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0. Introduction

A space is called a *Lašnev space* if it is the image of a metric space under a closed continuous mapping.

First we are concerned with products of Lašnev spaces. In spite of the recent simple and useful characterization of Lašnev spaces given by L. Foged [1], it is well known that Lašnev spaces behave quite badly with respect to product operations. Even a product of two Lašnev spaces does not preserve Fréchet property and countable tightness. An interesting general problem is to determine which spaces can be embedded in a finite or countable product of Lašnev spaces. We show that a subspace of the product of countably many Lašnev spaces is a metrizable space (resp. a Lašnev space) if and only if it is strongly Fréchet (resp. Fréchet).

Secondly we study the κ-metrizable property of Lašnev spaces. κ-metrizable spaces were introduced in 1976 by E. V. Ščepin (see [8]) as a generalization of metric spaces and locally compact topological groups. We show that a Lašnev space is metrizable if and only if it is κ-metrizable.

All spaces are assumed to be regular T_1. The letter N denotes the set of positive integers.

1. Subspaces of the Product of Countability Many Lašnev Spaces

A collection $\mathcal{P}$ of subsets of a space X is a *k-network* for X if, given a compact subset C of X and any neighborhood

U of C, there is a finite subcollection $\mathcal{Q}$ of $\mathcal{P}$ so that $C \subset \cup\mathcal{Q} \subset U$.

A collection $\mathcal{P}$ of subsets of a space X is *closure-preserving* if $\mathrm{Cl}(\cup\{P: P \in \mathcal{Q}\}) = \cup\{\mathrm{Cl}\, P: P \in \mathcal{Q}\}$ for any subcollection $\mathcal{Q}$ of $\mathcal{P}$. A collection $\mathcal{P}$ of subsets of X is *hereditarily closure-preserving* if, whenever a subset $R(P) \subset P$ is chosen for each $P \in \mathcal{P}$, the resulting collection $\mathcal{R} = \{R(P): P \in \mathcal{P}\}$ is closure-preserving.

A space X is a *Fréchet space* if for every $A \subset X$ and every $x \in \mathrm{Cl}\, A$ there exists a sequence $\{x_n\}$ of points of A converging to x. A space X is *strongly Fréchet* if whenever $\{A_n: n \in N\}$ is a decreasing sequence of sets in X and $x \in \cap\{\mathrm{Cl}\, A_n: n \in N\}$, there exists an $x_n \in A_n$ such that the sequence $\{x_n\}$ converges to x. A space is strongly Fréchet if and only if it is countably bisequential (see [3]). Note that every first countable space is strongly Fréchet.

For the product $X = \prod_{i=1}^{\infty} X_i$ of spaces, let $\pi_n: X \to X_n$ be the natural projection for each n.

L. Foged recently proved the following:

Theorem 1.1 [1]. *The following are equivalent for a space* X*:*

(a) X *is a Lašnev space;*

(b) X *is a Fréchet space with a* k*-network* $\mathcal{P} = \cup\{\mathcal{P}_n: n \in \mathbb{N}\}$ *where each* $\mathcal{P}_n$ *is a hereditarily closure-preserving collection;*

(c) X *is a Fréchet space with a* k*-network* $\mathcal{P} = \cup\{\mathcal{P}_n: n \in \mathbb{N}\}$ *where each* $\mathcal{P}_n$ *is a hereditarily closure-preserving and point finite collection;*

(d) X *is a Fréchet space with a* k-*network* $\mathcal{P} = \cup\{\mathcal{P}_n : n \in N\}$ *where each* $\mathcal{P}_n$ *is a hereditarily closure-preserving closed collection.*

The following theorem due to E. Michael is a generalization of the theorem of K. Morita, S. Hanai [4] and A. Stone [9] that a Lašnev space is metrizable if and only if it is first countable.

Theorem 1.2 [3]. *A Lašnev space is metrizable if and only if it is strongly Fréchet.*

In this section we prove:

Theorem 1.3. *Let* $\{X_n : n \in N\}$ *be a collection of Lašnev spaces. Then a subspace* X *of* $\prod\{X_n : n \in N\}$ *is Lašnev if and only if it is Fréchet.*

Combining Theorem 1.2 and Theorem 1.3, we have the following theorem which generalizes Theorem 1.2:

Theorem 1.4. *A subspace of the product of countably many Lašnev spaces is metrizable if and only if it is strongly Fréchet.*

Remark 1.5. Theorem 1.3 and Lemma 1.8 was suggested by the referee. Formerly the author proved Theorem 1.4 directly and it was an open question whether Theorem 1.3 is true or not. The author takes this opportunity to thank the referee for his many helpful comments and suggestions which much improve and simplify the results in this section.

Corollary 1.6. *A subspace of the product of countably many Lašnev spaces is metrizable if and only if it is first countable.*

K. Nagami [5] defined the class of free L-spaces as a generalization of Lašnev spaces. He asked: Does there exist a free L-space which cannot be embedded in the product of countably many Lašnev spaces? It is known that there exists a first countable non-metrizable free L-space [6]. It follows from the above corollary that the space is also a counterexample to his question as well as the example in [10].

We need two lemmas to prove Theorem 1.3. The following lemma is implicitly stated in [7].

Lemma 1.7. (a) *Let* $\mathcal{P}$ *be a* k-*network for a space* X, *and* Y *a subspace of* X. *Then* $\mathcal{Q} = \{P \cap Y: P \in \mathcal{P}\}$ *is a* k-*network for* Y.

(b) *Let* $\mathcal{P}_i$ *be a* k-*network for a space* X_i *for each* $i \in N$. *Define* $\mathcal{Q} = \{(\prod_{i=1}^{n} P_i) \times (\prod_{i=n+1}^{\infty} X_i): n \in N, P_i \in \mathcal{P}_i$ *for each* $i = 1,2,\cdots,n\}$. *Then* $\mathcal{Q}$ *is a* k-*network for the product* $\prod_{i=1}^{\infty} X_i$.

Lemma 1.8. *Let* $X_1, X_2, \cdots, X_n$ *be spaces, and* $\mathcal{P}_i$ *a hereditarily closure-preserving and point finite collection of* X_i *for each* $i = 1,2,\cdots,n$. *Suppose that a subspace* $X \subset \prod_{i=1}^{n} X_i$ *is a Fréchet space. Then the collection* $\mathcal{Q} = \{X \cap (\prod_{i=1}^{n} P_i): P_i \in \mathcal{P}_i$ *for each* $i = 1,2,\cdots,n\}$ *is hereditarily closure-preserving in* X.

Proof. Suppose $\mathcal{Q}$ is not hereditarily closure-preserving in X. Then by the Fréchet property of X, there is a sequence $\{x_j: j \in N\}$ converging to $x \in X$, with $x_j \in X \cap (\prod_{i=1}^n P_i^j)$, where the $\prod_{i=1}^n P_i^j$ are distinct members of $\prod_{i=1}^n \mathcal{P}_i$. We may assume that for some $m \leq n$, $\{P_m^j: j \in N\}$ are distinct members of $\mathcal{P}_m$.

For every $k \in N$, the collection $\{P_m^j: j \geq k\}$ is hereditarily closure-preserving in X_m. So $\{\pi_m(x_j): j \geq k\}$ is closed in X_m. Since $x \in \text{Cl}\ \{x_j: j \geq k\}$, $\pi_m(x) \in \text{Cl}\ \{\pi_m(x_j): j \geq k\} = \{\pi_m(x_j): j \geq k\}$. Thus $\pi_m(x) = \pi_m(x_j)$ for infinitely many $j \in N$, violating the point finiteness of $\mathcal{P}_m$.

Remark 1.9. In general, the product of hereditarily closure-preserving collections need not be hereditarily closure-preserving. For example, let X be an arbitrary space satisfying that there are a hereditarily closure-preserving collection $\{P_n: n \in N\}$ of X and a point x with $x \in \cap\{P_n: n \in N\}$. Let $Y = [0,1]$ be the unit interval. Define $\mathcal{P}_X = \{P_n: n \in N\}$ and $\mathcal{P}_Y = \{Y\}$. Then the product $\mathcal{P}_X \times \mathcal{P}_Y$ is not hereditarily closure-preserving. In fact, although $(x,1/n) \in P_n \times Y$ for each $n \in N$, the collection $\{\{(x,1/n)\}: n \in N\}$ is not closure-preserving.

Proof of Theorem 1.3. It is well known that every Lašnev space is Fréchet. So suppose that X is a Fréchet space. By Theorem 1.1, let $\mathcal{P}_i = \cup\{\mathcal{P}_{ij}: j \in N\}$ be a k-network for X_i so that each $\mathcal{P}_{ij}$ is hereditarily closure-preserving and point finite.

For a string $\sigma = \langle \sigma(1), \sigma(2), \cdots, \sigma(n) \rangle$ of length n in N, let $\mathcal{Q}(\sigma) = \{X \cap [(\prod_{i=1}^n P_i) \times (\prod_{i=n+1}^\infty X_i)]: P_i \in \mathcal{P}_{i\sigma(i)}\}$.

By Lemma 1.7, $\mathcal{Q} = \cup\{\mathcal{Q}(\sigma)$: σ is a finite string in N$\}$ is a k-network for X.

On the other hand, define $Y = \prod_{i=n+1}^{\infty} X_i$ and $\mathcal{P}_{n+1} = \{Y\}$. Apply Lemma 1.8 to the spaces $X_1, \cdots, X_n$, Y, the hereditarily closure-preserving and point finite collections $\mathcal{P}_{1\sigma(1)}, \cdots, \mathcal{P}_{n\sigma(n)}, \mathcal{P}_{n+1}$, and the Fréchet subspace $X \subset (\prod_{i=1}^{n} X_i) \times Y$. Then each $\mathcal{Q}(\sigma)$ is hereditarily closure-preserving in X.

Thus X is a Fréchet space with a σ-hereditarily closure-preserving k-network. Hence, by Theorem 1.1, X is a Lašnev space. The proof is completed.

Problem 1.10. Find an internal characterization of subspaces of the product of countably many Lašnev spaces.

Note that a space with a σ-hereditarily closure-preserving k-network need not be able to be embedded in a product of countably many Lašnev spaces. For example, let p denote a free ultrafilter on $\mathbb{N}$. Then $X = N \cup \{p\}$ cannot be embedded in any countable product of Lašnev spaces [10]. But since every compact subspace of X is a finite set, $\{\{x\}: x \in X\}$ is a k-network consisting of countable members.

Problem 1.11. Does a subspace of the product of countably many Lašnev spaces have a σ-hereditarily closure-preserving k-network?

2. κ-Metrizability of Lašnev Spaces

A κ-*metric* on a completely regular space X is a non-negative real-valued function $\rho(x,C)$ of two variables, a point $x \in X$ and a regular closed set C of X, satisfying the following conditions:

K1 (membership axiom) $\rho(x,C) = 0$ if and only if $x \in C$;

K2 (monotonicity axiom) If $C \subset C'$, then $\rho(x,C) \geq \rho(x,C')$ for every $x \in X$;

K3 (continuity axiom) $\rho(x,C)$ is continuous with respect to x for every C; and

K4 (union axiom) $\rho(x,\mathrm{Cl}(\cup_\alpha C_\alpha)) = \inf_\alpha \rho(x,C_\alpha)$ for every increasing transfinite sequence $\{C_\alpha\}$.

A space X is *monotonically normal* if to each pair (H,K) of disjoint closed subsets of X, one can assign an open set D(H,K) such that

(i) $H \subset D(H,K) \subset \mathrm{Cl}\ D(H,K) \subset X - K$;

(ii) if $H \subset H'$ and $K \supset K'$, then $D(H,K) \subset D(H',K')$.

The function D is called a *monotone normality operator* for X. Gruenhage [2] contains a brief survey on monotonically normal spaces, stratifiable spaces and the other generalized metric spaces.

The *sequential fan* is the space $S = N \times N \cup \{\infty\}$, where each point of $N \times N$ is an isolated point, and the collection $\{\{(m,n): n \geq \phi(m)\} \cup \{\infty\}: \phi$ is a function from N into N$\}$ forms a basis of neighborhoods of the point ∞.

Lemma 2.1. *The sequential fan* S *cannot be embedded in any monotonically normal* κ*-metrizable spaces.*

Proof. Suppose the contrary. Let X be a monotonically normal κ-metrizable spaces and $f: S \to X$ be an embedding. Let D and $\rho(x,C)$ be a monotone normality operator and a κ-metric for X respectively.

Define $x_{mn} = f(m,n)$, $x = f(\infty)$, and $B_{mn} = \mathrm{Cl}\ D(\{x_{mn}\},\{x\})$. Then $\{B_{mn}: m, n \in N\}$ is a collection of regular closed subsets of X satisfying the following:

(a) for each m, $x \in \mathrm{Cl}(\cup\{B_{mn}: n \in N\})$; and

(b) for each function $\phi: N \to N$, $x \notin \mathrm{Cl}(\cup\{B_{mn}: m, n \in N, n \leq \phi(m)\})$.

It is easy to see (a). (b) follows from the property of the monotone normality operator D. Whenever $n \leq \phi(m)$, $D(\{x_{mn}\},\{x\}) \subset D(\mathrm{Cl}\{x_{mn}: m, n \in N, n \leq \phi(m)\}, \{x\})$. So we have $\cup\{B_{mn}: m, n \in N, n \leq \phi(m)\} \subset \mathrm{Cl}\, D(\mathrm{Cl}\{x_{mn}: m, n \in N, n \leq \phi(m)\}, \{x\}) \subset X - \{x\}$. Hence $x \notin \mathrm{Cl}(\cup\{B_{mn}: m, n \in N, n \leq \phi(m)\})$. Now define $F_{mn} = \cup\{B_{mk}: k \leq n\}$. For each $m \in N$, by (a) and K1, $\rho(x, \mathrm{Cl}(\cup\{F_{mn}: n \in N\})) = \rho(x, \mathrm{Cl}(\cup\{B_{mn}: n \in N\})) = 0$. Since $\{F_{mn}: n \in N\}$ is an increasing sequence, by K4, there exists n_m with $\rho(x,F_{mn_m}) < 1/m$.

Let $C_m = \cup_{k=1}^{m} F_{kn_k}$ and $C = \mathrm{Cl}(\cup\{C_m: m \in N\})$. Observe that $\{C_m: m \in N\}$ is an increasing sequence and by K2, $\rho(x,C_m) \leq \rho(x,F_{mn_m}) < 1/m$. Thus by K4, $\rho(x,C) = \inf\{\rho(x,C_m): m \in N\} \leq 0$. But by (b), $x \notin C$ and hence by K1, $\rho(x,C) > 0$. Contradiction.

The following theorem gives a partial negative answer to Problem 6 of [8]: Is a quotient group (more generally, a quotient space) of a κ-metrizable topological group κ-metrizable?

Indeed, a non-metrizable image of a metrizable topological group under a closed continuous mapping is a quotient space of a κ-metrizable topological group and it is not κ-metrizable.

Theorem 2.2. *If a Lašnev space* X *is* κ*-metrizable, then it is metrizable.*

Proof. Let X be a non-metrizable Lašnev space. By Morita, Hanai and Stone's theorem ([4],[9]), the sequential fan S can be embedded in X. It is well known that every Lašnev space is stratifiable, hence monotonically normal. Using Lemma 2.1, X is not κ-metrizable, which completes the proof.

Remark 2.3. In the proof of Lemma 2.1 and Theorem 2.2, axiom K3 of the κ-metric is not needed.

Problem 2.4. Is a stratifiable space metrizable if it is κ-metrizable?

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Added in Proof. Recently the author gave a negative answer to Problem 2.4.

References

[1] L. Foged, *A characterization of closed images of metric spaces*, Proc. Amer. Math. Soc. 95 (1985), 487-490.

[2] G. Gruenhage, *Generalized metric spaces*, Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984.

[3] E. Michael, *A quintuple quotient quest*, Topology Appl. 2 (1972), 91-138.

[4] K. Morita and S. Hanai, *Closed mappings and metric spaces*, Proc. Japan Acad. 32 (1956), 10-14.

[5] K. Nagami, *Dimension of free* L*-spaces*, Fund. Math. 108 (1980), 211-224.

[6] A. Okuyama, *An example of an* L-, *Non*-D-*space*, Math. Japonica 28 (1983), 139-142.

[7] P. O'Meara, *On paracompactness in function spaces with the compact-open topology*, Proc. Amer. Math. Soc. 29 (1971), 183-189.

[8] E. V. Ščěpin, *On* κ-*metrizable spaces*, Math. USSR Izvestija 14 (1980), 407-440.

[9] A. Stone, *Metrizability of decomposition spaces*, Proc. Amer. Math. Soc. 7 (1956), 690-700.

[10] K. Tamano, *On a local property of products of Lašnev spaces*, Topology Appl. 14 (1982), 105-110.

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