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by

Bradd E. Clark, Mary C. Dooley, and Victor P. Schneider

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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1. Preliminaries

Let G be a group and H a proper subgroup of G. If H is a topological group, a natural question to ask is when can the topology on H be extended to a topology on G that makes G into a topological group? While answering a different question, Sharma in [6] has shown that any topology which makes the center of G into a topological group can be extended in such a way as to make all of G into a topological group. Little else has been written about this question.

For the purposes of this paper we shall assume that G is a group and that H is a proper subgroup of G that is also a topological group with topology τ. Finally we shall assume that $U = \{U_\alpha\}_{\alpha \in \Gamma}$ is a basis for the topology of H at the identity element e. If G is a topological group, then a basis for G can be obtained by multiplying each element of U by each element of G. Thus there is a natural way to extend the topology from H to G. The collection $U_L = \{gU_\alpha | U_\alpha \in U \text{ and } g \in G\}$ is called the *left translation basis* and τ_L, the topology induced by U_L, the *left translation topology*. Likewise, we define $U_R = \{U_\alpha g | U_\alpha \in U \text{ and } g \in G\}$ to be the *right translation basis* with τ_R the associated *right translation topology*. There may be other ways of extending a topology from H to G. However, if H is normal in G we shall note that either G with the translation topology is a topological group or that it is impossible to make G a topological group by extending the topology from H.

Obviously G need not be algebraically a product group. However, it is interesting to note that the translation topologies make G topologically a product.

Theorem 1. *The spaces* (G,τ_L) *and* (G,τ_R) *are both homeomorphic to* $H \times G/H$ *where* H *is endowed with the topology* τ *and* G/H *is the set of left cosets of* H *in* G *endowed with the discrete topology.*

Proof. For each coset in G/H we pick a fixed representation of the form g_*H. We define $f: G \to H \times G/H$ by $f(g) = (h,gH)$ where $g_*h = g$ and $g_*H = gH$. Since g_* is uniquely determiend by g, f is both a one-to-one and onto function.

Let $V \times \{g_*H\}$ be a basic open set in $H \times G/H$. We have that $f^{-1}(V \times \{g_*H\}) = g_*V$. Also, if gV is a basic open set in G then $f(gV) = g_*^{-1}gV \times \{g_*H\}$. Thus f is a homeomorphism between (G,τ_L) and $(H,\tau) \times (G/H$, discrete topology). A similar argument shows that (G,τ_R) is homeomorphic to the same product space.

2. Recognizing Topological Group Extensions

Although we can always find left and right translation topologies for G we can be less sure that G will be a topological group with such a topology. Thus it would be useful to find conditions under which G will be a topological group. As we shall see, when G is a topological group we need only speak of the translation topology as $\tau_L = \tau_R$.

Theorem 2. *The group* G *with a translation topology is a topological group if and only if* $\tau_L = \tau_R$.

Proof. If G is a topological group with a translation topology, then either U_L or U_R can be used as a basis for the topology of G. Thus $\tau_R = \tau_L$.

Suppose on the other hand that $\tau_L = \tau_R$. Let $a,b \in G$ and let abW be a basic open neighborhood of ab. Since H is a topological group we can find a neighborhood V of e in H such that $V^2 \subset W$. Also since $\tau_L = \tau_R$ we can find another neighborhood U of e such that $Ub \subset bV$. So $aUbV = a(Ub)V \subset abV^2 \subset abW$. Thus multiplication is continuous.

Let $a \in G$ and let $a^{-1}W$ be a neighborhood of a^{-1}. Since $\tau_L = \tau_R$ we can find a neighborhood V of e in H such that $Va^{-1} \subset a^{-1}W$. So we have $(aV^{-1})^{-1} = Va^{-1} \subset a^{-1}W$. Thus $(G,\tau_L) = (G,\tau_R)$ is a topological group.

Corollary 3. *If a neighborhood basis at* e *is contained in the center of* G, *then* G *is a topological group with the translation topology.*

Not every translation topology will yield a topological group. The obstruction can be either algebraic or topological in nature. Shelah in [7] has given an example of an algebraic obstruction. H is said to be *mal-normal* if and only if for every $g \in G - H$, $gHg^{-1} \cap H = \{e\}$. If G is a topological group with the translation topology then both H and gHg^{-1} are open in G and thus G has the discrete topology. Thus only the discrete topology on H can be extended to make G into a topological group when H is a mal-normal subgroup of G.

For an example of a topological obstruction suppose that H is connected. If $\tau_L = \tau_R$ then H, gH, and Hg are all components of G. But this cannot be true unless gH = Hg. If H is normal in G then for all $g \in G$ we have an isomorphism $C_g\colon H \to H$ defined by $C_g(h) = ghg^{-1}$. Certainly if G is a topological group then C_g is a homeomorphism for all $g \in G$ since C_g is the restriction to H of conjugation on G. But conjugation on G will be a homeomorphism. As the next theorem points out either the translation topology makes G into a topological group or there is no way to make G a topological group while extending the topology from H.

Theorem 4. *Let* H *be normal in* G. *Then* G *is a topological group with the translation topology if and only if* C_g *is a homeomorphism for all* $g \in G$.

Proof. Suppose that $C_g\colon H \to H$ is a homeomorphism for all $g \in G$, that $a,b \in G$ and that abW is a basic open set. We can find an open neighborhood V of e in H such that $V^2 \subset W$. Let $U = bVb^{-1}$. Since conjugation by b is a homeomorphism we know that U is a neighborhood of e in H. So $aUbV = abb^{-1}UbV = abV^2 \subset abW$. Therefore multiplication is continuous. A similar argument shows that the inverse operation is continuous.

3. Group Isomorphisms and Group Homeomorphisms

Even without knowledge of G, Theorem 4 can be used to tell much about extending topologies from normal subgroups. For example, suppose that $H \approx \mathbb{Z}$. The only group isomorphisms on $\mathbb{Z}$ are the identity map and the map that sends x to -x. No matter what topology is placed on H to make H a topological group, the translation topology on G will make G into a topological group whenever H is normal in G.

Of course, it is too much to expect every topology on every normal subgroup to extend properly to all of G. As an example of this let G be the group that has a presentation of the form $\{a,b,c \mid ab = ba,\ cac^{-1} = c^{-1}ac = b\}$, and let H be the normal subgroup of G that has $\{a,b \mid ab = ba\}$ for a presentation. Since H is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, we can topologize H by placing a p-adic topology on the first factor of H and a q-adic topology on the second factor with $p \neq q$. Conjugation of H by c is an isomorphism of H to itself that switches the generators a and b. Clearly this fails to be a homeomorphism and hence the translation topologies will fail to make G into a topological group.

Markov in [5] asked which groups admit nontrivial Hausdorff topologies which make them into topological groups. Kertesz and Szale [4] showed that every infinite Abelian group admits such a topology while Comfort and Ross [1] showed that in fact every infinite Abelian group admits a nontrivial metric topology that makes it a topological group. Sharma has shown that every group with infinite center can be made into a topological group with a non-trivial metric topology. Theorem 4 can be used to advance these results.

Corollary 5. *If* H *is a finitely generated Abelian group of infinite order and* H *is normal in* G, *then* G *can be made into a topological group with a nontrivial metric topology.*

Proof. We can find a subgroup H' normal in H which is isomorphic to $\mathbb{Z}^m$ for some integer $m \geq 1$ and which is invariant under isomorphisms from H to H. We place the p-adic topology on each factor of $\mathbb{Z}^m$ and note that this topology is automorphism invariant [2].

Corollary 6. *Let* $\mathbb{Q}^m$ *be the Cartesian product of* $m \geq 1$ *copies of the rationals and suppose that* $\mathbb{Q}^m$ *is a normal subgroup of* G. *Then* G *can be made into a topological group with a nontrivial metric topology.*

Proof. $\mathbb{Q}^m$ can be made into a topological group by placing the usual topology on $\mathbb{Q}^m$.

Corollary 7. *Let* F *be a free group and suppose that* F *is normal in* G. *Then* G *can be made into a topological group with a nontrivial Hausdorff topology.*

Proof. Let $\{U_\alpha\}_{\alpha \in \Gamma}$ be the collection of subgroups of F of finite index. Hall [3] has shown that this collection of subgroups along with their cosets form a basis for a nontrivial Hausdorff topology on F that makes F into a topological group.

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The University of Southwestern Louisiana

Lafayette, Louisiana 70504-1010