1}{n})$, $q_n = (1,\frac{-1}{n})$, for n = 1,2,3,$\cdots$ and $P_0 = (0,0)$, $q_0 = (1,0)$. Let $\overline{P_n q_0}$ and $\overline{q_n P_0}$ be segments joining P_n to q_0 and q_n to P_0 respectively for n = 1,2,$\cdots$ and $\overline{P_0 q_0} = M$. Let $X = \cup_{n=1}^{\infty}(\overline{P_n q_0} \cup \overline{q_n P_0}) \cup M$. Then it is easy to check that X is T-admissible and M is the $\mathcal{M}$-set of X. For each $x \in M$, every element $A \in F(x) \cap C(M)$ is admissible at x in X. Therefore by Corollary 3.6, C(X) is contractible.

Example 5.2. Let X_1 be the closure of the graph of $\text{Sin}\,\frac{1}{x}$, $0 < x \leq 1$, and X_2 the graph of $\frac{1}{2}\,\text{Sin}\,\frac{1}{x}$, $-1 \leq x < 0$, and $X = X_1 \cup X_2$. Let $P_i = (0,i)$, $q_i = (0,\frac{-i}{2})$, $i = \pm 1$. Let M be the line segment joining P_1 to P_{-1}, and N the line segment joining q_1 to q_{-1}. Since the first coordinates of points of M are all 0, we will use the following notations. Denote the point (0,z) by z in M, and [z,w] denotes the closed segment in M joining the point (0,z) and (0,w), $z < w$.

Since X is locally connected at each point $z \in X\backslash M$, each element of F(z) is admissible at z. If $z \in M$, there are elements in F(z) which are not admissible at z. Thus M is the $\mathcal{M}$-set of X. Let α: $M \to C(M)$ be a set-valued function defined as follows:

$$\alpha'(z) = \begin{cases} (F(z) \cap C(M)) \cup \{[-\varepsilon,\varepsilon] \mid \frac{1}{2} \le \varepsilon \le 1\}, & z \in N \\ \{[z,\varepsilon] \mid z \le \varepsilon \le -z\} \cup \{[z-\varepsilon,z+\varepsilon] \mid 0 \le \varepsilon \le |1+z|\}, & \\ \qquad z \in M\backslash N, -1 \le z \le \frac{1}{2}, & \\ \{[\varepsilon,z] \mid -z \le \varepsilon \le z\} \cup \{[-z-\varepsilon,z+\varepsilon] \mid 0 \le \varepsilon \le |1-z|\}, & \\ \qquad \frac{1}{2} \le z \le 1 & \end{cases}$$

One can easily check that X is T-admissible and α' monotone-connected connected fiber function. Thus by Corollary 3.6, X admits a c-function.

Example 5.3. Let X_n be the closure in the plane of the set $\{(x,y+4n) \mid y = \sin \frac{1}{x}, 0 < X \le 1\}$, $n = 0,1,2,\cdots$ and let X be the one-point compactification of $\cup_{n=0}^{\infty} X_n$. Let $P \in X$ be the point at ∞, $q = (0,-1)$ and let M_n be the line segment joining the points $(0,4n-1)$ and $(0,4n+1)$, and Z the segment joining P and q, and let $M_0' = M_0 \backslash \{q\}$. Then M_0', M_n, $n = 1,2,\cdots$, are the components of the $\mathcal{M}$-set $M = (\cup_{n=1}^{\infty} M_n) \cup M_0'$ of X, and $\overline{M} = \cup_{n=0}^{\infty} M_n$. We note that P and q are K-points. To check the T-admissibility of X, it suffices to check $a(x)$, when $x \in M_n$. Since each element of $F(x) \cap C(M_n)$ is admissible at x, and every subcontinuum of X containing M_n is admissible at X, it is clear that $a(x)$ satisfies the T-admissibility condition. We define α: $\overline{M} \to C(Z)$ by, for $x \in M_n$

$$\alpha(x) = (F(x) \cap C(M_n)) \cup \{A \in F(x) \cap C(Z) \mid M_n \subset A\}.$$

Then α is a monotone-connected lower semicontinuous fiber function. Hence by Theorem 3.5, C(X) is contractible.

Example 5.4. Let $X = X_1 \cup X_2$, where X_1 is the closure of the graph of $\sin \frac{1}{x}$, $0 < x \leq 1$, and X_2 is the closure of the graph of $\frac{1}{2} + \sin \frac{1}{x}$, $-1 \leq x < 0$. Then the line segment joining the points $(0,1)$ and $(0,-\frac{1}{2})$ is the S-set of X.

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