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by

S. W. Davis

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
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0. Introduction

In this article, we survey the so called "strict p-space problem" and present some recent partial results.* Among the new results given is a characterization of paracompact p-spaces which is reminiscent of the Alexandroff-Urysohn Metrization Theorem.

Definition 0.1. [A_1] Suppose X is a Tychonoff space. A *pluming* of X in its Stone-Čech compactification βX is a sequence $\langle \gamma_n : n \in \omega \rangle$ of open collections in βX, each covering X, such that for each $x \in X$, $\bigcap_{n\in\omega} st(x,\gamma_n) \subseteq X$.

A *strict pluming* of X in βX is a pluming $\langle \gamma_n : n \in \omega \rangle$ with the additional property that for each $x \in X$ and each $n \in \omega$, there exists $m \in \omega$ such that $\overline{st(x,\gamma_m)} \subseteq st(x,\gamma_n)$.

A space X which has a pluming in βX is called a p-*space*, [A_1]. A space which has a strict pluming is called a *strict* p-*space*, [A_2].

Since a locally compact space X must be open in βX, it is clear that all locally compact spaces are p-spaces. It is also clear that all spaces X which are Čech complete (i.e. X is a G_δ-set in βX [Č]) are p-spaces. It is also

*After this paper was submitted, Jiang Shouli, a graduate student at the University of Wisconsin, solved this problem by answering Question 1.1 (and hence also Questions 1.4 and 1.7) in the affirmative.

true that all metrizable spaces are p-spaces, although it is not quite as immediate from the definition.

Properties which are defined in terms of both a space and some external structure on the space are often awkward to apply and to understand. Fortunately, in the late 1960's, D. K. Burke gave us internal characterizations of both p-space and strict p-space, [B_1], [BS]. These have become the working definitions of these properties. Due to the theme of this article, we will state the characterization only of strict p-space.

Theorem 0.2. [BS] *A Tychonoff space* X *is a strict* p-*space if and only if there exists a sequence* $\langle \mathcal{G}_n : n \in \omega \rangle$ *of open covers of* X *such that* $\mathcal{G}_{n+1}$ *refines* $\mathcal{G}_n$, *for each* $n \in \omega$, *and for* $x \in X$, *the set* $P_x = \bigcap_{n\in\omega} st(x,\mathcal{G}_n)$ *is compact and* $\{st(x,\mathcal{G}_n) : n \in \omega\}$ *is a local base at* P_x, *i.e. if* U *is open and* $P_x \subseteq U$, *then there is* $n \in \omega$ *with* $st(x,\mathcal{G}_n) \subseteq U$.

From Burke's criterion, it is clear that all metrizable spaces and all Moore spaces are strict p-spaces. Indeed, if $\langle \mathcal{G}_n : n \in \omega \rangle$ is a development for X, then for each $x \in X$, we have $P_x = \{x\}$. For lack of a better term, we shall call the sequence of open covers in Burke's criterion a *strict* p-*sequence*. In the sequel, we shall also use Burke's P_x notation as above.

From the outset, discussion of p-spaces and strict p-spaces has been involved with covering properties. In fact, probably the major reason for the interest in these classes came from the attempts in the early 1960's to characterize the spaces which are preimages of metrizable spaces under perfect mappings, i.e. mappings which are closed, continuous, and inverses of points are compact. This was done by Arhangel'skiĭ.

Theorem 0.3. [A_1] *A Tychonoff space* X *is the perfect preimage of a metrizable space if and only if* X *is a paracompact* p-*space*.

In this discussion, the main covering property with which we will be concerned is θ-refinability.

Definition 0.4. [WW] A space X is said to be θ-*refinable* if and only if for every open cover $\mathcal{U}$ of X there exists a sequence $\langle \mathcal{V}_n: n \in \omega \rangle$ of open covers of X, each refining $\mathcal{U}$, such that for each $x \in X$, there exists $n_x \in \omega$ with $\text{ord}(x,\mathcal{V}_{n_x}) = |\{V: x \in V \in \mathcal{V}_{n_x}\}| < \omega$.

If the collections $\mathcal{V}_n$ are not required to cover X, but merely be open partial refinements of $\mathcal{U}$ and for each $x \in X$, there exists $n_x \in \omega$ with $0 < \text{ord}(x,\mathcal{V}_{n_x}) < \omega$, then we say X is *weakly* θ-*refinable*, [BL].

It is not difficult to see that all metacompact spaces [AD] and all subparacompact spaces [B_2] are θ-refinable, and that all perfect (= closed sets are G_δ-sets) weakly θ-refinable spaces are subparacompact.

It can be argued that θ-refinable is related to subparacompact in essentially the same way that metacompact is related to paracompact. For this reason, it has become fairly common in recent years to use the term *submetacompact* for θ-refinable. We will use the traditional terminology, but the reader should be aware of the new term.

1. The Problem

In this section, we state the main question. In addition, there are two auxillary questions which are intimately related and independently interesting. We are therefore inclined to lump the three of them together as the "strict p-space problem."

Question 1.1. Is every strict p-space θ-refinable?

The earliest published paper of which this author is aware which specifically states this question is [CJ]. However, it is essentially stated in [B_3].

It is shown in [B_1] that every θ-refinable p-space is a strict p-space. So an affirmative answer to this question would provide a very pretty characterization of strict p-spaces.

Part of what makes this question so tantalizing is contained in the following characterization of θ-refinable. First, let us make one more definition. We say a collection $\mathcal{V}$ of subsets of a space X is an F-*refinement* of a collection $\mathcal{U}$ of subsets of X if and only if for each $V \in \mathcal{V}$ there is a *finite* $\mathcal{W} \subseteq \mathcal{U}$ with $V \subseteq \cup\mathcal{W}$.

Theorem 1.2. [J] *A regular space* X *is* θ-*refinable if and only if every open cover of* X *has a* σ-*closure preserving* F-*refinement.*

It is an easy exercise to show the following:

Theorem 1.3. *Every open cover of a strict* p-*space has a* σ-*cushioned* F-*refinement.*

Dating back to the landmark work on paracompactness by Michael in the 1950's, most covering properties which can be characterized by closure preserving collections can also be characterized in exactly the same way by replacing "closure preserving" by "cushioned." If that is true of θ-refinable, (and it may be, we simply don't know), then the strict p-space problem is solved.

Interestingly, the "closure preserving" versus "cushioned" relationship is at the heart of the "M_1 versus M_3" problem as well, [G].

Question 1.4. Is every strict p-space with a G_δ-diagonal developable?

This would be answered by an affirmative answer to Question 1.1, in view of the following theorem.

Theorem 1.5. [K] *A space* X *is developable if and only if* X *is a θ-refinable* p-*space with a* G_δ-*diagonal.*

We remind the reader that X is said to have a G_δ-diagonal if and only if the diagonal of $X \times X$ is a G_δ-set in $X \times X$. If we increase the strength slightly of the diagonal condition, then the question has an affirmative answer.

Theorem 1.6. *If* X *is a strict* p-*space with a* $\overline{G}_\delta$-*diagonal* [K], [A_3], *or a* G_δ^*-*diagonal* [H], *then* X *is developable.*

This result eliminates many popular construction techniques, since any submetrizable space, i.e. a space with a weaker metrizable topology, has these strong diagonal conditions simply because the metric topology has them. So the search for a potential counterexample is likely to be difficult.

Question 1.7. Is every perfect image of a strict p-space also a strict p-space?

Once again, this question would be answered by an affirmative answer to Question 1.1, in view of the following theorem.

Theorem 1.8. [W] *Every perfect image of a* θ-*refinable* p-*space is a* θ-*refinable* p-*space.*

2. Results

At the heart of this problem lies the understanding (or lack thereof) of the covering properties of strict p-spaces. There is definitely something at work here. The two usual tests for the existence of covering properties for a class of spaces are (1) "Are all countably compact members of the class compact?" and (2) "Are all $\aleph_1$-compact members of the class Lindelöf?" To both of these questions the answer is "yes" for the class of strict p-spaces. We need to discover exactly what covering properties are present in strict p-spaces.

The following is an easy first step. See [D_1] for definitions.

Theorem 2.1. [D_2] *If* X *is a strict* p-*space, then* X *satisfies property* $|X|L$.

On the other hand, Burke has given an example in [B_2] of a strict p-space which is not subparacompact. In 1979, Chaber and Junnila solved the problem for locally compact spaces.

Theorem 2.2. [CJ] *Every locally compact strict p-space is* θ-*refinable.*

The author used a very similar approach to obtain the following. See [Au] for definition.

Theorem 2.3. [D_2] *Every locally Lindelöf strict p-space is* $\delta\theta$-*refinable.*

K. Wagner subsequently improved this.

Theorem 2.4. [Wa] *Every locally* $\aleph_1$-*compact strict p-space is* θ-*refinable.*

It is not known if strict p-spaces are even weakly θ-refinable. Wagner has investigated this quite a lot, and she has several nice results about when weakly θ-refinable strict p-spaces must be θ-refinable [Wa].

Before discussing the results related to the G_δ-diagonal condition, we state a lemma due to Cedar.

Theorem 2.5. [C] *A space* X *has a* G_δ-*diagonal if and only if there is a sequence* $\langle \mathcal{U}_n : n \in \omega \rangle$ *of open covers of* X *such that for each* $x \in X$, $\bigcap_{n\in\omega} st(x, \mathcal{U}_n) = \{x\}$.

Remark. Obviously, we can arrange to have $\mathcal{U}_{n+1}$ refine $\mathcal{U}_n$, for each $n \in \omega$. From now on, we will make that assumption.

It is tempting to assert that if we have a strict p-sequence and a G_δ-diagonal sequence as in Ceder's characterization, we need only blend them by the usual pairwise-intersection-at-each-level technique to obtain a development. Alas, this fails. Still, in the presence of G_δ-diagonal the sets P_x are very nice, in fact compact and metrizable by Sneider's Theorem.

Theorem 2.6. [S] *Every compact* T_2 *space with a* G_δ*-diagonal is metrizable.*

In the hope of using the nice structure of these sets, we prove the following result.

Theorem 2.7. *Suppose* $\langle \mathcal{G}_n : n \in \omega \rangle$ *is a strict* p*-sequence for* X*, and* $x \in X$. *If* $\langle U_n : n \in \omega \rangle$ *is a sequence of open sets in* X *such that for each* $n \in \omega$ *and* $k \in \omega$ *there exists* $m \in \omega$ *with* $m \geq k$ *and* $x \in U_m \subseteq U_n$ *and* $\langle \overline{U}_n \cap P_x : n \in \omega \rangle$ *is a neighborhood base at* x *in the subspace* P_x, *then* $\langle U_n \cap st(x,\mathcal{G}_n) : n \in \omega \rangle$ *is a neighborhood base at* x *in the space* X.

Proof. Suppose U is open and $x \in U$. Choose $n \in \omega$ such that $\overline{U}_n \cap P_x \subseteq U \cap P_x$. Let $W = X \setminus \overline{U}_n$, and note that $P_x \subseteq U \cup W$. Choose $k \in \omega$ such that $st(x,\mathcal{G}_k) \subseteq U \cup W$. Now choose $m \geq k$ with $U_m \subseteq U_n$, then $x \in U_m \cap st(x,\mathcal{G}_m) \subseteq U_n \cap st(x,\mathcal{G}_k) \subset U_n \cap (U \cup W) \subseteq U_n \cap U \subseteq U$.

If $\langle U_n : n \in \omega \rangle$ is a decreasing sequence of open sets with $\bigcap_{n\in\omega} \overline{U}_n = \{x\}$, then by the compactness of P_x the hypothesis of 2.7 is satisfied. Hence if X has a G_δ-diagonal sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ with $\{x\} = \bigcap_{n\in\omega} \overline{st(x,\mathcal{U}_n)}$ (then X is said to have a G_δ^*-diagonal, [H]) and if X is a strict p-space, then 2.7 gives us the result that X is developable.

Toward obtaining the above situation, we define the following properties.

Definition 2.8. We say a space X satisfies *Property (*)* if and only if for every open cover $\mathcal{U}$ of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of open covers of X, each refining $\mathcal{U}$, such that for each $x \in X$ there is $n_x \in \omega$ with $\overline{st(x,\mathcal{V}_{n_x})} \subseteq st(x,\mathcal{U})$.

If the terms $\mathcal{V}_n$ are not required to cover X, we say X satisfies *weak (*)*.

If the terms $\mathcal{V}_n$ are all the same, we say X satisfies *strong (*)*.

The following is an easy exercise.

Theorem 2.9. *If* X *is a regular space, then the following are true:*

i) *if* X *is metacompact, then* X *satisfies strong (*).*

ii) *if* X *is* θ*-refinable, then* X *satisfies (*).*

iii) *if* X *is weakly* θ*-refinable, then* X *satisfies weak (*).*

We now see that (*) is exactly what we need in the G_δ-diagonal situation.

Theorem 2.10. *If a space* X *has a* G_δ*-diagonal and satisfies (*), then* X *has a* G_δ^**-diagonal.*

Proof. To any G_δ-diagonal sequence for X, we apply (*) at each level. Now order the resulting $\omega \times \omega$ sequence by ω, and it will be a G_δ^*-diagonal sequence.

Corollary 2.10.1. *If* X *is a strict* p-*space with a* G_δ-*diagonal which satisfies* (*), *then* X *is developable.*

We can make similar use of 2.7 in the weak (*) case to prove the following result.

Theorem 2.11. *If* X *is a strict* p-*space with a* G_δ-*diagonal which satisfies weak* (*), *then* X *is quasi-developable.*

Proof. As in 2.10, we apply weak (*) to each term of a G_δ-diagonal sequence, and then reorder by ω to obtain a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of open collections where individual terms may not cover X, but for each $x \in X$, $\cap\{st(x,\mathcal{V}_n) : st(x,\mathcal{V}_n) \neq \emptyset\} = \{x\}$.

Now if $\langle \mathcal{G}_n : n \in \omega \rangle$ is a strict p-sequence for X, then by 2.7, for each $x \in X$, $\{st(x,\mathcal{V}_n) \cap st(x,\mathcal{G}_n) : n \in \omega\}$ is a neighborhood base at x. Hence, letting $\mathcal{W}_n = \{V \cap G : V \in \mathcal{V}_n, G \in \mathcal{G}_n\}$, we have that $\langle \mathcal{W}_n : n \in \omega \rangle$ is a quasidevelopment.

Corollary 2.11.1. *Every weakly* θ-*refinable strict* p-*space with a* G_δ-*diagonal is quasidevelopable.*

A similar result is given by Bennett and Berney in [BB] where it is shown that every hereditarily weakly θ-refinable p-space with a G_δ-diagonal is quasidevelopable. We note that the word "hereditarily" is omitted in [BB], but it seems to be needed for that approach to the result.

It seems natural at this point to ask the following question.

Question 2.12. Does every strict p-space satisfy (*)?

An example giving a negative response will, of course, answer 1.1. Also, by 2.10.1, a positive answer gives a positive answer to 1.4. In either case, an answer to this question would be interesting. In fact, better understanding of (*) would be interesting. By the following result, it is not truly a covering property.

Theorem 2.13. *The ordinal space* ω_1 *satisfies strong (*).*

Proof. Suppose $\mathcal{U}$ is an open cover of ω_1. For each $\alpha \in \omega_1$, choose $f(\alpha) < \alpha$ such that for some $U \in \mathcal{U}$, $(f(\alpha), \alpha] \subseteq U$. By the Pressing Down Lemma, there exists $\gamma < \omega_1$ such that $(\gamma, \omega_1) \subseteq st(\alpha, \mathcal{U})$ for every $\alpha > \gamma$. Choose a finite set $\{V_1, V_2, \cdots, V_n\}$ of open subsets of ω_1 whose closures refine $\mathcal{U}$ such that $\cup_{k=1}^{n} V_k = [0, \gamma]$. Now we define $\mathcal{V} = \{V_1, V_2, \cdots, V_n\} \cup \{U \cap [\gamma + 1, \omega_1) : U \in \mathcal{U}\}$. Now $\mathcal{V}$ is an open cover of ω_1 and refines $\mathcal{U}$. If $\alpha \leq \gamma$, $\overline{st(\alpha, \mathcal{V})} = \cup\{\overline{V}_i : 1 \leq i \leq n, \alpha \in V_i\} \subseteq st(\alpha, \mathcal{U})$. If $\alpha > \gamma$, $\overline{st(\alpha, \mathcal{V})} \subseteq (\gamma, \omega_1) \subseteq st(\alpha, \mathcal{U})$.

For our last result, we shall need the neighborhood metrization theorem of Nagata. We state the form given in the text by Willard [W].

Theorem 2.14. [N] *A* T_o *space* X *is metrizable if and only if each* $x \in X$ *possesses a countable neighborhood base* $\{U_{x,n} : n \in \omega\}$ *with the following properties:*

(a) *if* $y \in U_{x,n}$, *then* $U_{y,n} \subseteq U_{x,n-1}$

(b) *if* $y \notin U_{x,n-1}$, *then* $U_{y,n} \cap U_{x,n} = \emptyset$.

We now give a characterization of paracompact strict p-spaces (which are the same as paracompact p-spaces). This result is very similar in formulation to the Alexandroff-Urysohn Metrization Theorem which is, of course, a characterization of paracompact developable spaces. The author has been informed by H. H. Wicke that this result was known to J. M. Worrell in 1967, and, while never published, it is alluded to in [Wo_1].

Theorem 2.15. *A completely regular* T_2*-space* X *is a paracompact* p*-space if and only if* X *has a strict* p*-sequence* $\langle \mathcal{G}_n : n \in \omega \rangle$ *such that if* $U, V \in \mathcal{G}_{n+1}$ *and* $U \cap V \neq \emptyset$*, then there exists* $W \in \mathcal{G}_n$ *such that* $U \cup V \subseteq W$.

Proof. Suppose X is a paracompact p-space. By Arhangel'skiĭ's theorem (0.3), there is a metric space (M,d) and a perfect mapping f of X onto M. For $y \in M$ and $\varepsilon > 0$, we let $B(y,\varepsilon) = \{z \in M: d(y,z) < \varepsilon\}$. For each $n \in \omega\backslash\{0\}$, let $\mathcal{G}_n = \{f^{-1}(B(y,2^{-n})): y \in M\}$, and let $\mathcal{G}_0 = \{X\}$. It is straightforward to show that $\langle \mathcal{G}_n : n \in \omega \rangle$ is the desired strict p-sequence.

Now suppose X has a strict p-sequence of the type indicated. We shall construct a metrizable space Y and a perfect mapping of X onto Y. We define a relation $\sim$ on X by $x \sim y$ if and only if $x \in P_y$. The relation $\sim$ is clearly reflexive and symmetric. Suppose $x \sim y$ and $y \sim z$. For each $n \in \omega$, choose $U_n, V_n \in \mathcal{G}_{n+1}$ with $\{x,y\} \subseteq U_n$ and

$\{y,z\} \subseteq V_n$. Hence there exists $W_n \in \mathcal{G}_n$ with $\{x,z\} \subseteq W_n$. Thus $x \sim z$, and we have that $\sim$ is transitive. Hence $\sim$ is an equivalence relation, and for $x \in X$ the equivalence class determined by x is the set P_x. Let $Y = X/\sim$, and let f be the natural quotient mapping. To see that f is a perfect mapping, we have only to check that f is closed. Suppose $A \subseteq X$ is closed. Let $F = \cup_{x \in A} P_x = f^{-1}(f(A))$. Suppose $x \in X \setminus F$. Thus $P_x \cap A = \emptyset$, so there exists $n \in \omega$ such that $st(x,\mathcal{G}_n) \cap A = \emptyset$. We will show that $st(x,\mathcal{G}_{n+1}) \cap F = \emptyset$. If not, choose $y \in st(x,\mathcal{G}_{n+1}) \cap F$. Since $y \in F$, $P_y \cap A \neq \emptyset$, so there exists $U \in \mathcal{G}_{n+1}$ with $y \in U$ and $U \cap A \neq \emptyset$. Also choose $V \in \mathcal{G}_{n+1}$ with $\{x,y\} \subseteq V$. Since $y \in U \cap V$, there exists $W \in \mathcal{G}_n$ with $U \cup V \subseteq W$. Thus $W \subseteq st(x,\mathcal{G}_n)$ and $W \cap A \neq \emptyset$, a contradiction. Thus F is a closed set in X, and since Y has the quotient topology, then $f(A)$ is closed in Y. Hence f is a perfect mapping. We now use 2.14 to show that Y is metrizable; completing the proof. For each $y \in Y$, we can chose $x \in X$ with $y = P_x$. We assume that such a choice has been made, and for the remainder of the proof we refer to the point as P_x. For each $n \in \omega$, let $U_{P_x,n} = f(st(x,\mathcal{G}_{2n}))$. Since f is closed and $P_x \subseteq st(x,\mathcal{G}_{2n})$, we have that $U_{P_x,n}$ is a neighborhood of P_x in Y. If $P_y \in U_{P_x,n}$, then there exists $z \in P_y$ such that $z \in st(x,\mathcal{G}_{2n})$. So there is $G_1 \in \mathcal{G}_{2n}$ such that $\{x,z\} \subseteq G_1$ and there is $G_2 \in \mathcal{G}_{2n}$ such that $\{z,y\} \subseteq G_2$. Thus there is $G_3 \in \mathcal{G}_{2n-1}$ such that $\{x,y\} \subseteq G_3$. Hence $y \in st(x,\mathcal{G}_{2n-1})$. Now if $P_z \in U_{P_y,n}$, then, as above, $z \in st(y,\mathcal{G}_{2n-1})$. Hence $z \in st(x,\mathcal{G}_{2n-2})$.

So $P_z \in f(st(x,\mathcal{G}_{2n-2})) = U_{P_x,n-1}$. So condition (a) of Nagata's theorem is satisfied. Now suppose $U_{P_x,n} \cap U_{P_y,n} \neq \emptyset$, say $P_z \in U_{P_x,n} \cap U_{P_y,n}$. Then we have $P_z \cap st(x,\mathcal{G}_{2n}) \neq \emptyset$ and $P_z \cap st(y,\mathcal{G}_{2n}) \neq \emptyset$. Choose $r \in P_z \cap st(x,\mathcal{G}_{2n})$ and $t \in P_z \cap st(y,\mathcal{G}_{2n})$. Choose G_1, G_2, G_3, G_4 elements of $\mathcal{G}_{2n}$ such that $\{x,r\} \subseteq G_1$, $\{r,z\} \subseteq G_2$, $\{z,t\} \subseteq G_3$, and $\{t,y\} \subseteq G_4$. Choose $U_1,U_2 \in \mathcal{G}_{2n-1}$ with $G_1 \cup G_2 \subseteq U_1$ and $G_3 \cup G_4 \subseteq U_2$. Now $z \in U_1 \cap U_2$, so there exists $V \in \mathcal{G}_{2n-2}$ with $U_1 \cup U_2 \subseteq V$. Thus we have $\{x,y\} \subseteq V$, so $y \in st(x,\mathcal{G}_{2n-2})$. Hence $P_y \in U_{P_x,n-1}$. Thus condition (b) of Nagata's theorem is satisfied, so Y is metrizable, and the proof is complete.

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Miami University

Oxford, Ohio 45056