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## STABLE POINTS IN HYPERSPACES OF PEANO CONTINUA

by

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## STABLE POINTS IN HYPERSPACES OF PEANO CONTINUA

Doug Curtis

### 1. Introduction

For  $X$  a nondegenerate Peano continuum, let  $C(X)$  denote the hyperspace of subcontinua, topologized by the Hausdorff metric.  $M \in C(X)$  is *stable* if, for every map  $f: C(X) \rightarrow C(X)$  sufficiently close to the identity map,  $M \in f(C(X))$ . Thus,  $M$  is *unstable* if and only if  $\{M\}$  is a  $Z$ -set in  $C(X)$ . If  $X$  does not contain a free arc then  $C(X)$  is homeomorphic to the Hilbert cube [2], hence each point is unstable. On the other hand, if  $X$  contains a free arc then  $C(X)$  has stable points (for instance, every subarc in the interior of a free arc). We obtain the following surprisingly simple set of conditions for determining which points of  $C(X)$  are stable.

1.1. *Theorem.*  $M \in C(X)$  is stable if and only if:

- a)  $M$  is a finite union of free arcs in  $X$ ; and
- b) the quotient space  $M/bd_X M$  (or  $M$  if  $M = X$ ) has no cut points.

In particular,  $X \in C(X)$  is stable if and only if  $X$  is a finite graph with no cut points.

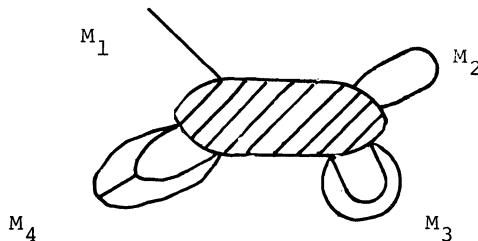


Figure 1

To illustrate the Theorem, consider  $X$  as in Figure 1. The elements  $M_1$  and  $M_3$  are unstable in  $C(X)$ , whereas  $M_2$  and  $M_4$  are stable.

The only previous work on stable points in hyperspaces is due to Neil Gray [5], [6], who showed that every point in the hyperspace  $2^X$  is unstable, and that for  $X$  a finite polyhedron, every point in  $C(X)$  is unstable if and only if  $X$  has no free arcs. (This was done before the Hilbert cube characterization of such hyperspaces.) For  $X$  a finite graph, Roman Duda [3], [4] described and analyzed polyhedral models for  $C(X)$ . Such models were very useful for suggesting an initial version of the above condition b).

The consideration of stable points in  $C(X)$  is simplified by the fact that  $C(X)$  is an AR, and therefore locally equiconnected. This implies that stability is determined locally, i.e., if  $\mathcal{U}$  is any neighborhood of  $M$  in  $C(X)$ , then  $M$  is stable in  $C(X)$  if and only if  $M$  is stable in  $\mathcal{U}$ .

## 2. Stable Points in $C(X)$

In this section we demonstrate the necessity of the conditions a) and b) for stable  $M \in C(X)$ . For the argument that  $M$  is a finite union of free arcs, with  $\text{bd } M$  not separating  $M$ , we use just two basic types of maps in  $C(X)$ , namely, "local channelization" maps and "global expansion" maps.

To describe the first type of map, we find it convenient to use partitions of  $X$ . A *partition* of a space is a finite collection  $\mathcal{P}$  of mutually disjoint connected open subsets whose closures cover the space. It will be assumed that

each partition element is the interior of its closure. Given a partition  $\mathcal{P}$  of  $X$  and  $x \in X$ , we may obtain a partition which has an element containing  $x$  by replacing (if necessary) the subcollection  $\mathcal{P}_x = \{P \in \mathcal{P} : x \in \bar{P}\}$  with a single larger element  $P_x = \text{int}(\text{cl}(U\mathcal{P}_x))$ . The basic result needed concerning partitions of a Peano continuum  $X$  is due to Bing [1] and Moise [8]: for every  $\epsilon > 0$ , there exists a partition  $\mathcal{P}$  of  $X$  such that  $\text{mesh } \mathcal{P} < \epsilon$  and the closure of each partition element is a Peano continuum. For each element  $P$  of such a partition, it is easily seen that there exists a finite graph  $T \subset \bar{P}$  such that the space  $U\{\bar{Q} : Q \in \mathcal{P} \setminus \{P\} \text{ with } \bar{Q} \cap \bar{P} \neq \emptyset\} \cup T$  is connected (and therefore a Peano continuum). We call  $T$  a *P-connecting graph*.

2.1. *Lemma.* For every  $x \in X$  and  $\epsilon > 0$ , there exists a neighborhood  $P$  of  $x$ , a Peano continuum  $K \subset X$ , and a finite graph  $T \subset K$  such that:

- i)  $\text{diam } (K \cup P) < \epsilon$ ;
- ii)  $\text{bd } P \subset K$ ;
- iii)  $K \cap P \subset T$ .

*Proof.* Choose a partition  $\mathcal{P}$  of  $X$  such that  $x \in P \in \mathcal{P}$ , the closure of each partition element is a Peano continuum, and  $\text{mesh } \mathcal{P} < \epsilon/3$ . Choose a  $P$ -connecting graph  $T$ , and take  $K = U\{\bar{Q} : Q \in \mathcal{P} \setminus \{P\} \text{ with } \bar{Q} \cap \bar{P} \neq \emptyset\} \cup T$ . Then  $P, K$ , and  $T$  have the required properties.

2.2. *Proposition.* Every stable  $M \in C(X)$  is a finite graph.

*Proof.* Choose  $\epsilon > 0$  such that for every map  $f: C(X) \rightarrow C(X)$  with  $d(f, \text{id}) < \epsilon$ ,  $M \in f(C(X))$ . (For

simplicity of notation, we let  $d$  denote both the metric on  $X$  and the corresponding Hausdorff metric on  $C(X)$ ). It suffices to show that at each point  $M$  is locally a finite graph. Given  $x \in M$ , choose a neighborhood  $P$  of  $x$  in  $X$ , a Peano continuum  $K \subset X$ , and a finite graph  $T \subset K$  satisfying the conditions of (2.1). Since  $C(K)$  is an AR, there exists a map  $\phi: \bar{P} \rightarrow C(K)$  with  $\phi(y) = \{y\}$  for each  $y \in \text{bd } P$ . Extend to a map  $\phi: X \rightarrow C(X)$  by setting  $\phi(y) = \{y\}$  for each  $y \in X \setminus \bar{P}$ . Then  $\phi$  induces a map  $f: C(X) \rightarrow C(X)$ , defined by  $f(A) = \cup\{\phi(a) : a \in A\}$ . Since  $d(f, \text{id}) < \varepsilon$ ,  $M = f(A)$  for some  $A \in C(X)$ . Since  $f(A) \cap P \subset K \cap P \subset T$ ,  $M$  is locally a finite graph.

2.3. *Proposition.* For every stable  $M \in C(X)$ ,  $\text{int}_X M$  is a dense connected subset of  $M$ .

*Proof.* Assume the metric  $d$  is convex. Given  $\varepsilon > 0$ , let  $\phi: X \rightarrow C(X)$  be the map defined by  $\phi(x) = \{y \in X : d(x, y) < \varepsilon\}$ . Then  $\phi$  induces an expansion map  $f: C(X) \rightarrow C(X)$ , with  $d(f, \text{id}) < \varepsilon$ . If  $M = f(A)$  for some  $A \in C(X)$ , then  $A \subset \text{int } M$  and  $d(A, M) < \varepsilon$ . Letting  $\varepsilon \rightarrow 0$ , we see that  $\text{int } M$  must be dense and connected.

Clearly, (2.2) and (2.3) together imply that for  $M$  stable,  $\text{bd } M$  is a finite set. It follows that the finite graph  $M$  is a finite union of free arcs in  $X$ .

2.4. *Proposition.* For every stable  $M \in C(X)$ ,  $M/\text{bd } M$  (or  $M$  if  $M = X$ ) has no cut points.

*Proof.* Suppose there exists a cut point  $p$ . By (2.3),  $p \in \text{int } M$ . Then there exists a nondegenerate finite graph

$G \subset \text{int } M$  and a nondegenerate subcontinuum  $Y \subset X$  such that  $X = G \cup Y$  and  $G \cap Y = \{p\}$ . (Some component  $F$  of  $M \setminus \{p\}$  lies in  $\text{int } M$ ; take  $G = F \cup \{p\}$  and  $Y = M \setminus F$ ). It follows that  $C_p(X) = \{A \in C(X) : p \in A\}$  is a neighborhood of  $M$  in  $C(X)$ , hence  $M$  is stable in  $C_p(X)$ . Let  $h: C_p(X) \rightarrow C_p(G) \times C_p(Y)$  be the natural homeomorphism defined by  $h(A) = (A \cap G, A \cap Y)$ . Then  $h(M) = (G, M \cap Y)$  is stable in  $C_p(G) \times C_p(Y)$ , and  $G$  must be stable in  $C_p(G)$ . However, it has been shown by Mark Lynch [7] that  $G$  is always unstable in  $C_p(G)$ . (In fact, Lynch shows that  $X$  is unstable in  $C_p(X)$ , for every continuum  $X$ .) Thus there are no cut points in  $M/\text{bd } M$ .

*Note.* If  $X$  is a non-locally connected continuum, the situation regarding stable points in  $C(X)$  can be quite different from that above. For example, let  $X$  be the union of two copies of the  $\sin(1/x)$ -continuum, meeting only at a common endpoint  $p$  of their limit arcs. Then it can be shown that  $\{p\}$  is stable in  $C(X)$ ; in fact, for every map  $f: C(X) \rightarrow C(X)$  sufficiently close to the identity map,  $f(\{p\}) = \{p\}$ . Moreover, by adding a countable collection of stickers to  $X$ , we may obtain a continuum  $Y$  which has no free arcs, and yet  $\{p\}$  is stable in  $C(Y)$ .

### 3. Reduction to $X = G$

In this section and the next two, we give the proof of sufficiency of the conditions a) and b) for  $M \in C(X)$  to be stable. The following result reduces the general problem to the special case  $X = G$ , a finite graph.

3.1. *Proposition.* Let  $M \in C(X)$  be a finite union of free arcs, with  $M \neq X$ . Let  $\{J_b : b \in \text{bd } M\}$  be a collection of mutually disjoint arcs in  $X$  such that  $b$  is an endpoint of  $J_b$  and  $J_b \cap M = \{b\}$ , for each  $b$ . Consider the finite graph  $G = \cup\{J_b : b \in \text{bd } M\} \cup M$ , and suppose that  $M$  is stable in  $C(G)$ . Then  $M$  is stable in  $C(X)$ .

*Proof.* Choose a collection  $\{K_b : b \in \text{bd } M\}$  of mutually disjoint continua in  $X$  such that:

- i)  $J_b \subset K_b$ ;
- ii)  $K_b \cap M = \{b\}$ ; and
- iii)  $K_b \cup M$  is a neighborhood of  $b$  in  $X$ .

Set  $Y = \cup\{K_b : b \in \text{bd } M\} \cup M$ . We show first that  $M$  is stable in  $C(Y)$ . For each  $b$ , choose a retraction  $r_b : K_b \rightarrow J_b$  such that  $r_b(x) = b$  only if  $x = b$ , and let  $r : Y \rightarrow G$  be the retraction defined by the collection  $\{r_b\}$ . Let  $R : C(Y) \rightarrow C(G)$  denote the induced hyperspace retraction. Note that  $R(B) = M$  only if  $B = M$ . If  $f : C(Y) \rightarrow C(Y)$  is a map sufficiently close to the identity, then the map  $g : C(G) \rightarrow C(G)$  defined by  $g(A) = R(f(A))$  will also be close to the identity, and by hypothesis we may assume  $M = g(A)$  for some  $A \in C(G)$ . But this implies  $M = f(A)$ . Thus  $M$  is stable in  $C(Y)$ . Since  $M \subset \text{int } Y$ ,  $C(Y)$  is a neighborhood of  $M$  in  $C(X)$ , and it follows that  $M$  is stable in  $C(X)$ .

#### 4. Cone Neighborhoods in $C(G)$

Since the hyperspace  $C(G)$  of a finite graph is homeomorphic to a polyhedron, every  $M \in C(G)$  has a neighborhood homeomorphic to the cone over some polyhedron  $S$ , with  $M$  corresponding to the cone vertex. It is easily shown that

$M$  is stable in such a cone neighborhood (and therefore in  $C(G)$ ) if and only if the cone base  $S$  is non-contractible. Our objective in this section is the construction, for nondegenerate  $M$ , of a particular cone neighborhood for which the Euler characteristic  $\chi(S)$  of the base is given in terms of a certain index  $\lambda(M/\text{bd } M)$  of the quotient graph  $M/\text{bd } M$ . Analysis of this index in section 5 will show that if  $M/\text{bd } M$  has no cut points then  $\chi(S) \neq 1$ , hence  $S$  is non-contractible, and  $M$  is stable in  $C(G)$ .

4.1. *Proposition.* For  $S$  a compact metric ANR, consider  $\text{Cone } S = S \times [0,1]/S \times \{0\}$ , with cone vertex  $\theta$  corresponding to  $S \times \{0\}$ . Then  $\theta$  is an unstable point in  $\text{Cone } S$  if and only if  $S$  is contractible.

*Proof.* If  $S$  is contractible, there exists for every  $t > 0$  a retraction of  $S \times [0,t]/S \times \{0\}$  onto  $S \times \{t\}$ . Let  $r_t: \text{Cone } S \rightarrow S \times [t,1]$  be the extension of such a retraction by the identity. Then  $d(r_t, \text{id}) \rightarrow 0$  as  $t \rightarrow 0$ , and  $\theta$  is unstable in  $\text{Cone } S$ .

Conversely, suppose there exist maps  $f: \text{Cone } S \rightarrow S \times (0,1]$  arbitrarily close to the identity. By local equiconnectedness of  $S$ , we may assume that  $f|S \times \{1\} = \text{id}$ . Then  $f$  followed by the projection onto  $S$  defines a contraction of  $S$  to a point.

Let  $G$  be a nondegenerate finite graph, considered with respect to a specific subdivision. We say that a subgraph  $L$  (possibly degenerate) is *large* if  $L$  meets each edge of  $G$ . Let  $\mathcal{L}(G)$  denote the collection of large subgraphs of  $G$ , and for each vertex  $v$  of  $G$ , let  $\mathcal{L}_v(G) = \{L \in \mathcal{L}(G) :$



$v \in L$ }. For each  $L \in \mathcal{L}(G)$  let  $\text{def } L$  be the number of edges in  $G \setminus L$ . We define the *stability index*  $\lambda(G) = \sum \{(-1)^{\text{def } L} : L \in \mathcal{L}(G)\}$ . Similarly, for each vertex  $v$  define  $\lambda_v(G) = \sum \{(-1)^{\text{def } L} : L \in \mathcal{L}_v(G)\}$ .

We will show in section 5 that  $\lambda_v(G) = 0$  for all  $v$ ;  $\lambda(G)$  is a topological invariant of  $G$  (provided the subdivision of  $G$  has more than one edge); and  $\lambda(G) = 0$  if and only if  $G$  has a cut point.

Referring to Figure 1, with  $G_i = M_i/\text{bd } M_i$  for  $i = 1, 2, 3, 4$ , it's not difficult to see that  $\lambda(G_1) = 0 = \lambda(G_3)$  and  $\lambda(G_2) = 1 = \lambda(G_4)$ .

For nondegenerate  $M \in C(G)$ , let  $M^* = M/\text{bd } M$  if  $M \neq G$ , and  $M^* = M$  if  $M = G$ . Thus  $M^*$  is topologically a finite graph.

**4.2. Theorem.** *For every nondegenerate  $M \in C(G)$ , there exists a finite polyhedron  $S$  and an imbedding  $h: \text{Cone } S \rightarrow C(G)$  such that:*

- i)  $h(\theta) = M$ ;
- ii)  $h(\text{Cone } S)$  is a neighborhood of  $M$  in  $C(G)$ ; and
- iii)  $\chi(S) = 1 - \lambda(M^*)$ .

*Proof.* We assume that  $M$  is a subgraph of  $G$ . Since  $h(\text{Cone } S)$  is a neighborhood of  $M$  in  $G$ , we may in fact assume that  $M$  is a large subgraph of  $G$ . Let  $\tilde{M}$  denote a barycentric subdivision of  $M$ , and consider the subdivision  $\text{sd } G = \tilde{M} \cup (G \setminus M)$  of  $G$ . Let  $E = \{e_1, \dots, e_k\}$  be the collection of edges of  $\text{sd } G$ . For each  $L \in \mathcal{L}(M)$ , set  $e(L) = \{e_i : e_i \text{ meets } L \text{ in a single point}\} \subset E$ . Note that  $e(L) \neq \emptyset$ , unless  $L = M = G$ .

Let  $T = \{(t_i) \in \prod_1^k [0,1]_i : \sum t_i \leq 1 \text{ and } \text{supp}(t_i) \subset e(L)\}$  for some  $L \in \mathcal{L}(M)$ , where  $\text{supp}(t_i) = \{e_i : t_i > 0\}$ .  $T$  can be considered as the cone over the polyhedron  $S = \{(t_i) \in T : \sum t_i = 1\}$ , with cone vertex  $\theta = (0, \dots, 0)$ . We will construct an imbedding  $h: T \rightarrow C(G)$  such that  $h(\theta) = M$  and  $h(T)$  is a neighborhood of  $M$  in  $C(G)$ .

Assume that  $\{e_{2i-1}, e_{2i}\}$  is the subdivision of an edge  $\alpha_i$  of  $M$ , for each  $1 \leq i \leq j$ , and that  $\{e_{2j+1}, \dots, e_k\}$  is the collection of edges of  $G \setminus M$ . Let  $d$  be the shortest path-length metric on  $G$ , with each edge of  $sd G$  having unit length. Thus for each  $i \leq j$ , the arc  $\alpha_i = e_{2i-1} \cup e_{2i}$  has length 2. Let  $C_i$  denote the quotient space obtained from the hyperspace  $J \in C(\alpha_i) : \text{diam } J \leq 1$  by collapsing the subspace of degenerate elements to a point. Then  $C_i$  is a 2-cell, and for  $\Delta_i = \{(s,t) \in [0,1]_{2i-1} \times [0,1]_{2i} : s + t \leq 1\}$ , there exists a homeomorphism  $g_i: \Delta_i \rightarrow C_i$  such that:

- 1)  $g_i(\{0\} \times [0,1]_{2i}) \subset C_{v_i}(\alpha_i)$ , where  $v_i$  is the endpoint of  $\alpha_i$  lying in  $e_{2i-1}$ ;
- 2)  $g_i([0,1]_{2i-1} \times \{0\}) \subset C_{w_i}(\alpha_i)$ , where  $w_i$  is the endpoint of  $\alpha_i$  lying in  $e_{2i}$ ;
- 3)  $\text{diam } g_i(s,t) = s + t$ .

For each edge  $e_i$  of  $G \setminus M$ ,  $2j < i \leq k$ , let  $w_i$  be the endpoint of  $e_i$  not lying in  $M$ , and define a homeomorphism  $g_i: [0,1]_i \rightarrow C_{w_i}(e_i)$  such that

- 4)  $\text{diam } g_i(t) = 1 - t$ .

The desired imbedding  $h: T \rightarrow C(G)$  is defined by the formula

$$h((t_i)) = \cup_{i \leq j} \{cl(\alpha_i \setminus g_i(t_{2i-1}, t_{2i}))\} \cup \cup_{i > 2j} \{cl(e_i \setminus g_i(t_i))\}.$$

Note that  $h(\emptyset) = M$ . Since  $\text{supp}(t_i) \subset e(L)$  for some  $L \in \mathcal{L}(M)$ , it is clear that  $L \subset h((t_i))$  and that  $h((t_i)) \in C(G)$ . For any  $A \subset G$ , let  $\delta[A] = \sum_1^k \text{diam}(A \cap e_i)$ . Then 3) and 4) imply that for each  $(t_i) \in T$ ,

$$\delta[(h((t_i)) \setminus M) \cup (M \setminus h((t_i)))] = \sum t_i.$$

And in fact,  $h(T) = \{H \in C(G) : \delta[(H \setminus M) \cup (M \setminus H)] \leq 1\}$ .

(Since each edge of  $M$  has length 2, each  $H$  in the above set must contain a large subgraph of  $M$ ). Thus  $h(T)$  is a neighborhood of  $M$  in  $C(G)$ .

It remains to be shown that  $\chi(S) = 1 - \lambda(M^*)$ . The collection  $\{F : F \text{ is a face of } S\}$  is in 1-1 correspondence with the collection  $\{A : \emptyset \neq A \subset e(L) \text{ for some } L \in \mathcal{L}(M)\}$ , the correspondence being given by  $F_A = \{(t_i) \in S : \text{supp}(t_i) \subset A\}$ . For  $F = F_A$  we write  $\text{supp } F = A$ . Then  $\dim F = \# \text{supp } F - 1$ . For each face  $F$ , let  $L_F$  denote the largest  $L \in \mathcal{L}(M)$  for which  $\text{supp } F \subset e(L)$ . For each  $L \in \mathcal{L}(M)$  we will compute  $\sigma(L) = \sum \{(-1)^{\dim F} : F \text{ is a face with } L_F = L\}$ . Then  $\chi(S) = \sum \{\sigma(L) : L \in \mathcal{L}(M)\}$ .

For  $L \in \mathcal{L}(M)$ , the set  $e(L)$  may contain the following types of edges or pairs of edges of  $sd G$ :

- a) pairs of edges  $\{e_{2i-1}, e_{2i}\}$  in  $\tilde{M}$ ,  $i \leq j$ ;
- b) edges  $e_i$  in  $\tilde{M}$ ,  $i \leq 2j$ , which do not occur as members of pairs in a);
- c) edges  $e_i$  in  $G \setminus M$ ,  $i > 2j$ .

Let  $m$  be the number of pairs of edges in  $e(L)$  of type a);  $n$  the number of edges of type b); and  $p$  the number of edges of type c). Note that  $\text{def } L = m + n$ .

For each face  $F$  of  $S$  with  $L_F = L$ , we have  $\text{supp } F \subset e(L)$ , with  $\text{supp } F$  containing all edges in  $e(L)$  of type b) and at least one member from each pair of type a). In other words, for some  $0 \leq \ell \leq m$  and  $0 \leq q \leq p$ ,  $\text{supp } F$  contains  $\ell$  pairs of edges of type a), and  $m - \ell$  single members of the remaining pairs; all  $n$  edges of type b); and  $q$  edges of type c). We must have  $m + n + q > 0$ , and  $\dim F = \#\text{supp } F - 1 = \ell + m + n + q - 1$ .

We now compute  $\sigma(L) = \sum \{(-1)^{\dim F} : L_F = L\}$ . There are four cases to consider.

*Case i):*  $L \neq M$  and  $L \cap \text{bd } M = \emptyset$ . Then  $m + n > 0$  and  $p = 0$ , and

$$\begin{aligned} \sigma(L) &= \sum_{\ell=0}^m \binom{m}{\ell} 2^{m-\ell} (-1)^{\ell+m+n-1} \\ &= (-1)^{m+n-1} \sum_{\ell=0}^m \binom{m}{\ell} 2^{m-\ell} (-1)^\ell \\ &= (-1)^{m+n-1} (2 - 1)^m \\ &= -(-1)^{\text{def } L}. \end{aligned}$$

*Case ii):*  $L \neq M$  and  $L \cap \text{bd } M \neq \emptyset$ . Then  $m + n > 0$  and  $p > 0$ , and

$$\begin{aligned} \sigma(L) &= \sum_{\ell=0}^m \sum_{q=0}^p \binom{m}{\ell} 2^{m-\ell} \binom{p}{q} (-1)^{\ell+m+n+q-1} \\ &= (-1)^{m+n-1} \sum_{\ell=0}^m \binom{m}{\ell} 2^{m-\ell} (-1)^\ell \sum_{q=0}^p \binom{p}{q} (-1)^q \\ &= (-1)^{m+n-1} (2 - 1)^m (1 - 1)^p \\ &= 0. \end{aligned}$$

*Case iii):*  $L = M \neq G$ . Then  $m + n = 0$  and  $p > 0$ , and

$$\begin{aligned} \sigma(L) &= \sum_{q=1}^p \binom{p}{q} (-1)^{q-1} \\ &= -(\sum_{q=0}^p \binom{p}{q} (-1)^q - 1) \\ &= -((1 - 1)^p - 1) \\ &= 1. \end{aligned}$$

*Case iv):*  $L = M = G$ . Then  $e(L) = \emptyset$ , there are no faces  $F$  associated with  $L$ , and  $\sigma(L) = 0$ .

Finally, we compute  $\chi(S) = \sum\{\sigma(L) : L \in \mathcal{L}(M)\}$ . Suppose first that  $M \neq G$ . Then  $\chi(S) = 1 - \sum\{(-1)^{\text{def } L} : L \in \mathcal{L}(M) \text{ with } L \cap \text{bd } M = \emptyset\}$ . If the quotient space  $M^* = M/\text{bd } M$  is given the natural subdivision induced by  $M$ , then the collection  $\{L \in \mathcal{L}(M) : L \cap \text{bd } M = \emptyset\}$  may be identified with the collection  $\mathcal{L}(M^*) \setminus \mathcal{L}_v(M^*)$ , where  $v$  is the vertex of  $M^*$  corresponding to  $\text{bd } M$ . For each  $L$  in these collections,  $\text{def } L$  has the same value with respect to either  $M$  or  $M^*$ . Thus,

$$\begin{aligned}\chi(S) &= 1 - \sum\{(-1)^{\text{def } L} : L \in \mathcal{L}(M^*) \setminus \mathcal{L}_v(M^*)\} \\ &= 1 - (\lambda(M^*) - \lambda_v(M^*)) \\ &= 1 - \lambda(M^*).\end{aligned}$$

Now suppose  $M = G$ . Then  $\text{bd } M = \emptyset$ , and

$$\begin{aligned}\chi(S) &= -\sum\{(-1)^{\text{def } L} : L \in \mathcal{L}(M) \text{ with } L \neq M\} \\ &= 1 - \lambda(M).\end{aligned}$$

This completes the proof of the theorem.

### 5. The Stability Index $\lambda(G)$

We show in this section that every graph without cut points has non-zero stability index. By (4.1) and (4.2), this implies that a nondegenerate  $M \in C(G)$  is stable if  $M/\text{bd } M$  (or  $M$  if  $M = G$ ) has no cut points. By the reduction (3.1), this completes the proof of the stable point characterization theorem (1.1).

In considering  $\lambda(G)$ , we will always assume that the graph  $G$  has at least two edges. With this proviso, we obtain the following invariance property.

5.1. *Proposition.* *The stability index  $\lambda(G)$  is invariant under subdivision, and therefore a topological invariant of  $G$ .*

*Proof.* Given an edge  $e = \langle v, w \rangle$  of  $G$ , and an interior point  $z$  of  $e$ , consider the subdivision  $\tilde{G}$  whose edges are  $\langle v, z \rangle$ ,  $\langle w, z \rangle$ , and the edges of  $G \setminus e$ . For each  $L \in \mathcal{L}(G)$  there are four cases to consider.

*Case i):*  $L \supset e$ . Then  $L \in \mathcal{L}(\tilde{G})$ , and  $\text{def } L$  has the same value with respect to either  $G$  or  $\tilde{G}$ .

*Case ii):*  $L \cap e = \{v\}$ . Then  $L \cup \langle v, z \rangle \in \mathcal{L}(\tilde{G})$ , and  $\text{def } L = \text{def}(L \cup \langle v, z \rangle)$ .

*Case iii):*  $L \cap e = \{w\}$ . This is analogous to ii).

*Case iv):*  $L \cap e = \{v, w\}$ . Then  $L$ ,  $L \cup \langle v, z \rangle$ , and  $L \cup \langle w, z \rangle$  are all members of  $\mathcal{L}(\tilde{G})$ , and their net contributions to  $\lambda(\tilde{G})$  is the same as the contribution  $(-1)^{\text{def } L}$  of  $L$  to  $\lambda(G)$ .

Since each member of  $\mathcal{L}(\tilde{G})$  is uniquely associated with some  $L \in \mathcal{L}(G)$ , under one of the above cases, we have  $\lambda(G) = \lambda(\tilde{G})$ . By iteration,  $\lambda(G)$  is invariant under all subdivisions.

5.2. *Proposition.* *For each vertex  $v$  of  $G$ ,  $\lambda_v(G) = 0$ .*

*Proof.* By induction on the number  $n$  of edges in  $G$ . The statement is easily seen to be true for  $n = 2$ . Assume it is true for  $n = k$ , and consider a graph  $G$  with  $k + 1$  edges. Given a vertex  $v$  of  $G$ , let  $e = \langle v, w \rangle$  be an edge containing  $v$ . We may assume that  $w$  is not an endpoint of  $G$ , since otherwise it is clear that  $\lambda_v(G) = 0$ . Let  $G/e$  be the quotient graph obtained by collapsing  $e$  to a point. For each  $L \in \mathcal{L}_v(G)$ , there are two possibilities:

1) if  $e \in L$ , then  $L/e \in L_V(G/e)$ , and every member of  $L_V(G/e)$  is so obtained;

2) if  $e \notin L$ , then  $G \setminus \text{int } e$  is a graph,  $L \in L_V(G \setminus \text{int } e)$ , and every member of  $L_V(G \setminus \text{int } e)$  is so obtained.

Since the inductive hypothesis applies to both  $G/e$  and  $G \setminus \text{int } e$  (assuming the latter is a graph), this shows that the contribution to  $\lambda_V(G)$  made by those members of  $L_V(G)$  containing  $e$  is 0, as is the contribution made by those members not containing  $e$ . Hence  $\lambda_V(G) = 0$ .

5.3. *Theorem.* Let  $G$  be a finite graph with  $m$  vertices and  $n$  edges. Then  $\lambda(G) = (-1)^{m-n} |\lambda(G)|$ , and  $\lambda(G) = 0$  if and only if  $G$  has a cut point.

*Proof.* By induction on  $m$ . If  $m = 3$ , then either  $n = 2$  or  $n = 3$ . For  $n = 2$ ,  $G \approx \text{arc}$  and  $\lambda(G) = (-1)^0 + 2(-1)^1 + (-1)^2 = 0$  (there is a degenerate large subgraph). For  $n = 3$ ,  $G \approx \text{circle}$  and  $\lambda(G) = (-1)^0 + 3(-1)^1 + 3(-1)^2 = 1$ . Thus the theorem holds for  $m = 3$ .

Assume the theorem holds for all  $m < k$ ; we verify that it holds for  $m = k$ . Consider a graph  $G$  with  $k$  vertices and  $n$  edges. If  $G$  has a cut point  $v$ , then  $L(G) = L_V(G)$ , and  $\lambda(G) = \lambda_V(G) = 0$  by (5.2). Now suppose  $G$  does not have a cut point. It is easily seen that there exists a finite sequence of graphs  $G_1, \dots, G_\lambda = G$  such that  $G_1$  is a cycle with 3 vertices, and each  $G_{i+1}$  is obtained from  $G_i$  either by subdivision or by addition of an edge between a pair of nonadjacent vertices. For each  $i$ , let  $k_i$  and  $n_i$  denote the number of vertices and the number of edges, respectively, of  $G_i$ . We show inductively that the theorem holds for each  $G_i$  (note that each  $G_i$  is without cut points).

As shown in the first paragraph, the theorem holds for  $G_1$ . Suppose it holds for some  $G_i$ ,  $i < \ell$ . Thus,  $\lambda(G_i) = (-1)^{k_i - n_i} |\lambda(G_i)| \neq 0$ . If  $G_{i+1}$  is obtained from  $G_i$  by subdivision, then  $k_{i+1} - n_{i+1} = k_i - n_i$ ,  $\lambda(G_i) = \lambda(G_{i+1})$  by (5.1), and the theorem holds for  $G_{i+1}$ . Thus, we have only to consider the case that  $G_{i+1} = G_i \cup e$ , where  $e$  is an edge of  $G_{i+1}$  with endpoints  $v$  and  $w$  in  $G_i$ .

Let  $G_i^* = G_{i+1}/e = G_i/\{v,w\}$  denote the quotient graph, with vertex  $z$  corresponding to  $\{v,w\}$ . Consider the partition  $\{L \in \mathcal{L}(G_i) : L \cap \{v,w\} = \emptyset\} \cup \{L \in \mathcal{L}(G_i) : L \cap \{v,w\} \neq \emptyset\}$  of the collection  $\mathcal{L}(G_i)$ . The subcollection  $\{L \in \mathcal{L}(G_i) : L \cap \{v,w\} = \emptyset\}$  corresponds to  $\mathcal{L}(G_i^*) \setminus \mathcal{L}_z(G_i^*)$ , hence the contribution made by this subcollection to the index  $\lambda(G_i)$  is  $\lambda(G_i^*) - \lambda_z(G_i^*) = \lambda(G_i^*)$ , by (5.2). We have  $\{L \in \mathcal{L}(G_i) : L \cap \{v,w\} \neq \emptyset\} = \mathcal{L}(G_{i+1}) \setminus \mathcal{L}_e(G_{i+1})$ , where  $\mathcal{L}_e(G_{i+1}) = \{L \in \mathcal{L}(G_{i+1}) : e \subset L\}$ . Since  $\mathcal{L}_e(G_{i+1})$  corresponds to  $\mathcal{L}_z(G_i^*)$ , its contribution to the index  $\lambda(G_{i+1})$  is 0. Thus, the contribution made to  $\lambda(G_{i+1})$  by the subcollection  $\{L \in \mathcal{L}(G_i) : L \cap \{v,w\} \neq \emptyset\}$  is  $\lambda(G_{i+1})$ ; since  $n_{i+1} = n_i + 1$ , its contribution to  $\lambda(G_i)$  is  $-\lambda(G_{i+1})$ . This shows that  $\lambda(G_i) = \lambda(G_i^*) - \lambda(G_{i+1})$ .

Since  $G_i^*$  has  $k_i - 1$  vertices, and  $k_i \leq k$ , our induction hypothesis gives  $\lambda(G_i^*) = (-1)^{k_i - 1 - n_i} |\lambda(G_i^*)|$ . In other words, if  $\lambda(G_i^*)$  is non-zero, it has the opposite sign from  $\lambda(G_i)$ . It follows that  $\lambda(G_{i+1}) = \lambda(G_i^*) - \lambda(G_i)$  is non-zero, with opposite sign from  $\lambda(G_i)$ . Since  $k_{i+1} - n_{i+1} = k_i - n_i - 1$ , the theorem holds for  $G_{i+1}$ . This completes the inductive proof.



## 6. Stable Points in $C_V(X)$

For a nonempty finite subset  $V$  of the Peano continuum  $X$ , let  $C_V(X) = \{M \in C(X) : V \subset M\}$ .

6.1. *Theorem.*  $M \in C_V(X)$  is stable if and only if:

a)  $M$  is a finite union of free arcs;

b)  $\text{bd } M$  does not separate  $M$ ;

c)  $V \subset \text{int } M$ ; and

d) for each  $p \in \text{int } M$ , each component of  $M \setminus \{p\}$  meets  $V \cup \text{bd } M$ .

*Proof.* We first show the conditions are necessary. Suppose  $M$  is stable in  $C_V(X)$ . Then  $M$  must be a finite graph, by the same arguments used for (2.1) and (2.2). (We may assume that the finite graph  $T \subset K \cap \bar{P}$  contains the finite set  $V \cap P$ , and that  $\phi: \bar{P} \rightarrow C(K)$  is a map with  $\phi(y) = \{y\}$  for each  $y \in T \cup \text{bd } P$ . Then the induced map  $f: C(X) \rightarrow C(X)$  sends  $C_V(X)$  into itself.)

Likewise, the proof of (2.3) shows that  $\text{int } M$  is a dense connected subset of  $M$ . Thus  $\text{bd } M$  is a finite set which does not separate  $M$ , and  $M$  is a finite union of free arcs in  $X$ . And, since  $V \subset \text{int } f(A)$  for each  $A \in C_V(X)$ , we must have  $V \subset \text{int } M$ .

Finally, suppose there exists  $p \in \text{int } M$  such that some component  $N$  of  $M \setminus \{p\}$  is disjoint from  $V \cup \text{bd } M$ . Then  $N$  is also a component of  $X \setminus \{p\}$ , and  $X = G \cup Y$ , where  $G = \bar{N} \subset M$  is a nondegenerate graph,  $Y = X \setminus N$  is a continuum containing  $V$ , and  $G \cap Y = \{p\}$ . If  $p \in V$ ,  $C_V(X)$  is homeomorphic to  $C_p(G) \times C_V(Y)$ , under a homeomorphism sending  $M$  to  $(G, M \cap Y)$ . Then the stability of  $M$  in  $C_V(X)$  implies

stability of  $G$  in  $C_p(G)$ , contradicting the result of Lynch referred to in the proof of (2.4). If  $p \notin V$ , set  $W = V \cup \{p\}$ . Then  $C_W(X)$  is a neighborhood of  $M$  in  $C_V(X)$ , thus  $M$  is stable in  $C_W(X)$ , and this leads to a contradiction as above.

Now suppose  $M \in C_V(X)$  satisfies the conditions a) through d). For  $V = \{v_1, \dots, v_n\}$ , consider the Peano continuum  $\hat{X} = X \cup I_1 \cup \dots \cup I_n$ , where each  $I_i$  is an arc with endpoints  $\{v_i, w_i\}$ ,  $I_i \cap X = \{v_i\}$ , and  $I_i \cap I_j = \emptyset$  for  $i \neq j$ . Set  $\hat{M} = M \cup J_1 \cup \dots \cup J_n$ , where each  $J_i$  is a proper subarc of  $I_i$  containing the endpoint  $v_i$ . Since  $C_V(\hat{X})$  is homeomorphic to  $C_V(X) \times \prod_1^n C_{V_i}(I_i)$ , with  $\hat{M}$  corresponding to  $(M, J_1, \dots, J_n)$ , it suffices to show that  $\hat{M}$  is stable in  $C_V(\hat{X})$ . And since  $C_V(\hat{X})$  is a neighborhood of  $\hat{M}$  in  $C(\hat{X})$ , this is equivalent to showing that  $\hat{M}$  is stable in  $C(\hat{X})$ .

We verify that  $\hat{M}$  satisfies the conditions for stability given in (1.1). Clearly,  $\hat{M}$  is a finite union of free arcs in  $\hat{X}$ . We have  $bd \hat{M} = bd_X M \cup \{w_1, \dots, w_n\}$ . Since  $bd_X M$  does not separate  $M$  and is disjoint from  $V$ ,  $bd \hat{M}$  does not separate  $\hat{M}$ . To complete the argument that the quotient space  $\hat{M}/bd \hat{M}$  has no cut points, we show that for each  $p \in \text{int } \hat{M}$ , each component  $\hat{N}$  of  $\hat{M} \setminus \{p\}$  meets  $bd \hat{M}$ . If  $p \in J_i \setminus \{v_i\}$  for some  $i$ , this is clear (note that in the case  $M = X$ , condition d) forces  $V$  to be nondegenerate). On the other hand, if  $p \in \text{int}_X M$ , then each component  $\hat{N}$  of  $\hat{M} \setminus \{p\}$  either contains a component  $N$  of  $M \setminus \{p\}$  or is of the form  $J_i \setminus \{v_i\}$  for some  $i$ . In the latter case,  $w_i \in \hat{N}$ , and we are done. In the former case, condition d) guarantees that the component  $N$

of  $M \setminus \{p\}$  meets either  $\text{bd}_X M$  or  $V$ . If  $N$  meets  $\text{bd}_X M \subset \text{bd } \hat{M}$ , we are done. And if  $v_i \in N$  for some  $i$ , then  $w_i \in \hat{N}$ . Thus in any case  $\hat{N}$  meets  $\text{bd } \hat{M}$ .

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