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COUNTABILITY PROPERTIES OF FUNCTION SPACES WITH SET-OPEN TOPOLOGIES

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1. Introduction and Definitions

This paper organizes and extends some of the ideas found in the papers listed in the references. The general problem is to characterize topological properties of the space, $C(X)$, of continuous real-valued functions on a topological space X . The topologies considered on $C(X)$ will be "set-open" topologies generated by sets of functions which carry a member of a certain family of compact subsets of X into some open set in the space of real numbers R . These families of subsets of X can be studied in their own right, and can be used to decide when $C(X)$ has certain properties.

Throughout this paper all spaces, X, Y, Z, \dots , will be Tychonoff spaces. If α is a family of compact subsets of a space X , then the space $C_\alpha(X)$ is the set $C(X)$ with the following topology. The subbasic open sets are of the form

$$[A, V] = \{f \in C(X) : f(A) \subset V\},$$

where $A \in \alpha$ and V is open in R . This is a completely regular topology on $C(X)$. In fact $C_\alpha(X)$ is a topological vector space, and as a consequence $C_\alpha(X)$ is a homogeneous space.

In order to obtain characterizations of certain properties of $C_\alpha(X)$, we will need to impose some restriction on α .

We will say that α is *admissible* if it satisfies the following two conditions:

(U) For every $A, B \in \alpha$, there exists a $C \in \alpha$ such that $A \cup B \subset C$.

(R) For every $A \in \alpha$ and finite open cover \mathcal{U} of A in X , there exists a finite $\beta \subset \alpha$ such that $A \subset \cup\beta$ and β refines \mathcal{U} .

From the comment after Corollary 2.3, we see that there is no loss of generality in assuming (U); so that (U) is included here for convenience of notation. An incidental consequence of (R) is that $C_\alpha(X)$ will then be a Hausdorff space.

For example, some admissible α on \mathbb{R}^2 with the usual topology include the following: all compact sets, all finite sets, all compact countable sets, all Cantor sets, all arcs, and all closed squares. Except for the first and the last, these all generate different topologies on $C(\mathbb{R}^2)$. This can be seen from Corollary 2.3.

The family of all compact subsets of X generates the compact-open topology, denoted by $C_k(X)$. Also the family of all finite subsets of X generates the topology of point-wise convergence, denoted by $C_p(X)$.

Given a family α of compact subsets of X , there are two concepts which play a key role in characterizing the "local countability" properties of $C_\alpha(X)$. First, an α -cover of X is a family \mathcal{C} of subsets of X such that for every $A \in \alpha$, there exists a $C \in \mathcal{C}$ satisfying $A \subset C$. Secondly, an α -sequence in X is a sequence $\{C_n\}$ of subsets of X such

that for every $A \in \alpha$, there exists a number M with the property that for all positive integers $n \geq M$, $A \subset C_n$.

We list some of the "local countability" properties of $C_\alpha(X)$ which can be characterized in terms of properties of α (see section 3). For each of the following, α is an admissible family of compact subsets of X .

(1) Points of $C_\alpha(X)$ are G_δ -subsets iff α contains a countable subfamily whose union is dense in X .

(2) $C_\alpha(X)$ is first countable iff α contains a countable α -cover of X .

(3) $C_\alpha(X)$ is a Fréchet space iff every open α -cover of X contains an α -sequence in X .

(4) $C_\alpha(X)$ has countable tightness iff every open α -cover of X contains a countable α -cover of X .

A few "global countability" properties of $C_\alpha(X)$, such as second countable, can also be characterized in terms of properties of α .

In the next section we will be comparing different families of compact subsets of a space. To this end, we introduce the following concept. If α and β are families of compact subsets of X , then α can be approximated by β provided that for every $A \in \alpha$ and every open U in X containing A , there exist $B_1, \dots, B_n \in \beta$ with $A \subset B_1 \cup \dots \cup B_n \subset U$.

2. Basic Tools

A useful concept for studying function spaces is the "induced function." Every continuous function $\phi: X \rightarrow Y$ induces a function $\phi^*: C(Y) \rightarrow C(X)$ defined by $\phi^*(f) = f \circ \phi$

for each $f \in C(Y)$. If ϕ maps onto a dense subspace of Y , then ϕ^* will be one-to-one.

To determine the topological properties of ϕ^* , we need topologies on $C(X)$ and $C(Y)$; so let α and β be families of compact subsets of X and Y , respectively. We can now compare α and β by using ϕ . Let Z be the closure of $\phi(X)$ in Y . The notation $\phi(\alpha)$ will be used for the family $\{\phi(A) : A \in \alpha\}$ of compact subsets of Z . Also the notation $\beta \cap \phi$ will be used for the family $\{B \cap Z : B \in \beta\}$ of compact subsets of Z .

Theorem 2.1. *Let $\phi : X \rightarrow Y$ be a continuous function, and let α and β be families of compact subsets of X and Y , respectively.*

(a) *Then $\phi^* : C_\beta(Y) \rightarrow C_\alpha(X)$ is continuous if and only if $\phi(\alpha)$ can be approximated by β .*

(b) *Also $\phi^* : C_\beta(Y) \rightarrow C_\alpha(X)$ is open (onto its image) if and only if $\beta \cap \phi$ can be approximated by $\phi(\alpha)$.*

Proof. Suppose first that $\phi(\alpha)$ can be approximated by β . To see that ϕ^* is continuous, let $f \in C_\beta(Y)$, and let $A \in \alpha$ and V be open in R with $\phi^*(f) \in [A, V]$. Then there exist $B_1, \dots, B_n \in \beta$ such that $\phi(A) \subset B_1 \cup \dots \cup B_n \subset f^{-1}(V)$. Define $T = [B_1, V] \cap \dots \cap [B_n, V]$, which is a neighborhood of f in $C_\beta(Y)$. Now it is easy to see that $\phi^*(T) \subset [A, V]$.

Conversely, suppose that $\phi(\alpha)$ cannot be approximated by β . Then there exist $A \in \alpha$ and open W in Y containing $\phi(A)$ such that for every $B_1, \dots, B_n \in \beta$, either $\phi(A) \not\subset B_1 \cup \dots \cup B_n$ or $B_1 \cup \dots \cup B_n \not\subset W$. Now choose $f \in C_\beta(Y)$ such that $f(\phi(A)) = 1$ and $f(Y \setminus W) = 0$, and let $V = R \setminus [-1/2, 1/2]$.

Then $[A, V]$ is a neighborhood of $\phi^*(f)$ in $C_\alpha(X)$. To see that ϕ^* is not continuous at f , let $G = [B_1, V_1] \cap \dots \cap [B_n, V_n]$ be any basic open set in $C_\beta(Y)$ containing f . We may assume that the V_i 's are bounded open intervals.

Case 1. Suppose $\phi(A) \not\subset B_1 \cup \dots \cup B_n$. Then there exists a $a \in A$ such that $\phi(a) \notin B_1 \cup \dots \cup B_n$. In this case choose $g \in C_\beta(Y)$ so that $g(x) = f(x)$ for each $x \in B_1 \cup \dots \cup B_n$ and $g(a) = 0$. Clearly $g \in G$, while $\phi^*(g) \notin [A, V]$.

Case 2. Suppose $\phi(A) \subset B_1 \cup \dots \cup B_n$. Define $\beta = \{B_1, \dots, B_n\}$. By induction, we may choose a subfamily β' of β so that each member of $\beta \setminus \beta'$ is not contained in W (so its corresponding V contains 0) and so that there is a $B \in \beta'$ with $B \cap \phi(A) \not\subset \cup(\beta' \setminus \{B\})$. Let $a \in A$ such that $\phi(a) \in B \setminus \cup(\beta' \setminus \{B\})$. Then define $f': Y \rightarrow R$ by $f'(x) = f(x)$ if $x \notin a$ and $f'(a) = 0$. Let $F_k(Y)$ be the set of all real-valued functions on Y with the compact-open topology (see [10]). Let G' be the basic open subset of $F_k(Y)$ defined by $G' = [B_1, V_1] \cap \dots \cap [B_n, V_n] \cap [\{\phi(a)\}, (-1/2, 1/2)]$, which is nonempty since it contains f' . Since $C_k(Y)$ is dense in $F_k(Y)$ (see [10]), then there exists a $g \in G' \cap C_k(Y)$. In particular, $g \in G$. But since $g(\phi(a)) \in (-1/2, 1/2)$, then $\phi^*(g) \notin [A, V]$.

We omit the proof of part (b), since it can be done in a similar manner.

Corollary 2.2. Let $\phi: X \rightarrow Y$ be a continuous function such that $\phi(X)$ is dense in Y , and let α and β be families of compact subsets of X and Y , respectively. Then

$\phi^*: C_\beta(Y) \rightarrow C_\alpha(X)$ is an embedding if and only if each of $\phi(\alpha)$ and β can be approximated by the other.

Corollary 2.3. Let α and β be families of compact subsets of X . Then $C_\alpha(X) = C_\beta(X)$ if and only if each of α and β can be approximated by the other.

From the previous corollary, we see that if α is any family of compact subsets of X and if β is the family of all finite unions of members of α , then $C_\alpha(X) = C_\beta(X)$.

A second useful thing for studying function spaces is the fact that the function space of a topological sum is homeomorphic to the product of the function spaces of the summands. To make this precise, suppose $\{X_i: i \in I\}$ is a pairwise disjoint family of spaces, and let $X = \Sigma_i X_i$. Also for each $i \in I$, let α_i be a family of compact subsets of X_i , and let $\alpha = \cup\{\alpha_i: i \in I\}$ as a family of compact subsets of X .

Define $S: C_\alpha(X) \rightarrow \prod_i C_{\alpha_i}(X_i)$ by $\pi_i(S(f)) = f|_{X_i}$ for each $f \in C_\alpha(X)$ and each $i \in I$. Now S is easily seen to be a bijection. To see that S is a homeomorphism, let $A \in \alpha_i$ and let V be open in R . If W_i is the set $[A, V]$ taken in $C_{\alpha_i}(X_i)$ and W is the set $[A, V]$ taken in $C_\alpha(X)$, then $S^{-1}(\pi_i^{-1}(W_i)) = W$.

In the next section we will consider the metrizable-ness of $C_\alpha(X)$. In anticipation of this, we end this section with a lemma relating the topology on $C_\alpha(X)$ to the supremum metric topology.

Let d be the usual metric on \mathbb{R} bounded by 1; that is, $d(x,y) = \min\{1, |x - y|\}$. Then $C_d(X)$ will denote $C(X)$ with the topology generated by the metric $d^*(f,g) = \sup\{d(f(x), g(x)) : x \in X\}$, which is a complete metric.

Lemma 2.4. *Let α be an admissible family of compact subsets of X , and let $A \in \alpha$. If $\gamma = \{A \cap B : B \in \alpha\}$, then $C_\gamma(A) = C_d(A)$.*

Proof. Since $C_d(A) = C_k(A)$, then every open set in $C_\gamma(A)$ is open in $C_d(A)$. For the other direction, let $f \in C_d(A)$ and let $\epsilon > 0$. Now $\{f^{-1}((f(a) - \epsilon/3, f(a) + \epsilon/3)) : a \in A\}$ is an open cover of A , and thus has a finite sub-cover \mathcal{U} . By property (R), there exists a finite $\beta \subset \alpha$ such that $A \subset \cup\beta$ and β refines \mathcal{U} . Then for each $B \in \beta$, there exists an $a_B \in A$ such that $f(B) \subset V_B$ where $V_B = (f(a) - \epsilon/3, f(a) + \epsilon/3)$. Define $W = \cap\{[A \cap B, V_B] : B \in \beta\}$, which is a neighborhood of f in $C_\gamma(A)$. One can easily check that for each $g \in W$, $d^*(f,g) \leq 2\epsilon/3 < \epsilon$, as desired.

In the sequel, $C_\alpha(A)$ will stand for $C_\gamma(A)$ as defined in Lemma 2.4.

As a consequence of Lemma 2.4, for admissible α , sets of the following form comprise a base for $C_\alpha(X)$:

$$\langle f, A, \epsilon \rangle = \{g \in C(X) : \text{for all } a \in A, \\ |f(a) - g(a)| < \epsilon\},$$

where $f \in C(X)$, $A \in \alpha$, and $\epsilon > 0$. We will use this when working with the zero function f_0 . In this case $\langle f_0, A, \epsilon \rangle = [A, (-\epsilon, \epsilon)]$; so that f_0 has as local base all sets of the form $[A, V]$, where $A \in \alpha$ and V is an open neighborhood of 0 in \mathbb{R} .

3. Countability Properties

Each of the theorems of this section is a characterization of a countability property on $C_\alpha(X)$ which is a generalization or analog of the characterization of that property on $C_p(X)$ or $C_k(X)$. Only the first few proofs will be given to illustrate how the appropriate modification can be made to handle the general case.

There are many "local countability" properties which can be investigated for $C_\alpha(X)$. Most of them fall between the extremes which we deal with in this section. We begin with the weak property of points being G_δ -subsets. This property is equivalent to submetrizability for function spaces; where a submetrizable space is a space from which there exists a continuous bijection onto a metric space.

Theorem 3.1. Let α be an admissible family of compact subsets of X . Then the following are equivalent:

- (a) $C_\alpha(X)$ is submetrizable;
- (b) points of $C_\alpha(X)$ are G_δ -subsets;
- (c) α contains a countable subfamily whose union is dense in X .

Proof. (a) \rightarrow (b) is immediate, so first consider (b) \rightarrow (c). If f_0 is the zero function, then $\{f_0\} = \cap \{W_n : n \in \mathbb{N}\}$, where $W_n = [A_n, V_n]$ are basic open neighborhoods of f_0 . Define $\beta = \{A_n : n \in \mathbb{N}\}$. Let Y be the closure of $\cup \beta$ in X , and suppose there is some x in $X \setminus Y$. Then we could find an $f \in C(X)$ so that $f(x) = 1$ and $f(Y) = 0$. But then $f \in W_n$ for each n , so that $f = f_0$. With this contradiction, we see that $\cup \beta$ is dense in X .

To see (c) \rightarrow (a), let $\{A_n : n \in \mathbb{N}\}$ be a subfamily of α whose union is dense in X . Let $Z = \sum_n A_n$ be the topological sum of the A_n , and let $p: Z \rightarrow X$ be the natural projection. Now $p^*: C_\alpha(X) \rightarrow C_\beta(Z)$ is continuous and one-to-one, where $\beta = \{A_n \cap A : n \in \mathbb{N} \text{ and } A \in \alpha\}$. Also from the previous section we see that $C_\beta(Z)$ is homeomorphic to $\prod_n C_\alpha(A_n)$. But each $C_\alpha(A_n)$ is metrizable by Lemma 2.4.

One generalization of first countability is the concept of being a q -space. A q -space is a space such that for each point there exists a sequence $\{U_n\}$ of neighborhoods of that point so that if $x_n \in U_n$ for each n then $\{x_n\}$ has a cluster point in the space. There are several other properties which are intermediate between first countability and q -space, such as being of point countable type. The next theorem shows that these properties are all equivalent for $C_\alpha(X)$.

Theorem 3.2. Let α be an admissible family of compact subsets of X . Then the following are equivalent:

- (a) $C_\alpha(X)$ is metrizable;
- (b) $C_\alpha(X)$ is a q -space;
- (c) α contains a countable α -cover of X .

Proof. (a) \rightarrow (b) is immediate, so first consider (b) \rightarrow (c). Let $\{W_n : n \in \mathbb{N}\}$ be a sequence of neighborhoods of the zero function f_0 satisfying the definition of q -space at f_0 . We may suppose that each W_n is a basic neighborhood of f_0 as in the proof of (b) \rightarrow (c) in Theorem 3.1. Let each A_n and β be defined as in this proof. To see that $\cup\beta = X$,

suppose there is an $x \in X \setminus U\beta$. Then for each n , choose $f_n \in C(X)$ such that $f_n(x) = n$ and $f_n(A_n) = 0$. Since each $f_n \in W_n$, then the sequence $\{f_n\}$ must cluster in $C_\alpha(X)$. Since this is impossible, we see that $U\beta = X$. Then by Theorem 3.1, $C_\alpha(X)$ is submetrizable. Now a submetrizable q -space is first countable. So f_0 has a countable base $\{\{B_n, V_n\} : n \in \mathbb{N}\}$ for some $B_n \in \alpha$ and open V_n in R . It is easy to check that $\{B_n : n \in \mathbb{N}\}$ is an α -cover of X .

To see (c) \rightarrow (a), let $\{A_n : n \in \mathbb{N}\}$ be an α -cover of X taken from α . Let $Z = \sum_n A_n$ and $p: Z \rightarrow X$ be as in the proof of (c) \rightarrow (a) in Theorem 3.1. This proof also works here because $p^*: C_\alpha(X) \rightarrow C_\beta(Z)$ is an embedding by Corollary 2.2.

Note that statement (c) in Theorem 3.2 can be replaced by the statement that α contains an α -sequence.

Corollary 3.3. *Let α be an admissible family of compact subsets of X . Then $C_\alpha(X)$ is completely metrizable if and only if α contains a countable α -cover of X which generates the topology on X (in the sense that a subset of X is closed iff its intersection with each member of this family is closed).*

Proof. The sufficiency uses the same technique as (c) \rightarrow (a) in the previous theorem. Since p is a quotient map in this case, then $p^*(C_\alpha(X))$ will be closed in $C_\beta(Z)$, and therefore complete.

For the necessity, we know from the previous theorem that α contains a countable α -cover β . One can argue that β generates the topology on X by arguing as in [10] for

the compact-open topology case. The first step is to show that each real-valued function on X which has continuous restriction to each member of α is continuous. Then the last step is to argue as in showing that a hemicompact $k_{\mathbb{R}}$ -space is a k -space.

A space is a *Fréchet space* (has *countable tightness*, respectively) provided that for each subset S and point x in the closure of S , there exists a sequence $\{x_n\}$ in S such that $\{x_n\}$ converges to x (x is in the closure of $\{x_n\}$, respectively). Clearly every first countable space is a Fréchet space, and every Fréchet space has countable tightness.

Theorem 3.4. [4],[5],[6],[7] *Let α be an admissible family of compact subsets of X . Then $C_{\alpha}(X)$ is a Fréchet space if and only if every open α -cover of X contains an α -sequence in X .*

Theorem 3.5. [4],[6],[7] *Let α be an admissible family of compact subsets of X . Then $C_{\alpha}(X)$ has countable tightness if and only if every open α -cover of X contains a countable α -cover of X .*

For the "global countability" properties of $C_{\alpha}(X)$, let us begin by considering separability. Although the characterization of the separability of $C_{\alpha}(X)$ is not in terms of a property of α , it is useful for looking at other properties.

Theorem 3.6. [12],[13] *Let α be a family of compact subsets of X which approximates the family of finite subsets of X . Then $C_\alpha(X)$ is separable if and only if there exists a continuous bijection from X onto a separable metric space.*

By coupling Theorem 3.6 with Theorem 3.2, we see that $C_\alpha(X)$ has a countable base if and only if X is submetrizable and α contains a countable α -cover. We give the following reformulation.

Theorem 3.7. *Let α be an admissible family of compact subsets of X . Then the following are equivalent:*

- (a) $C_\alpha(X)$ is second countable;
- (b) α contains a countable α -cover consisting of metrizable spaces;
- (c) α contains a countable subfamily which approximates α .

The final property we consider in this section is that of having a countable network; that is, a countable family of subsets which generates the topology (in the sense that every open set is a union of members of the family). By an α -network in X we mean a family of subsets of X such that for each $A \in \alpha$ and open U containing A , there is a member of the α -network contained in U and containing A .

Theorem 3.8. [8],[10] *Let α be an admissible family of compact subsets of X . Then $C_\alpha(X)$ has a countable network if and only if X has a countable α -network.*

Most of the countability properties in this section can be expressed in terms of cardinal functions, so that these theorems then have higher cardinality analogs.

4. Gruenhagen Game

There is a property introduced in [5], called W-space, which is intermediate between the first countability and Fréchet space properties. This property is defined in terms of an infinite two person game.

The following is a description of the Gruenhagen game, $\Gamma_*(X,x)$, played on space X at point $x \in X$. On the n th play, player I chooses an open neighborhood U_n of x , and player II chooses a point x_n in U_n . Player I wins if $\{x_n\}$ converges to x in X , and otherwise player II wins. The space X is called a W-space if for each $x \in X$, player I has a winning strategy in $\Gamma_*(X,x)$. Every first countable space is a W-space, and every W-space is a Fréchet space.

If α is an admissible family of compact subsets of X , we will be interested in characterizing when $C_\alpha(X)$ is a W-space. Since $C_\alpha(X)$ is a topological vector space, it is homogeneous. Therefore the choice of point from $C_\alpha(X)$ is immaterial. So we will denote the Gruenhagen game on $C_\alpha(X)$ by $\Gamma_*(C_\alpha(X))$.

There is a dual game, $\Gamma'_\alpha(X)$, played on X as follows. On the n th play, player I chooses $A_n \in \alpha$, and player II chooses an open set U_n containing A_n . Player I wins if $\{U_n\}$ is an α -cover, and otherwise player II wins. We will also consider a modified version, called $\Gamma'_\alpha(X)$, where player I wins if $\{U_n\}$ is an α -sequence.

Lemma 4.1. If α is an admissible family of compact subsets of X , then player I has a winning strategy in $\Gamma_\alpha(X)$ if and only if player I has a winning strategy in $\Gamma'_\alpha(X)$.

Proof. Suppose that σ is a winning strategy for player I in $\Gamma_\alpha(X)$. That is, for each sequence (U_1, \dots, U_n) obtained as plays of player II, $\sigma(U_1, \dots, U_n)$ gives the $n + 1$ play for player I; also $\sigma(\emptyset)$ is the first play for player I. Then any sequence of plays where player I uses σ to choose the n th play will result in a win for player I. We say that (U_1, \dots, U_n) is compatible with σ if it consists of plays of player II in $\Gamma_\alpha(X)$ where player I uses σ .

By induction, define strategy σ' for player I in $\Gamma'_\alpha(X)$ as follows. First let $\sigma'(\emptyset) = \sigma(\emptyset)$. Suppose (U_1, \dots, U_n) is compatible with σ' . Then take $\sigma'(U_1, \dots, U_n) \in \alpha$ so that it contains $\sigma(\emptyset) \cup \{U\{\sigma(U_{i_1}, \dots, U_{i_k}) : 1 \leq i_1 < \dots < i_k \leq n \text{ and } (U_{i_1}, \dots, U_{i_k}) \text{ is compatible with } \sigma\}\}$. Now let $\{U_n\}$ be any sequence of plays made by player II in $\Gamma'_\alpha(X)$, where player I uses σ' . To see that $\{U_n\}$ is an α -sequence, let $A \in \alpha$. It suffices to show that every subsequence of $\{U_n\}$ has some member which contains A . So let $\{U_{n_i}\}$ be a subsequence of $\{U_n\}$. Since $\sigma(\emptyset) \subset U_{n_1}$, then (U_{n_1}) is compatible with σ . Suppose $(U_{n_1}, \dots, U_{n_i})$ is compatible with σ . Then $\sigma(U_{n_1}, \dots, U_{n_i}) \subset \sigma'(U_1, \dots, U_{n_{i+1}-1}) \subset U_{n_{i+1}}$, so that $(U_{n_1}, \dots, U_{n_{i+1}})$ is compatible with σ . Therefore by induction, $(U_{n_1}, \dots, U_{n_i})$ is compatible with

σ for each i . But then $\{U_{n_i}\}$ results from plays by player II in $\Gamma_\alpha(X)$ with player I using σ . Since σ is a winning strategy, some U_{n_i} contains A .

The dual nature of these games is now illustrated by our last theorem.

Theorem 4.2. Let α be an admissible family of compact subsets of X . Then player I has a winning strategy in $\Gamma_\star(C_\alpha(X))$ if and only if player I has a winning strategy in $\Gamma_\alpha(X)$.

Proof. We will take the zero function, f_0 , as base point in $\Gamma_\star(C_\alpha(X))$. First suppose σ is a winning strategy for player I in $\Gamma_\alpha(X)$. Define strategy Σ for player I in $\Gamma_\star(C_\alpha(X))$ as follows. Let $A_1 = \sigma(\emptyset)$, and for each n , let $V_n = (-1/n, 1/n)$. Define $\Sigma(\emptyset) = [A_1, V_1]$. Now suppose player II in $\Gamma_\star(C_\alpha(X))$ chooses f_1 in response to player I taking $\Sigma(\emptyset)$. Let $U_1 = f_1^{-1}(V_1)$, which contains A_1 . Then take $A_2 = \sigma(U_1)$, and define $\Sigma(f_1) = [A_2, V_2]$. Continue in this manner to define $\Sigma(f_1, \dots, f_n)$, where the f_1, \dots, f_n are the plays of player II. To see that $\{f_n\}$ converges to f_0 , let $[A, V]$ be a basic neighborhood of f_0 in $C_\alpha(X)$. Now the $\{U_n\}$ are the plays of player II in $\Gamma_\alpha(X)$ while player I is using σ . Then there exists an M such that for all $n \geq M$, $A \subset U_n$. So for all $n \geq M$, $f_n(A) \subset f_n(U_n) \subset V_n$. We may suppose that M is large enough that $V_n \subset V$. Then for all $n \geq M$, $f_n \in [A, V]$. Therefore $\{f_n\}$ converges to f_0 , and thus Σ is a winning strategy for player I in $\Gamma_\star(C_\alpha(X))$.

Conversely, suppose we are given a winning strategy Σ for player I in $\Gamma_*(C_\alpha(X))$. Define strategy σ for player I in $\Gamma_\alpha(X)$ as follows. Let $[A_1, V_1]$ be a basic neighborhood of f_0 contained in $\Sigma(\emptyset)$. Then define $\sigma(\emptyset) = A_1$. Now suppose player II in $\Gamma_\alpha(X)$ chooses U_1 . Then choose $f_1 \in C_\alpha(X)$ such that $f_1(A_1) = 0$ and $f_1(X \setminus U_1) = 1$. So let $[A_2, V_2]$ be a basic neighborhood of f_0 contained in $\Sigma(f_1)$, and define $\sigma(U_1) = A_2$. Continue in this manner to define $\sigma(U_1, \dots, U_n)$, where U_1, \dots, U_n are the plays of player II. To see that $\{U_n\}$ is an α -cover of X , let $A \in \alpha$. The $\{f_n\}$ are the plays of player II in $\Gamma_*(C_\alpha(X))$ while player I is using Σ . So $\{f_n\}$ converges to f_0 ; and thus there exists an n such that $f_n \in [A, (-1, 1)]$. Since $f_n(X \setminus U_n) = 1$, then $A \subset U_n$. Therefore σ is a winning strategy for player I in $\Gamma_\alpha(X)$.

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