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0. Introduction

Throughout this paper *uniformity* will mean separated diagonal uniformity and *proximity* will mean separated Efremovič proximity. If (X,δ) is a proximity space, then $\Pi(\delta)$ will denote the set of all uniformities on X which induce δ, and we shall call (X,δ) *completely uniformizable* when $\Pi(\delta)$ contains a complete uniformity. Also the Smirnov compactification of (X,δ) will be denoted by δX.

The purpose of this paper is to study properties of completely uniformizable proximity spaces. One known result [5, Theorem 2.2, p. 226] asserts that a completely uniformizable proximity space is Q-closed, but the converse of this assertion does not hold. In seeking a satisfactory characterization of completely uniformizable proximity spaces, one may consider the realcompact rich proximity spaces of [1]. A proximity space (X,δ) is *rich* if each realcompactification of X contained in δX can be realized as the uniform completion of a member of $\Pi(\delta)$. Thus, every realcompact rich proximity space is completely uniformizable. In section 1 we shall show that when a proximity space (X,δ) is completely uniformizable, every realcompactification of X contained in δX of the form $X \cup K$, where K is compact, can be obtained as the uniform completion of a member of $\Pi(\delta)$. The question of whether every completely uniformizable proximity space is rich remains unanswered.

Results on the cardinality of certain subsets of the outgrowths of Smirnov compactifications have appeared in [4], [5], and [6] where the notion of embedding uniformly discrete subspaces has played an important role. In section 2 we shall introduce a "local" version of this notion: compactifications with *locally* ω^**-embedded outgrowth*. We shall show that the Smirnov compactification of any completely uniformizable proximity space is of this type. Moreover, it will be shown that δX being a compactification of X with locally ω^*-embedded outgrowth is not sufficient for (X,δ) to be completely uniformizable.

The notion of locally ω^*-embedded outgrowth will be applied in section 3 to show that the Smirnov compactification of a noncompact, completely uniformizable proximity space (X,δ) contains as many nonrealcompact extensions of X as it does arbitrary extensions of X. This in turn provides a new result on the number of nonrealcompact extensions of a realcompact space contained in its Stone-Čech compactification.

Given a uniform space $(X,\mathcal{U})$ we shall let $\mathcal{U}X$ denote the set of all minimal $\mathcal{U}$-Cauchy filters on X. For $U \in \mathcal{U}$, we set

$$U^* = \{(\mathcal{F},\mathcal{G}) \in \mathcal{U}X \times \mathcal{U}X\colon \text{for some } F \in \mathcal{F} \cap \mathcal{G},\ F \times F \subset U\},$$

and we let $\mathcal{U}^*$ denote the uniformity on $\mathcal{U}X$ generated by the uniform base $\{U^*\colon U \in \mathcal{U}\}$. When we identify the points of X with their neighborhood filters in X, $(\mathcal{U}X,\mathcal{U}^*)$ becomes the canonical uniform completion of $(X,\mathcal{U})$.

Given a proximity space (X,δ) we shall let δX denote the set of all maximal δ-round filters on X. For $A \subset X$, we set

$$O(A) = \{\mathcal{F} \in \delta X: A \in \mathcal{F}\},$$

and we declare (for $E_1, E_2 \subset \delta X$) $E_1 \overline{\delta^*} E_2$ if and only if there are $A_1, A_2 \subset X$ with $A_1 \overline{\delta} A_2$ and $E_i \subset O(A_i)$ $(i = 1,2)$. When we identify the points of X with their neighborhood filters in X, $(\delta X, \delta^*)$ becomes the canonical proximity space underlying the Smirnov compactification of (X,δ). Moreover, if $\mathcal{U} \in \Pi(\delta)$, then the minimal $\mathcal{U}$-Cauchy filters coincide with the δ-round $\mathcal{U}$-Cauchy filters and every minimal $\mathcal{U}$-Cauchy filter is a maximal δ-round filter. Thus, $X \subset \mathcal{U}X \subset \delta X$; also the proximities $\delta(\mathcal{U}^*)$ and $\delta^*|_{\mathcal{U}X}$ agree (as do the topologies $\tau(\mathcal{U}^*)$ and $\tau(\delta^*)|_{\mathcal{U}X}$). We use $\mathcal{U}_\delta$ to denote the totally bounded member of $\Pi(\delta)$.

If Z_1 and Z_2 are Hausdorff extensions of a Tychonoff space X, we write $Z_1 =_X Z_2$ to mean that Z_1 and Z_2 are homeomorphic by a homeomorphism which fixes the points of X. We use βX to denote the Stone-Čech compactification of X. ω will denote the countable cardinal (least infinite ordinal), and c will denote 2^ω.

Some of the notions discussed in this paper were initially developed in [2].

1. $\hat{\sigma}$- completability

Rich proximity spaces were introduced in [1] as proximity spaces (X,δ) for which each realcompactification of X contained in δX, the Smirnov compactification of X, can be realized as the uniform completion of a uniformity on X belonging to $\Pi(\delta)$. More precisely, we have the following definition.

Definition 1.1. [1] Let δ be a compatible proximity on a Tychonoff space X.

(a) We say that X is δ-*completable* to a Tychonoff extension T of X if there is a compatible complete uniformity $\mathcal{V}$ on T such that $\delta(\mathcal{V}|_X) = \delta$.

(b) (X,δ) is a *rich* proximity space if X is δ-completable to every realcompactification of X contained in δX.

The proximity space induced on a Tychonoff space X by its Stone-Čech compactification βX is a rich proximity space. It is shown in [1] that there are realcompact, noncompact proximity spaces (X,δ) which are rich where δ is not induced by βX. However, the problem of finding an internal characterization of rich proximity spaces remains open.

It is clear that every realcompact rich proximity space must be completely uniformizable; so it is natural to ask if every completely uniformizable proximity space is rich. (Assuming the nonexistence of measurable cardinals, a completely uniformizable proximity space must be realcompact.) This is essentially a question about the realcompactifications to which a completely uniformizable proximity space (X,δ) is δ-completable.

Definition 1.2. Let T be a Tychonoff extension of a Tychonoff space X.

(a) T is a *finite-outgrowth (f.o.)* extension of X if $T = X \cup F$ where F is finite.

(b) T is a *relatively-compact-outgrowth (r.c.o.)* extension of X if T = X $\cup$ K where K is compact.

Note that in part (b) of the above definition the outgrowth T-X need not be compact.

It is shown in [6, Corollary 2.1.1, p. 32] that for a given uniform space $(X,\mathcal{U})$ any maximal $\delta(\mathcal{U})$-round filter may be added to the set of $\delta(\mathcal{U})$-round $\mathcal{U}$-Cauchy filters to obtain the set of $\delta(\mathcal{V})$-round $\mathcal{V}$-Cauchy filters for a uniformity $\mathcal{V}$ on X such that $\mathcal{V} \subset \mathcal{U}$ and $\delta(\mathcal{V}) = \delta(\mathcal{U})$. This result yields the following theorem.

Theorem 1.3. *Let* (X,δ) *be a completely uniformizable proximity space. Then* X *is* δ*-completable to every f.o. extension of* X *contained in* δX.

Proof. For a compatible uniformity $\mathcal{U}$ on X, the $\delta(\mathcal{U})$-round $\mathcal{U}$-Cauchy filters agree with the minimal $\mathcal{U}$-Cauchy filters. So the result follows from [1, Proposition 2.1, p. 322].

We now extend the above result to the r.c.o. extension case.

Theorem 1.4. *Let* (X,δ) *be a completely uniformizable proximity space. Then* X *is* δ*-completable to every r.c.o. extension of* X *contained in* δX.

Proof. Let $\mathcal{U}$ be a complete member of $\Pi(\delta)$ and let K be a compact subset of δX. Recall that the points of δX are the maximal δ-round filters and that we identify the points of X with the fixed maximal δ-round filters. Thus, X is the set of minimal $\mathcal{U}$-Cauchy filters. By [1, Proposition

2.1, p. 322] it suffices to find a uniformity $\mathcal{V}$ on X for which $\delta(\mathcal{V}) = \delta$ and $X \cup K$ is the set of minimal $\mathcal{V}$-Cauchy filters.

Now we may write K as $K = \{\mathcal{F}_i : i \in I\}$ where $\mathcal{F}_i$ is a maximal δ-round filter for each $i \in I$. For each $U \in \mathcal{U}$ and $F_i \in \mathcal{F}_i$ $(i \in I)$ set

$$B(U, \langle F_i \rangle_i) = U \cup (\cup_{i \in I} F_i \times F_i),$$

and let

$$\mathcal{B} = \{B(U, \langle F_i \rangle_i) : U \in \mathcal{U},\ F_i \in \mathcal{F}_i\ (i \in I)\}.$$

We claim that $\mathcal{B}$ is a uniform base on X. As in the proof of [6, Theorem 2.1, p. 31], the only difficult verification is that of the "square root" axiom. Let $U \in \mathcal{U}$ and $F_i \in \mathcal{F}_i$ $(i \in I)$ and set $B = B(U, \langle F_i \rangle_i)$. We must find an entourage $D \in \mathcal{B}$ for which $D \circ D \subset B$. To this end let $W_1 \in \mathcal{U}$ such that $W_1 \circ W_1 \subset U$. Now each $\mathcal{F}_i \in K$ is δ-round so that for $i \in I$ we may choose $G_i \in \mathcal{F}_i$ and $V_i \in \mathcal{U}_\delta$ with $V_i = V_i^{-1}$ and $V_i[G_i] \subset F_i$. (Recall that $\mathcal{U}_\delta$ denotes the totally bounded member of $\Pi(\delta)$.) Thus, $K \subset \cup_{i \in I} O(G_i)$. Since K is compact, there are $i_1, \cdots, i_n \in I$ such that $K \subset \cup_{j=1}^n O(G_{i_j})$. I.e., if $\mathcal{F} \in K$, then for some $j \in \{1, \cdots, n\}$, $G_{i_j} \in \mathcal{F}$. Now for each $i \in I$ choose $\sigma(i) \in \{i_1, \cdots, i_n\}$ such that $G_{\sigma(i)} \in \mathcal{F}_i$.

Set $W_2 = \cap_{j=1}^n V_{i_j}$. Then $W_2 \in \mathcal{U}_\delta$. Now each $\mathcal{F}_i \in K$ is $\mathcal{U}_\delta$-Cauchy. So for each $i \in I$ there is $H_i \in \mathcal{F}_i$ such that $H_i \times H_i \subset W_2$. Setting

$$D = B(W_1 \cap W_2, \langle G_{\sigma(i)} \cap H_i \rangle_i)$$

yields the desired entourage, as may be easily checked.

Now let $\mathcal{V}$ be the uniformity on X generated by $\mathcal{B}$. It is straightforward to verify that $\mathcal{U}_\delta \subset \mathcal{V} \subset \mathcal{U}$, so that $\delta(\mathcal{V}) = \delta$, and that each member of $X \cup K$ is $\mathcal{V}$-Cauchy. It remains to show that if $\mathcal{G} \in \delta X$ is $\mathcal{V}$-Cauchy, then $\mathcal{G} \in X \cup K$. Assume (by way of contradiction) that $\mathcal{G} \notin X \cup K$. Since K is compact, there is $G \in \mathcal{G}$ with $O(G) \cap K = \phi$. I.e., for each $i \in I$, $G \notin \mathcal{F}_i$. Let $H \in \mathcal{G}$ such that $H \bar{\delta} X-G$. Since all members of K are maximal δ-round filters, it follows that for all $i \in I$, $X-H \in \mathcal{F}_i$. Now since $\mathcal{G} \notin X$, $\mathcal{G}$ is not $\mathcal{U}$-Cauchy. Thus, there is $U \in \mathcal{U}$ such that whenever $S \in \mathcal{G}$, $S \times S \not\subset U$.

Let $V \in \mathcal{U}$ be symmetric with $V \circ V \subset U$. Now $B = B(V, \langle X-H \rangle_i) \in \mathcal{V}$ and, since $\mathcal{G}$ is $\mathcal{V}$-Cauchy, there is $z \in X$ such that $B[z] \in \mathcal{G}$. If we set $S = B[z] \cap H$, then we may conclude that $S \in \mathcal{G}$ and $S \times S \subset U$. This is the desired contradiction.

While the above result demonstrates that a completely uniformizable proximity space (X,δ) is δ-completable to many of its realcompactifications contained in its Smirnov compactification, the following question nevertheless remains unanswered: Do the completely uniformizable proximity spaces coincide with the realcompact rich proximity spaces?

2. Locally ω^*-embedded Outgrowth

If (X,δ) is a proximity space and $\mathcal{U}$ is a non-totally bounded member of $\Pi(\delta)$, then X must contain an infinite $\mathcal{U}$-uniformly discrete set (which is also an infinite σ-discrete subset of positive gauge for some pseudometric

σ compatible with δ). Thus, [6, proof of Theorem 3.2, p. 33] or [5, Theorem 3.1, p. 226] yields the following theorem which first appeared in [4, Theorem 3.3, p. 157].

Theorem 2.1. *If* (X,δ) *is a noncompact completely uniformizable proximity space, then* $|\delta X - X| \geq 2^c$.

[4] provides the same lower bound for the cardinality of a nonempty closed G_δ-subset of the Smirnov compactification δX of a completely uniformizable proximity space (X,δ) when that subset is disjoint from X. Also, according to [5], even when (X,δ) is not necessarily completely uniformizable, 2^c serves as a lower bound for the cardinality of any nonempty zero-set of δX disjoint from the realcompletion of X. We shall now extend the result in Theorem 2.1 to a "local" version. Let $D(\omega)$ denote the discrete topological space of cardinality ω, and let $\omega^* = \beta D(\omega) - D(\omega)$.

Definition 2.2. (a) If Z is a Hausdorff compactification of a Tychonoff space X and $X \subset Y \subset Z$, then Z is said to have *locally* ω^**-embedded outgrowth with respect to* Y if for each $p \in Z-Y$ and each neighborhood H of p in Z, there is a closed discrete subspace S of X such that $|S| = \omega$, $cl_Z S =_S \beta S$, and $cl_Z S \subset H$.

(b) A Hausdorff compactification Z of a Tychonoff space X has *locally* ω^**-embedded outgrowth* if Z has locally ω^*-embedded outgrowth with respect to X.

Note that if Z is a Hausdorff compactification of X with locally ω^*-embedded outgrowth, then every nonempty open subset of Z-X (with the relative topology induced by Z) contains a copy of ω^* and, hence, has cardinality of at least 2^c.

Theorem 2.3. *If* (X,δ) *is a proximity space and* $\mathcal{U} \in \Pi(\delta)$*, then the Smirnov compactification* δX *of* X *has locally* ω^**-embedded outgrowth with respect to* $\mathcal{U}X$.

Proof. Let $p \in \delta X - \mathcal{U}X$ and let H be an open subset of δX with $p \in H$. Let G be an open subset of δX with $p \in G$ and $cl_{\delta X}G \subset H$, and set $A = G \cap \mathcal{U}X$. Then $cl_{\delta X}A = cl_{\delta X}G \not\subset \mathcal{U}X$, and so $Y = cl_{\mathcal{U}X}A$ is not compact.

Now $\mathcal{U}^*$ is a complete uniformity on $\mathcal{U}X$ and $\delta(\mathcal{U}^*) = \delta^*|_{\mathcal{U}X}$. Also $\mathcal{U}^*|_Y$ is complete (since Y is closed in $\mathcal{U}X$) and non-totally bounded (since Y is not compact). Observing that $Y \cap X$ is dense in Y, we conclude that $\mathcal{U}|_{Y\cap X} = (\mathcal{U}^*|_Y)|_X$ is non-totally bounded. As in [6, proof of Theorem 3.2, p. 33], there is an entourage $U \in \mathcal{U}$ and a countably infinite set $S \subset Y \cap X$ such that

$$U \cap [(Y \cap X) \times (Y \cap X)] \cap (S \times S) = U \cap (S \times S) = \Delta_S.$$

$\mathcal{U}|_S$ is the discrete uniformity on S, and $\delta|_S = \delta(\mathcal{U}|_S)$ is the discrete proximity on S. Moreover, $cl_{\delta X}S$ is the Smirnov compactification of $(S,\delta|_S)$, whence $cl_{\delta X}S =_S \beta S$, and certainly $cl_{\delta X}S \subset H$. Now if V is a symmetric entourage in $\mathcal{U}$ such that $V \circ V \subset U$ and $y \in \mathcal{U}X$, then $|V^*[y] \cap S| \leq 1$. Thus, S is closed and discrete in $\mathcal{U}X$ (and, hence, S is a closed subset of X as well).

Corollary 2.4. *If* (X,δ) *is a completely uniformizable proximity space, then* δX *is a Hausdorff compactification of* X *with locally* ω^**-embedded outgrowth.*

The proof of Theorem 2.3 also yields the following corollary.

Corollary 2.5. *If* (X,δ) *is a proximity space and* $\mathcal{U} \in \Pi(\delta)$*, then* δX *is a Hausdorff compactification of* $\mathcal{U}X$ *with locally* ω^**-embedded outgrowth.*

Corollary 2.6. *If* Z *is a rich compactification of a Tychonoff space* X *and* $X \subset Y \subset Z$ *where* Y *is realcompact, then* Z *has locally* ω^**-embedded outgrowth with respect to* Y.

Note that the Stone-Čech compactification βX of a Tychonoff space X has locally ω^*-embedded outgrowth with respect to its Hewitt realcompactification υX. Thus, every βX-neighborhood of a point in $\beta X - \upsilon X$ contains a copy of $D(\omega)$ which is a subset of X and is C*-embedded in βX. According to [3, 9D1, p. 136], such a copy of $D(\omega)$ can be found which is actually C-embedded in X.

Also note that δX may fail to be a Hausdorff compactification of X with locally ω^*-embedded outgrowth when (X,δ) is not completely uniformizable. A trivial example is provided by the proximity induced on R, the real numbers with the usual topology, by its one-point compactification. A nontrivial example, where $\Pi(\delta)$ contains a non-totally bounded member, is given next.

Example 2.7. Let d denote the usual metric on the set Q of rational numbers, $\mathcal{U} = \mathcal{U}(d)$, and $\delta = \delta(d)$. Then $\mathcal{U}$ is non-totally bounded. By [7, Theorem 21.26, p. 202], since $\mathcal{U}$ is metrizable, $\mathcal{U}$ is the largest uniformity inducing δ, and since $\mathcal{U}$ is not complete, no complete uniformity induces δ.

Now $\mathcal{U}Q =_Q R$, the real numbers with the usual topology, which is locally compact. So $\mathcal{U}Q$ is an open subset of δQ and $\mathcal{U}Q \cap (\delta Q - Q) \neq \phi$. Since $|\mathcal{U}Q| = |R| = c$, $\mathcal{U}Q$ contains no copy of $\beta D(\omega)$.

We shall conclude this section with an example which demonstrates that a (noncompact and realcompact) proximity space (X,δ) need not be completely uniformizable when δX is a Hausdorff compactification of X with locally ω^*-embedded outgrowth.

Example 2.8. Let P denote the space of irrational numbers with the usual topology. Then P is noncompact, and every subspace of P is realcompact. Since P is a G_δ-set in R, by [8, Theorem 24.12, p. 179] there is a compatible complete metric d on P. Let $\mathcal{U} = \mathcal{U}(d)$ and $\gamma = \delta(d)$. Then $\mathcal{U}$ is a complete metrizable uniformity which induces γ, and so, by Corollary 2.4, γP is a Hausdorff compactification of P with locally ω^*-embedded outgrowth.

Now let $X = P - \{\pi\}$ and $\delta = \gamma|_X$. Then X is a noncompact and realcompact space, δ is a compatible proximity on X, and $\delta X =_X \gamma P$. Since $\mathcal{U}|_X$ is a metrizable uniformity inducing δ, $\mathcal{U}|_X$ is the largest uniformity inducing δ by [7, Theorem 21.26, p. 202]. Since $\mathcal{U}|_X$ is not complete, no complete uniformity can induce δ.

Let H be an open subset of γP (which we identify with δX) such that $H \cap (\gamma P - X) \neq \phi$. H is not a subset of P since $\text{int}_{\gamma P}P = \phi$. So $H \cap (\gamma P - P) \neq \phi$. Thus, there is a countably infinite, closed, discrete subspace S of P such that $\text{cl}_{\gamma P}S =_S \beta S$ and $\text{cl}_{\gamma P}S \subset H$. So $K = S - \{\pi\}$ is a countably infinite, closed, discrete subspace of X, $cl_{\gamma P}K = (cl_{\gamma P}S) - \{\pi\} =_K \beta K$, and $cl_{\gamma P}K \subset H$. So γP is a Hausdorff compactification of X with locally ω^*-embedded outgrowth.

3. Nonrealcompact Extensions

In this section we shall determine the number of non-realcompact extensions of a completely uniformizable proximity space contained in its Smirnov compactification.

Theorem 3.1. *Let* X *be a noncompact Tychonoff space. If* Z *is a Hausdorff compactification of* X *with locally* ω^**-embedded outgrowth, then there are exactly* $2^{|Z-X|}$ *nonreal-compact extensions of* X *contained in* Z.

Proof. Let G be a nonempty open subset of Z - X (with the relative topology induced by Z) such that

$$|(Z - X) - G| = |Z - X|,$$

and let H be an open subset of Z such that $G = H \cap (Z - X)$. Since Z is a Hausdorff compactification of X with locally ω^*-embedded outgrowth, there is a countably infinite, closed, discrete subspace S of X such that $cl_Z S =_S \beta S$ and $cl_Z S \subset H$. Thus, there is a nonrealcompact space T such that $S \subset T \subset cl_Z S$. For each $A \subset (Z - X) - G$ set $T_A = X \cup T \cup A$. T_A is nonrealcompact since $T = T_A \cap cl_Z S$ is a nonrealcompact closed subset of T_A. Also, if $A_i \subset (Z - X) - G$ $(i = 1,2)$ and $A_1 \neq A_2$, then $T_{A_1} \neq T_{A_2}$. So there are at least

$$|\mathcal{P}((Z - X) - G)| = 2^{|(Z-X)-G|} = 2^{|Z-X|}$$

nonrealcompact extensions of X contained in Z. Since there are exactly $2^{|Z-X|}$ extensions of X contained in Z, the proof is complete.

The following corollary follows immediately from Corollary 2.4 and Theorem 3.1.

Corollary 3.2. *If* (X,δ) *is a noncompact completely uniformizable proximity space, then* δX *contains exactly* $2^{|\delta X-X|}$ *nonrealcompact extensions of* X.

[3, 9D2, p. 136] yields a method for constructing nonrealcompact extensions of a noncompact, realcompact space X contained in its Stone-Čech compactification βX: in this case $|\beta X - X| \geq 2^c$ and if $\phi \neq S \subset \beta X - X$ with $|S| < 2^c$, then $T = \beta X - S$ is such an extension. Assuming the generalized continuum hypothesis, this construction guarantees only 2^c distinct such extensions when $|\beta X - X| = 2^c$. The following simple application of Corollary 3.2 guarantees that there are exactly 2^{2^c} such extensions in this case.

Corollary 3.3. *Let* X *be a noncompact, realcompact space. Then* βX *contains exactly* $2^{|\beta X-X|}$ *nonrealcompact extensions of* X.

Proof. The uniformity functionally determined on X by the real-valued continuous functions on X is a complete, compatible uniformity on X whose proximity is induced by βX.

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