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by

S. D. Iliadis

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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Introduction

All spaces in this paper are assumed to be metrizable and having a countable basis.

A space is said to be *rational* (resp. *rim-finite*) if it has a basis of open sets with countable (resp. finite) boundaries (a finite set is considered also as countable).

It is said that a space X has *rim-type* $\leq \alpha$, where α is an ordinal number, iff it has a basis B of open sets such that the α-derivative (see [Ku], v.I, § 24.IV) of the boundary of every element of B is empty. If α is the least such ordinal, then we say that the space X has rim-type α.

It is easy to see that if X has rim-type α then α is a countable ordinal number (a finite ordinal number is also considered as countable). In what follows all ordinal numbers considered are countable.

A space T is said to be *universal* for a family R of spaces iff T is an element of R and for every $X \in R$ there is an embedding $i: X \to T$.

We say that a space T has the *property of finite intersection* with respect to a family R of spaces iff for every element X of R there is a fixed embedding $i_X: X \to T$ such that if X and Y are different elements of R then the set $i_X(X) \cap i_Y(Y)$ is finite. The space T is not necessarily an element of R.

Nobeling (see [N]) proved that in the family of all rim-finite spaces, the family of all rim-finite compact spaces and the family of all rim-finite continua there does not exist a universal element.

Also, it is well-known (see [Ku], v.II, §51.IV) that in the family of all rational compact spaces and in the family of all rational continua there does not exist a universal element.

In $[I_1]$ it is proved that in the family of all (locally connected) compact spaces having rim-type $\leq \alpha$ and in the family of all (locally connected) continua having rim-type $\leq \alpha$ there does not exist a universal element.

On the other hand, in $[I_2]$, it is proved that in the family of all rational spaces there exists a universal element having the property of finite intersection with respect to the subfamily of all rational continua (more precisely, with respect to a given subfamily the power of which is less than or equal to the continuum).

In part I of this paper we give some lemmas about the type and the rim-type of spaces. These lemmas are used (in part III) in order to represent the spaces as quotient spaces of "good" partitions of subsets of the Cantor ternary set. The "good" means that the corresponding partitions have some properties of "countability."

In part II, starting with a family of pairs (S,D) where S is a subset of the Cantor ternary set and D is a "good" partition, we construct a space which is used (as a universal element) in part III where we give the main result: In the family of all spaces of rim-type $\leq \alpha$ there is a universal element having the property of finite intersection with respect to a given subfamily the power of which is less than or equal to the continuum (in particular, with respect to the subfamily of all compact spaces of rim-type $\leq \alpha$).

Let $\alpha = \beta + n$ where β is a limit ordinal number or 0 and n is a non-negative integer.

Using theorem 8 of [I-T] we also have:

There exists a continuum of rim-type $\leq \beta + 2n + \min\{\beta, 1\}$ having the property of finite intersection with respect to the family of all compact spaces of rim-type $\leq \alpha$.

In particular, there exists a continuum of rim-type 2 having the property of finite intersection with respect to the family of all rim-finite compact spaces.

This result gives affirmative answers to problems 8 and 9 (hence, negative answers to problems 10 and 11) of $[I_3]$.

By the main result it follows that for a given space X of rim-type $\leq \alpha$ there is a space of rim-type $\leq \alpha$ having the property of finite intersection with respect to the family of all closed subsets of X.

I.1. For every space X and for every subset Q of X by cl(Q), Bd(Q), Int(Q) and $|Q|$ we denote the closure, the boundary, the interior and the cardinality of Q, respectively. If X is metric then by diam(Q) we denote the diameter of Q. An open subset U of X is called a *regular* open subset iff $U = \mathrm{Int}(\mathrm{cl}(U))$. Obviously, a set U is regular iff $\mathrm{Bd}(U) = \mathrm{Bd}(X \setminus \mathrm{cl}(U))$.

A space Y is said to be an *extension* of X iff X is a dense subset of Y. A space Y is said to be a *compactification* of X iff Y is a compact extension of X.

For every space M and for every ordinal number α by $M^{(\alpha)}$ we denote the α-th derivative of M. The ordinal number 0 is considered as a limit ordinal. A non-limit ordinal number is called *isolated*.

We say that a space M has *type* α (resp. $\leq \alpha$) and write type(M) $= \alpha$ (resp. type(M) $\leq \alpha$) iff $M^{(\alpha)} = \emptyset$ and $M^{(\beta)} \neq \emptyset$ for every $\beta < \alpha$ (resp. $M^{(\alpha)} = \emptyset$).

We say that a space M *at the point* $x \in M$ has *type* α (resp. $\leq \alpha$) and write type(x,M) $= \alpha$ (resp. type(x,M) $\leq \alpha$) iff $x \in M^{(\alpha)}$ and $x \notin M^{(\beta)}$ for every $\beta < \alpha$ (resp. $x \notin M^{(\alpha)}$). Obviously, for every $M \neq \emptyset$ and for every $x \in M$, type(x,M) is an isolated ordinal number and type(M) $= \sup_{x \in M}\{\text{type}(x,M)\}$.

We observe that two countable locally compact spaces whose type is the same limit ordinal number are homeomorphic.

Indeed, let M and N be locally compact spaces and type(M) = type(N) $= \alpha$, where $\alpha > 0$ is a limit ordinal number. Let $M \cup \{a\}$ and $N \cup \{b\}$ be one-point compactifications of M and N, respectively. Then, type($M \cup \{a\}$) = type($N \cup \{b\}$) $= \alpha + 1$ and $|(M \cup \{a\})^{(\alpha)}| = |(N \cup \{b\})^{(\alpha)}| = 1$. Hence, by [M-S] there is a homeomorphism h of $M \cup \{a\}$ onto $N \cup \{b\}$. Obviously, $h(a) = b$. Hence, $h(M) = N$.

We also observe that by [M-S] it follows that the set of all non-homeomorphic compact spaces having type $\leq \alpha$ is countable.

2. Let α be a limit ordinal number and let $\beta = \alpha + m$ where m is a non-negative integer.

By $P(\alpha,m)$ we denote the set of all pairs (X,K) where K is a compact space having type $\alpha + n$ for some non-negative integer $n \geq m$ and X is a subset of K such that $\text{type}(X) = \beta = \alpha + m$ and $K \setminus K^{(\alpha)} \subseteq X$ (hence, $K \setminus K^{(\alpha)} = X \setminus X^{(\alpha)}$).

By $TP(\alpha,m)$, $m \geq 1$, we denote the set of all 3-tuples (a,X,K) such that $(X,K) \in P(\alpha,m)$ and $a \in X^{(\beta-1)} = X^{(\alpha+m-1)}$.

If $(X,K(X)) \in P(\alpha,m)$ then by $B(K(X))$ we denote the set of all open and closed subsets of $K(X)$. By $B(X)$ we denote the set of all subsets U' of X of the form $U' = U \cap X$ where $U \in B(K(X))$.

Let e be a subset of $TP(\alpha,m)$ and $(a,X,K(X)) \in e$. A neighbourhood $U' \in B(X)$ of a is called *standard* for the subset e iff for every $(b,Y,K(Y)) \in e$ there exist a neighborhood $V' \in B(Y)$ of b and a homeomorphism h of U' onto V' such that $h(a) = b$.

We observe that if $(a,X,K(X)) \in TP(\alpha,m)$ and $a \in U \in B(K(X))$ then $(a,U \cap X,U) \in TP(\alpha,m)$.

Lemma 1. *For a given* α *there exists a finite set* $ETP(\alpha,m) = \{e_1^m,\cdots,e_{k(m)}^m\}$ *whose elements are subsets of* $TP(\alpha,M)$ *such that:*

1) $e_1^m \cup \cdots \cup e_{k(m)}^m = TP(\alpha,m)$,

2) $e_i^m \cap e_j^m = \emptyset$ *if* $i \neq j$,

3) *if* $(a,X,K(X)) \in e_i^m$, $i = 1,\cdots,k(m)$, *and* $a \in U \in B(K(X))$ *then* $(a,U \cap X,U) \in e_i^m$,

4) *if* $(a,X,K(X)) \in e_i^m$, $i = 1,\cdots,k(m)$, *then* a *has a standard neighbourhood for the subset* e_i^m.

Proof. We may assume that all spaces are metric. We prove the lemma by induction on m.

Let m = 1. By e^1_1 (resp. e^1_2) we denote the set of all elements (a,X,K(X)) of TP(α,m) such that the space X is locally compact (resp. is not locally compact) at the point a.

We prove that the set ETP(α,m) = $\{e^1_1, e^1_2\}$ is the required set. Indeed, it is clear that the properties 1)-3) of the lemma are satisfied.

Suppose that (a,X,K(X)) $\in e^1_1$ and (b,Y,K(Y)) $\in e^1_1$. Since $X^{(\alpha)}$ and $Y^{(\alpha)}$ consist of isolated points (in the relative topology), there are U $\in$ B(K(X)) and v $\in$ B(K(Y)) such that:
1) the sets U' = U $\cap$ X and V' = V $\cap$ Y are compact and
2) U $\cap$ $X^{(\alpha)}$ = {a} and V $\cap$ $Y^{(\alpha)}$ = {b}. Obviously, type(U') = type(V') = α + 1, $(U')^{(\alpha)}$ = {a} and $(V')^{(\alpha)}$ = {b}. Hence, the sets U' and V' are homeomorphic (see [M-S]). Moreover, if h is an arbitrary homeomorphism of U' onto V' then h(a) = b.

Since the construction of the set U' is independent of the construction of the set V', the neighbourhood U' of a is standard for the element e^1_1.

Now, suppose that (a,X,K(X)) $\in e^1_2$ and (b,Y,K(Y)) $\in e^1_2$. For every i = 1,2,$\cdots$ there exists $U_i \in$ B(K(X)) (resp. $V_i \in$ B(K(Y))) such that: 1) $U_1 \cap X^{(\alpha)}$ = {a} (resp. $V_1 \cap Y^{(\alpha)}$ = {b}), 2) a $\in U_{i+1} \subseteq U_i$ (resp. b $\in V_{i+1} \subseteq V_i$), 3) the set $(U_i \setminus U_{i+1}) \cap X$ (resp. $(V_i \setminus V_{i+1}) \cap Y$) is not compact and 4) $\lim_{i\to\infty} \mathrm{diam}(U_i) = 0$ (resp. $\lim_{i\to\infty} \mathrm{diam}(V_i) = 0$).

Since $K(X) \setminus (K(X))^{(\alpha)} \subseteq X$ by properties 1) and 2) it follows that $(U_i \setminus U_{i+1}) \cap X = (U_i \setminus U_{i+1}) \cap (X \setminus X^{(\alpha)}) = (U_i \setminus U_{i+1}) \cap (K(X) \setminus (K(X))^{(\alpha)})$. Hence, the set $(U_i \setminus U_{i+1}) \cap X$ as an open subset of $K(X)$ is locally compact.

On the other hand, the set $U_i \setminus U_{i+1}$ is compact and the set $(U_i \setminus U_{i+1}) \cap X$ is not compact. Hence, $(U_i \setminus U_{i+1}) \cap (K(X))^{(\alpha)} \neq \emptyset$. It then follows that $\text{type}((U_i \setminus U_{i+1}) \cap X) = \alpha$.

Similarly, the set $(V_i \setminus V_{i+1}) \cap Y$ is locally compact and $\text{type}((V_i \setminus V_{i+1}) \cap Y) = \alpha$. Hence, there is a homeomorphism h_i of $(U_i \setminus U_{i+1}) \cap X$ onto $(V_i \setminus V_{i+1}) \cap Y$.

We construct a map h of $U_1' = U_1 \cap X$ onto $V_1' = V_1 \cap Y$ setting $h(a) = b$ and considering that h coincides with the map h_i on the set $(U_i \setminus U_{i+1}) \cap X$. By property 4) of the sets U_i and V_i it follows that the map h is a homeomorphism of U_1' onto V_1'.

As in the first case, since the construction of the set U_1' is independent of the construction of the set V_1', the neighbourhood U_1' of a is standard for the element e_2^1.

Suppose that the lemma is true for $m < m_0$, $m_0 \geq 2$.

For every element $e \in \bigcup_{i=1}^{m_0-1} \text{ETP}(\alpha,i)$ we consider a fixed 3-tuple $(a(e),X(e),K(X(e)))$ of e and a fixed standard neighbourhood $U(e)$ of $a(e)$ in $X(e)$ for the element e.

Let $p = (a,X,K(X)) \in \text{TP}(\alpha,m_0)$. Set $X_1 = X \setminus X^{(\beta_0-1)}$ where $\beta_0 = \alpha + m_0$. Obviously, $\text{type}(X_1) = \beta_0-1$, $X_1^{(\beta_0-2)} = X^{(\beta_0-2)} \setminus X^{(\beta_0-1)}$ and $(X_1,K(X)) \in P(\alpha,m_0-1)$.

In the set $\text{ETP}(\alpha,m_0-1)$ we define a subset $\text{ETP}(\alpha,m_0-1,p)$ as follows: an element e of $\text{ETP}(\alpha,m_0-1)$ belongs to

$ETP(\alpha, m_0-1, p)$ iff for every neighbourhood $W \in B(K(X))$ of a there exists a point $x \in W \cap X_1^{(\beta_0-2)}$ such that $(x, X_1, K(X)) \in e$ (hence, $(x, W \cap X_1, W) \in e$).

For every $e \in ETP(\alpha, m_0-1, p)$ by $X(e,p)$ we denote the set of all points $x \in X_1^{(\beta_0-2)}$ for which $(x, X_1, K(X)) \in e$.

Obviously, if $e_1 \neq e_2$, then $X(e_1,p) \cap X(e_2,p) = \emptyset$.

We observe that type $(\{a\} \cup X(e,p)) = 2$ and $\{a\} = (\{a\} \cup X(e,p))^{(1)}$.

We say that the element $e \in ETP(\alpha, m_0-1, p)$ is compact (resp. non-compact) if the space $\{a\} \cup X(e,p)$ is locally compact (resp. is not locally compact) at the point a.

A sequence $U_1^p, U_2^p, \cdots$ of elements of $B(K(X))$ is called a normal sequence of p if: 1) $a \in U_{i+1}^p \subseteq U_i^p$, $i = 1,2,\cdots$, 2) $U_1^p \cap X^{(\beta_0-1)} = \{a\}$, 3) $U_1^p \cap X_1^{(\beta_0-2)} \subseteq \cup_{e \in ETP(\alpha, m_0-1, p)} X(e,p)$, 4) $\lim_{i\to\infty} \operatorname{diam}(U_i^p) = 0$, 5) the set $\{a\} \cup (U_1^p \cap X(e,p))$ is a compact set if e is a compact element and 6) for every $i = 1,2,\cdots$ the set $(U_i^p \setminus U_{i+1}^p) \cap X(e,p)$ is infinite if e is a non-compact element.

Since $ETP(\alpha, m_0-1)$ is finite, the existence of a normal sequence is easily proved.

Let $U_1^p, U_2^p, \cdots$ be a normal sequence.

Let $U_1^p \cap X_1^{(\beta_0-2)} = \{x_1, x_2, \cdots\}$. If $\alpha > 0$ then for every i we consider a point c_i of U_1^p such that:
1) $c_i \in (U_1^p)^{(\alpha_i)} \setminus (U_1^p)^{(\alpha_i+1)}$ where $\alpha_i < \alpha$, 2) $\lim_{i\to\infty} \alpha_i = \alpha$ and 3) $\lim_{i\to\infty} d(c_i, x_i) = 0$ where $d(c_i, x_i)$ is the distance between c_i and x_i.

For every point $x \in U_1^p \cap X_1^{(\beta_0-2)} = \{x_1,x_2,\cdots\}$ we consider an element $U_1(x) \in B(K(X))$ such that: 1) $x \in U_1(x) \subseteq U_1^p$, 2) if $x_i \neq x_j$ then $U_1(x_i) \cap U_1(x_j) = \emptyset$, 3) if $x \in U_i^p \setminus U_{i+1}^p$ then $U_1(x) \subseteq U_i^p \setminus U_{i+1}^p$, 4) if $\alpha > 0$ then $c_i \notin U_1(x)$ for every $i = 1,2,\cdots$ and 5) $\lim_{i\to\infty} \operatorname{diam}(U_1(x_i)) = 0$.

Since the set $U_1^p \cap X_1^{(\beta_0-2)}$ is countable and consists of isolated points (in the relative topology) the existence of such elements of $B(K(X))$ is easily established.

We observe that if $x \in X(e,p)$ then by property 3) of the lemma $(x, U_1(x) \cap X, U_1(x)) \in e$. Hence, there exist a neighbourhood $U(x) \in B(U_1(x)) \subseteq B(K(X))$ of x and a homeomorphism $h(x)$ of $U(e)$ onto $U(x) \cap X$ such that $h(x)(a(e)) = x$.

Set $\tilde{U} = \{a\} \cup \bigcup_{x\in U_1^p\cap X_1^{(\beta_0-2)}} U(x)$ and $\hat{U} = \{a\} \cup (U_1^p \setminus \tilde{U})$. Obviously, $\hat{U} = \operatorname{cl}(\hat{U})$.

We prove that $(\hat{U} \cap X, \hat{U}) \in P(\alpha,i_0)$ where $i_0 \leq m_0-1$.

Indeed, if $\alpha > 0$ then by the choice of the points c_i, $i = 1,2,\cdots$ it follows that $\operatorname{type}(\hat{U}) \geq \alpha$. On the other hand, since $\hat{U} \cap X_1^{(\beta_0-2)} = \emptyset$ it follows that $\operatorname{type}(\hat{U} \cap X) \leq \alpha + m_0-1$.

Let $y \in \operatorname{Cl}(\tilde{U}) \cap \hat{U}$. By the construction of the points c_i, $i = 1,2,\cdots$ and the sets (Ux) where $x \in U_1^p \cap X_1^{(\beta_0-2)}$ it follows that $y \in \operatorname{cl}(\{c_1,c_2,\cdots\})$. This means that $y \in (\hat{U})^{(\alpha)}$. In particular, $a \in (\hat{U})^{(\alpha)}$.

Hence, if $z \in \hat{U}\setminus(\hat{U})^{(\alpha)}$ then $z \notin \operatorname{cl}(\tilde{U})$. This means that $z \in \hat{U} \cap X$. Thus, $\hat{U}\setminus(\hat{U})^{(\alpha)} \subseteq \hat{U} \cap X$.

There exists an integer i_0 such that $\operatorname{type}(\hat{U} \cap X) = \alpha + i_0$. Hence, $(\hat{U} \cap X, \hat{U}) \in P(\alpha,i_0)$ where $i_0 \leq m_0-1$.

There exists an integer $i_1 \leq i_0$ and a neighbourhood $\hat{U}_1(a) \in B(\hat{U})$ of a such that type$(\hat{U}_1(a) \cap X) = \alpha + i_1$ and $a \in (\hat{U}_1(a) \cap X)^{(\alpha+i_1-1)}$. Hence, $(a,\hat{U}_1(a) \cap X,\hat{U}_1(a)) \in TP(\alpha,i_1)$.

By $e(p)$ we denote the element of $ETP(\alpha,i_1)$ which contains the 3-tuple $(a,\hat{U}_1(a) \cap X,\hat{U}_1(a))$.

By induction it follows that the element $e(p)$ is not dependent on the choice of the neighbourhood $\hat{U}_1(a)$ of a in $\hat{U}$.

There exists a neighbourhood $\hat{U}(a) \in B(\hat{U}_1(a))$ of a and a homeomorphism $h(p)$ of $U(e(p))$ onto $\hat{U}(a) \cap X$ such that $h(p)(a(e(p))) = a$.

The set $\hat{U}(a) \cap cl(\tilde{U})$ is an open and closed subset of $cl(\tilde{U}) \cap \hat{U}$. There exists an open subset $\tilde{U}_1(a)$ of $cl(\tilde{U})$ such that $\tilde{U}_1(a) \cap \hat{U} = \hat{U}(a) \cap cl(\tilde{U})$.

By $\tilde{U}(a)$ we denote the union of all elements $U(x)$, $x \in U_1^p \cap X_1^{(\beta_0-2)}$ such that $U(x) \cap \tilde{U}_1(a) \neq \emptyset$. By the properties of the sets $U(x)$ we have $cl(\tilde{U}(a)) \setminus \tilde{U}(a) = cl(\tilde{U}(a)) \cap \hat{U} = \tilde{U}_1(a) \cap \hat{U} = \hat{U}(a) \cap cl(\tilde{U})$. From this follows that the set $U(a) = \tilde{U}(a) \cup \hat{U}(a)$ is an open and closed subset of $K(X)$. Obviously, $a \in U(a)$.

We observe that the element $e(p)$ is dependent on the choice of the normal sequence $U_1^p, U_2^p, \cdots$, the points c_i, $i = 1,2,\cdots$ and on the choice of the standard neighbourhoods $U(x)$, $x \in U_1^p \cap X_1^{(\beta_0-2)}$.

In the set $\cup_{i=1}^{m_0-1} ETP(\alpha,i)$ we define an order as follows: we set $e_{j_1}^{m_1} < e_{j_2}^{m_2}$ iff, either $m_1 < m_2$, or $m_1 = m_2$ and $j_1 < j_2$.

Choosing the suitable normal sequence $U_1^p, U_2^p, \cdots$, the points c_i and the neighbourhoods $U(x)$, we may consider that the element $e(p)$ is the least possible.

Now we define the set $ETP(\alpha, m_0)$. Let $p, q \in TP(\alpha, m_0)$. We say that the elements p and q of $TP(\alpha, m_0)$ belong to an element of $ETP(\alpha, m_0)$ iff: 1) $ETP(\alpha, m_0 - 1, p) = ETP(\alpha, m_0 - 1, q)$, 2) an element e of $ETP(\alpha, m_0 - 1, p)$ is compact iff this element is compact as an element of $ETP(\alpha, m_0 - 1, q)$ and 3) $e(p) = e(q)$.

Obviously, properties 1)-3) of the lemma are satisfied.

We prove property 4) of the lemma.

Let e be the element of $ETP(\alpha_0, m_0)$ which contains $(\alpha, X, K(X))$. We prove that the neighbourhood $U(\alpha) \cap X$ of a is a standard neighbourhood for the element e.

Indeed, let $q = (b, Y, K(Y))$ be an element of e. For the 3-tuple q we consider all sets which were constructed for the 3-tuple p. For these sets we use the same notations replacing only letters p,a,x,X and U by letters q,b,y,Y and V, respectively.

We prove that there is a homeomorphism h of $U(a) \cap X$ onto $V(b) \cap Y$ such that $h(a) = b$.

Let $\hat{h} = h(q) \circ h^{-1}(p)$. Then $\hat{h}$ is a homeomorphism of $\hat{U}(a) \cap X$ onto $\hat{V}(b) \cap Y$ such that $\hat{h}(a) = b$.

On the other hand, there exists an integer k such that $U_k^p \subseteq U(a)$ and $V_k^q \subseteq V(a)$.

Between the sets $U(a) \cap X_1^{(\beta_0 - 2)}$ and $V(b) \cap Y_1^{(\beta_0 - 2)}$ there exists a one-to-one correspondence such that: 1) if $x \in X(e,p)$ and x corresponds to y then $y \in Y(e,q)$ and 2) if e is a noncompact element of $ETP(\alpha, m_0 - 1, p)$

and $x \in (U_i^p \setminus U_{i+1}^p) \cap X(e,p)$, $i \geq k+1$, or $x \in (U(a) \setminus U_{k+1}^p) \cap X(e,p)$ then $y \in (V_i^q \setminus V_{i+1}^q) \cap Y(e,q)$ or $y \in (V(b) \setminus V_{k+1}^q) \cap Y(e,q)$, respectively.

Let x correspond to y. Then $(x, U_1(x) \cap X, U_1(x))$ and $(y, V_1(y) \cap Y, V_1(y))$ belong to the same element e of $ETP(\alpha, m_0-1)$. Hence, the map $h(y)h^{-1}(x)$ is a homeomorphism of $U(x) \cap X$ onto $V(y) \cap Y$ such that $(h(y)h^{-1}(x))(x) = y$.

Now, we construct the map h. First, we observe that $U(a) \cap X = (\tilde{U}(a) \cap X) \cup ((\hat{U}(a) \setminus (\hat{U}(a) \cap cl(\tilde{U}))) \cap X)$. This is true because each point $z \in (\hat{U}(a) \cap cl(\tilde{U})) \cap X$ is a limit point for the set $X_1^{(\beta_0-2)}$, that is, z belongs to the set $X^{(\beta_0-1)}$ which is impossible by the choice of the set U_1^p.

Also, we have $V(b) \cap Y = (\tilde{V}(b) \cap Y) \cup ((\hat{V}(b) \setminus (\hat{V}(a) \cap cl(\tilde{V}))) \cap Y)$.

Let $z \in U(a) \cap X$. If $Z \in \tilde{U}(a)$ then there is $x \in X_1^{(\beta_0-2)}$ such that $z \in U(x)$. In this case we set $h(z) = (h(y)h^{-1}(x))(z)$. If $z = a$ then we set $h(a) = b$.

If $z \in \hat{U}(a) \cap X$ then we set $h(z) = \hat{h}(z)$. By the above properties of the sets $U(a) \cap X$ and $V(b) \cap Y$ it follows that h is a homeomorphism of $U(a) \cap X$ onto $V(b) \cap Y$.

Thus, $U(a) \cap X$ is a standard neighbourhood of a for the element e.

The proof of the lemma is completed.

Remark. Let $G(\alpha,m)$, $m \geq 0$, be the set of all spaces X for which there exists a compact space K such that $(X,K) \in P(\alpha,m)$. By $PG(\alpha,m)$, $m \geq 1$, we denote the set of all pairs (a,X) where $X \in G(\alpha,m)$ and $a \in X^{(\alpha+m-1)}$.

We say that the elements (a,X) and (b,Y) of PG(α,m) are *equivalent* (write (a,X) $\sim$ (b,Y)) iff there exist open and closed neighbourhoods U and V of a and b in X and Y, respectively, and a homeomorphism h of U onto V such that h(a) = b. Obviously, the relation "$\sim$" is an equivalence relation.

By EPG(α,m) we denote the set of all equivalence classes.

Obviously, if (a,X) $\in$ PG(α,m) then for some compact space K, (a,X,K) $\in$ TP(α,m) and, conversely, if (a,X,K) $\in$ TP(α,m) then (a,X) $\in$ PG(α,m).

By property 4) of Lemma 1 it follows that if (a,X,K(X)) and (b,Y,K(Y)) belong to an element of ETP(α,m) then (a,X) $\sim$ (b,Y).

Hence, the set EPG(α,m) is finite. Moreover, every element of EPG(α,m) is a finite union of elements of ETP(α,m). We observe that the set ETP(α,m) is not defined uniquely.

Let e be a subset of PG(α,m) and (a,X) $\in$ e. An open and closed neighbourhood U of a in X is called *standard* for e iff for every (b,Y) $\in$ e there are an open and closed neighbourhood V of b in Y and a homeomorphism h of U onto V such that h(a) = b.

Since every element EPG(α,m) is a finite union of elements of ETP(α,m), by property 4) of Lemma 1 it follows that for every element e of EPG(α,m) and for every (a,X) $\in$ e, the point a has a standard neighbourhood in X.

3. *Lemma* 2. *The set* $G(\alpha,m)$ *is countable.*

Proof. We prove the lemma by induction on m.

Let $m = 0$. If $\alpha = 0$ then $G(0,0) = \{\emptyset\}$. If $\alpha > 0$ then every element of $G(\alpha,0)$ is a locally compact space having type α. Since every two locally compact spaces having type α are homeomorphic, the set $G(\alpha,0)$ is a singleton.

Suppose that the lemma is true for $m < m_0$, $m_0 \geq 1$.

We prove that the set $G(\alpha,m_0)$ is countable.

Let $X \in G(\alpha,m_0)$. By $K(X)$ we denote a compact space such that $(X,K(X)) \in P(\alpha,m_0)$.

Suppose that $X^{(\alpha+m_0-1)} = \{x_1,x_2,\cdots\}$ is an infinite set. Then, for every $i = 1,2,\cdots$ there exists a point $c_i \in X$ such that: 1) $c_i \in X^{(\alpha_i)} \setminus X^{(\alpha_i+1)}$, where $\alpha_i < \alpha$, 2) $\lim_{i\to\infty} \alpha_i = \alpha$ and 3) $\lim_{i\to\infty} d(c_i,x_i) = 0$, where $d(c_i,x_i)$ is the distance between c_i and x_i.

If the set $X^{(\alpha+m_0-1)}$ is finite and $X \neq K(X)$ then by c_i, $i = 1,2,\cdots$ we denote a point of X such that: 1) $c_i \in X^{(\alpha_i)} \setminus X^{(\alpha_i+1)}$, where $\alpha_i < \alpha$, 2) $\lim_{i\to\infty} \alpha_i = \alpha$ and 3) there is a point $c \in K(X) \setminus X$ such that $\lim_{i\to\infty} c_i = c$.

For every $x \in X^{(\alpha+m_0-1)}$ we consider a neighbourhood $U_1(x) \in B(K(X))$ of x such that: 1) if $x_i \neq x_j$ then $U_1(x_i) \cap U_1(x_j) = \emptyset$, 2) if the set $X^{(\alpha+m_0-1)}$ is infinite then $\lim_{i\to\infty} \operatorname{diam}(U_1(x_i)) = 0$ and 3) if $X^{(\alpha+m_0-1)}$ is infinite or $X^{(\alpha+m_0-1)}$ is finite and $X \neq K(X)$ then $c_j \notin U_1(x)$ for every $j = 1,2,\cdots$ and $x \in X^{(\alpha+m_0-1)}$.

For every element $e \in ETP(\alpha,m_0)$ (see Lemma 1) by $X(e)$ we denote the set of all points $x \in X^{(\alpha+m_0-1)}$ for which $(x,X,K(X)) \in e$. Obviously, $X^{(\alpha+m_0-1)} = \cup_{e\in ETP(\alpha,m_0)} X(e)$.

If $x \in X(e)$ then $(x,U_1(x) \cap X,U_1(x)) \in e$. Hence, there are a neighbourhood $U(x) \in B(K(X))$ of x and a homeomorphism $h(x)$ of $U(e)$ (see Lemma 1) onto $U(x) \cap X$ such that $U(x) \subseteq U_1(x)$ and $h(x)(a(e)) = x$.

Set $X' = K(X) \setminus (\cup_{x\in X^{(\alpha+m_0-1)}} U(x))$. Obviously, X' is a compact space and if $X = K(X)$ then $\text{type}(X' \cap X) \leq \alpha+m_0-1$.

If $X \neq K(X)$, in particular if $X^{(\alpha+m_0-1)}$ is an infinite set, then by the choice of the points c_i, we have $\text{type}(X' \cap X) \geq \alpha$. In this case, if $X^{(\alpha+m_0-1)}$ is finite then it is clear that $X' \setminus (X')^{(\alpha)} \subseteq X' \cap X$. This means that $X \cap X' \in G(\alpha,i)$, $i \leq m_0-1$, because $\text{type}(X \cap X') \leq \alpha+m_0-1$.

Let $X^{(\alpha+m_0-1)}$ be infinite. Set $X'' = \cup_{x\in X^{(\alpha+m_0-1)}} U(x)$. Obviously, if $z \in (X' \setminus (X')^{(\alpha)}) \setminus cl(X'')$ then $z \in X' \cap X$.

If $z \in X' \cap cl(X'')$ then by the choice of the sets $U_1(x)$, $x \in X^{(\alpha+m_0-1)}$ it follows that $z \in cl(X^{(\alpha+m_0-1)})$ and, hence, $z \in cl(\{c_1,c_2,\cdots\})$. This means that $z \in (X')^{(\alpha)}$.

Hence, $X' \setminus (X')^{(\alpha)} \subseteq X' \cap X$ and since $\text{type}(X' \cap X) \geq \alpha$ we have $X' \cap X \in G(\alpha,i)$, $i \leq m_0-1$.

Let Y be an element of $G(\alpha,m_0)$. For the space Y we construct the sets $Y(e)$, $e \in ETP(\alpha,m_0)$, the neighbourhoods $U(y)$, $y \in Y^{(\alpha+m_0-1)}$, the homeomorphisms $h(y)$, $y \in Y^{(\alpha+m_0-1)}$, and the space $Y' \cap Y$ as we constructed the corresponding sets and homeomorphisms for the space X.

Now, we prove that the spaces X and Y are homeomorphic if: 1) for every $e \in ETP(\alpha,m_0)$, $|X(e)| = |Y(e)|$ and 2) the spaces $X' \cap X$ and $Y' \cap Y$ are homeomorphic.

Indeed, let h' be a homeomorphism of $X' \cap X$ onto $Y' \cap Y$. By the condition $|X(e)| = |Y(e)|$ it follows that between the sets $X^{(\alpha+m_0-1)}$ and $Y^{(\alpha+m_0-1)}$ there is a one-to-one correspondence such that if $x \in X^{(\alpha+m_0-1)}$ corresponds to $y \in Y^{(\alpha+m_0-1)}$ then $(x,X,K(X))$ and $(y,Y,K(Y))$ belong to the same element of the set $ETP(\alpha,m_0-1)$.

A homeomorphism h of X onto Y is constructed as follows: 1) if $z \in U(x) \cap X$ where $x \in X^{(\alpha+m_0-1)}$ then we set $h(z) = h(y)((h(x))^{-1}(z))$ where y is the point of $Y^{(\alpha+m_0-1)}$ which corresponds to x and 2) if $z \in X' \cap X$ then $h(z) = h'(z)$.

From the above it follows that since the set $ETP(\alpha,m_0)$ is finite, the sets $G(\alpha,m)$, $0 \leq m < m_0$, are countable and the set of all compact spaces having type less than α is countable, the set $G(\alpha,m_0)$ is countable. The proof of the lemma is completed.

Remark. From Lemma 2 it follows that the set of all spaces having a finite type is countable. On the other hand it is easy to prove that the set of all spaces whose type is a given infinite ordinal number has power greater than or equal to the continuum.

4. *Lemma* 3. *Let* X *be a space having type* $\alpha+m$ *where* α *is a limit ordinal number and* m *is a non-negative integer.*

If Y *is a metric compactification of* X *and* $\dim Y \leq 0$ *then there exists a compactification* K *of* X *such that:* 1) *natural projection* π *of* Y *onto* K *exists,* 2) $\text{type}(K) \leq \alpha + 2m + \min\{\alpha,1\}$, 3) $\text{type}(X \cup (K \setminus K^{(\alpha)})) = \alpha + m$, 4) *if* $K = \{z_1, z_2, \cdots\}$ *then* $\lim_{i\to\infty} \text{diam}(\pi^{-1}(z_i)) = 0$ *and* 5) *for a given* $\varepsilon > 0$, $\text{diam}(\pi^{-1}(z)) < \varepsilon$ *for every* $z \in K$.

The proof of this lemma is similar to the proof of Theorem 3 of [I-T]. This lemma is used in the proof of Lemma 4.

5. *Lemma* 4. *Let* X *be a metric space of rim-type* $\leq \alpha + n$ *where* α *is a limit ordinal number and* n *is a non-negative integer. Then, there exist an extension* Z *of* X *and a basis* $B(Z) = \{T_1, T_2, \cdots\}$ *of open sets of* Z *such that:* 1) *the set* $\text{Bd}(T_i)$, $i = 1,2,\cdots$ *is a compact set,* 2) $\text{type}(\text{Bd}(T_i)) \leq \alpha + 2n + \min\{\alpha,1\}$, $i = 1,2,\cdots$, 3) $T_i = \text{Int}(\text{cl}(T_i))$, $i = 1,2,\cdots$, 4) $\text{Bd}(T_i) \cap \text{Bd}(T_j) = \emptyset$ *if* $i \neq j$, *and* 5) $\text{type}((\text{Bd}(T_i) \cap X) \cup (\text{Bd}(T_i) \setminus (\text{Bd}(T_i))^{(\alpha)})) \leq \alpha + n$.

The proof of this lemma is similar to the proof of Theorem 8 of [I-T].

The extension Z is constructed in the same way as the space Z is constructed in the proof of Theorem 8 of [I-T]. Instead of Theorem 3 of [I-T] which is used in the proof of Theorem 8 of [I-T] we use here Lemma 3.

The basis $B(Z) = \{T_1, T_2, \cdots\}$ of the theorem is a subbasis of $\{Q_1, Q_2, \cdots\}$ (see Theorem 8 of [I-T]) such that $\text{Bd}(T_i \cap \text{Bd}(T_j) = \emptyset$ if $i \neq j$.

We observe that in the present paper X is a subset of its compactification or extension, while in [I-T] a compactification of X is a pair (K,h), where K is a compact space and h is a homeomorphism of X into a dense subset of K. Therefore, in Lemma 3 we consider the compactification K of X as the quotient space of the partition $\{\pi^{-1}(z), z \in K\}$ of Y and we identify x of X with the point {x} of the above partition. Also, in Lemma 4 we identify a point x of X with the point f(x) of $[0,1]^N$ (see Theorem 8 of [I-T]).

II. 1. By L_n, $n = 1,2,\cdots$, we denote the set of all ordered n-tuples $i_1\cdots i_n$, where $i_k = 0$ or 1, $k = 1,\cdots,n$. Set $L_0 = \{\emptyset\}$ and $L = \cup_{n=0}^{\infty} L_n$. For n = 0, by $i_1\cdots i_n$ we denote the element $\emptyset$ of L. We say that the element $i_1\cdots i_n$ of L is a *part* of the element $j_1\cdots j_m$ if, either n = 0, or $1 \leq n \leq m$ and $i_k = j_k$ for every $k \leq n$. The elements of L are denoted also by $\bar{i},\bar{j},\bar{i}_1$ etc. If $\bar{i} = i_1\cdots i_n$ then by $\bar{i}0$ (resp. $\bar{i}1$) we denote the element $i_1\cdots i_n 0$ (resp. $i_1\cdots i_n 1$) of L.

By Λ_n, $n = 1,2,\cdots$, we denote the set of all ordered n-tuples $i_1\cdots i_n$, where i_k, $k = 1,\cdots,n$ is a positive integer. Set $\Lambda = \cup_{n=1}^{\infty} \Lambda_n$. The elements of Λ are denoted by $\bar{\alpha},\bar{\beta}$ etc. Let $\bar{\alpha} \in \Lambda_n$, $\bar{\beta} \in \Lambda_m$, $\bar{\alpha} = i_1\cdots i_n$, $\bar{\beta} = j_1\cdots j_m$. We write $\bar{\beta} \geq \bar{\alpha}$ if $1 \leq n \leq m$ and $i_k = j_k$ for every $k \leq n$. Obviously, if $\bar{\alpha},\bar{\beta} \in \Lambda_n$ and $\bar{\beta} \geq \bar{\alpha}$ then $\bar{\beta} = \bar{\alpha}$. Also, for every $\bar{\alpha} \in \Lambda_n$ the set of all elements $\bar{\beta} \in \Lambda_{n+1}$ such that $\bar{\beta} \geq \bar{\alpha}$, is a countable non-finite set.

By C we denote the Cantor ternary set. By $C_{\bar{i}}$, where $\bar{i} = i_1 \cdots i_n \in L$, $n \geq 1$, we denote the set of all points of C for which the k-th digit in the ternary expansion, $k = 1, \cdots, n$, coincides with 0 if $i_k = 0$ and with 2 if $i_k = 1$. Also, set $C_\emptyset = C$. For every subset s of L_k, $k = 0, 1, \cdots$, we set $C_s = \cup_{\bar{i} \in s} C_{\bar{i}}$. For every point a of C and for every integer $n \geq 0$ by $\bar{i}(a,n)$ we denote the element $\bar{i} \in L_n$ for which $a \in C_{\bar{i}}$. Obviously, this element is uniquely determined. For every subset F of C and for every integer $n = 0, 1, 2, \cdots$ by st(F,n) (it is called *the* n-*star of* F *in* C) we denote the union of all sets $C_{\bar{i}}$, where $\bar{i} \in L_n$, such that $C_{\bar{i}} \cap F \neq \emptyset$. If $F = \{a\}$ then we set $st(F,n) = st(a,n)$.

Let D be a partition of a subset S of C, $\bar{i}$ an element of L_k and t an arbitrary subset of L_n, where $k,n = 0, 1, 2, \cdots$. Set $D(1) = \{d \in D: d$ is not singleton$\}$, $D_{\bar{i}} = \{d \in D: d \cap C_{\bar{i}0} \neq \emptyset,\ d \cap C_{\bar{i}1} \neq \emptyset,\ d \subseteq C_{\bar{i}0} \cup C_{\bar{i}1}\}$, $D_k = \cup_{\bar{i} \in L_k} D_{\bar{i}}$ and $D(k,t) = \{d \in D_k: d \cap C_{\bar{j}} \neq \emptyset,\ \bar{j} \in t$ and $d \subseteq \cup_{\bar{j} \in t} C_{\bar{j}}\}$. Obviously, 1) $D(1) = \cup_{k=0}^{\infty} D_k$, $D_{\bar{i}} \cap D_{\bar{j}} = \emptyset$ if $\bar{i}, \bar{j} \in L_k$ and $\bar{i} \neq \bar{j}$, 3) $D_{k_1} \cap D_{k_2} = \emptyset$ if $k_1 \neq k_2$ and 4) $D(k,t_1) \cap D(k,t_2) = \emptyset$ if $t_1 \subseteq L_n$, $t_2 \subseteq L_n$ and $t_1 \neq t_2$. We say that the *degree* of an element d of D is k and we write $\deg(d) = k$ iff $d \in D_k$.

For every integer $k \geq 0$ by D_k^* we denote the subset of S which is the union of all elements of D_k.

2. By $M(\alpha)$, where α is an ordinal number we denote a countable family of spaces for which the α-derivative is empty. We suppose that two different elements of $M(\alpha)$ are not homeomorphic.

A partition D of a subset S of C is called $M(\alpha)$-*partition* iff 1) D is an upper semi-continuous partition, 2) every element of D is a singleton or consists of two points and 3) for every pair of integers k and n, $k,n = 0,1,2,\cdots$, and for every subset t of L_n there exists an element $M(D,k,t) \in M(\alpha)$ which is homeomorphic to the subset $D(k,t)$ of the quotient space D.

By $h(D,k,t)$ we denote a homeomorphism of $M(D,k,t)$ onto $D(k,t)$. In the future we suppose that for a given $M(\alpha)$-partition the homeomorphisms $h(D,k,t)$ are fixed.

By A we denote a family of pairs (S,D), where S is a non-empty subset of C and D is a $M(\alpha)$-partition of S (it is supposed that $\emptyset \notin D$). We suppose that the power of A is less than or equal to the continuum.

By S(A) we denote the set of all subsets S of C such that there exists a pair $(S,D) \in A$. If (S_1,D_1) and (S_2,D_2) are different elements of A then we consider S_1 and S_2 as different elements of S(A) (though $S_1 \equiv S_2$).

If $S \in S(A)$ then by D(S) we denote the corresponding $M(\alpha)$-partition of S such that $(S,D(S)) \in A$.

Since the power of A is less than or equal to the continuum, for every element $\bar{i} \in L$ there exists a subset $S(\bar{i})$ of S(A) such that: 1) $S(\emptyset) = S(A)$, 2) $S(\bar{i}) \cap S(\bar{j}) = \emptyset$ if $\bar{i},\bar{j} \in L_k$, $\bar{i} \neq \bar{j}$, 3) $S(\bar{i}) = S(\bar{i}0) \cup S(\bar{i}1)$ and 4) for every $S_1,S_2 \in S(A)$, $S_1 \neq S_2$ there exists an integer $k \geq 0$ and elements $\bar{i},\bar{j} \in L_k$, $\bar{i} \neq \bar{j}$ such that $S_1 \in S(\bar{i})$ and $S_2 \in S(\bar{j})$.

Let (S_1,D_1), $(S_2,D_2) \in A$. We say that the elements S_1 and S_2 of S(A) are (k,n)-*equivalent*, where $k,n = 0,1,\cdots$, if 1) there exists an element $\bar{i} \in L_k$ such that $S_1, S_2 \in S(\bar{i})$ and 2) for every subset t of L_n we have $M(D_1,k,t) = M(D_2,k,t)$.

By S(k,n) we denote the set of all classes of the (k,n)-equivalence. It is easy to see that the set S(k,n) is countable.

Let Q be a subset of an element of S(k,n). If t is a subset of L_n and $S \in Q$ then the element M(D(S),k,t) of $M(\alpha)$ is independent of S. This element is denoted by M(Q,k,t).

Consider the set $C \times S(A)$. For every subset Q of S(A) we set $J(Q) = \{(a,S) \in C \times S(A): a \in S,\ S \in Q\}$. The set J(S(A)) is also denoted by J(A).

A subset y of J(Q) is called a D-*set* of J(Q) with respect to (k,n)-equivalence iff there exist a subset t(y) of L_n and an element z(y) of M(Q,k,t(y)) such that

$$y = \cup_{S\in Q}(h(D(S),k,t(y))(z(y)) \times \{S\}).$$

For every $S \in Q$ we denote by y(S) the element h(D(S),k, t(y))(z(y)) of D(S). Obviously, $y(S) \times \{S\} = y \cap (C \times \{S\})$ and $y(S) \in D_k(S) \equiv (D(S))_k$. Also, for every $S \in Q$ and for every $d \in D_k(S)$ there exists a uniquely determined D-set y of J(Q) with respect to (k,n)-equivalence such that $d = y(S)$.

The ordinal number type$(y(S),D_k(S))$ is independent of S. This ordinal number is called the *type* of y with respect to Q (denoted by type(y)). The number k is called the *degree* of the D-set y (denoted by deg(y)).

It is easy to see that if y_1 is a D-set of $J(Q_1)$ with respect to (k_1,n_1)-equivalence, y_2 is a D-set of $J(Q_2)$ with respect to (k_2,n_2)-equivalence, $S \in Q_1 \cap Q_2$ and $y_1(S) \neq y_2(S)$ then $y_1 \cap y_2 = \emptyset$.

3. As it is mentioned in the introduction, in part II of this paper we construct a space denoted by T(A) which will be used in part III as a universal element. The points of the space T(A) are the elements of some partition of the set J(A). The elements of this partition which are not singletons, are countable. These elements (denoted by $d(\bar{\alpha},m)$) and some subsets of J(A) (denoted by $U(\bar{\gamma},m,k-1)$) are constructed by induction in the next "long" lemma. The subsets $U(\bar{\gamma},m,k-1)$ define some neighbourhoods of the points $d(\bar{\alpha},m)$ in the space T(A). The topology of the space T(A) is not quotient because in the set J(A) there is no topology. Moreover, the topology of T(A) is not quotient with respect to the "natural" topology which may be defined in the set J(A).

Lemma 5. *For every integer* $k = 0,1,2,\cdots$ *and for every element* $\bar{\alpha}$ *of* Λ_{k+1} *there exist:*

i) *an integer* $n(\bar{\alpha}) \geq k+1$,

ii) *a set* $S(\bar{\alpha})$ *which is a subset of an element of the set* $S(k,n(\bar{\alpha}))$ *(it is possible that* $S(\bar{\alpha}) = \emptyset$*),*

iii) *an ordering* $y(\bar{\alpha},0),y(\bar{\alpha},1),y(\bar{\alpha},2),\cdots$ *of the set of all* D*-sets of* $J(S(\bar{\alpha}))$ *with respect to* $(k,n(\bar{\alpha}))$*-equivalence (it is possible that for some integers* m, $y(\bar{\alpha},m) = \emptyset$*),*

iv) *a finite sequence* $d(\bar{\alpha},0),d(\bar{\alpha},1),\cdots,d(\bar{\alpha},k)$ *of subsets of* $J(S(\bar{\alpha}))$ *(it is possible that for some integers* m, $d(\bar{\alpha},m) = \emptyset$*),*

v) *a subset* $U(\overline{\gamma},m,k-1)$ *of* $J(S(\overline{\gamma}))$, *for every* $\overline{\gamma} \in \Lambda_q$, $1 \leq q \leq k$ *and* $0 \leq m \leq q - 1$ *(it is possible that for some* $\overline{\gamma}$ *and* m, $U(\overline{\gamma},m,k-1) = \emptyset$*) and*

vi) *a subset* $s(\overline{\gamma},m,\overline{\alpha})$ *of* $L_{n(\overline{\alpha})}$ *for every* $\overline{\gamma} \in \Lambda_q$, $1 \leq q \leq k$, $0 \leq m \leq q - 1$ *and* $\overline{\gamma} \leq \overline{\alpha}$ *such that:*

1) $n(\overline{\alpha}) \geq n(\overline{\beta})$ *if* $\overline{\alpha} \geq \overline{\beta}$

2) $S(A) = \cup_{\overline{\alpha}\in\Lambda_1} S(\overline{\alpha})$.

3) *If* $\overline{\alpha}_1, \overline{\alpha}_2 \in \Lambda_{k+1}$, $\overline{\alpha}_1 \neq \overline{\alpha}_2$ *then* $S(\overline{\alpha}_1) \cap S(\overline{\alpha}_2) = \emptyset$.

4) *If* $\overline{\beta} \in \Lambda_k$ *then* $S(\overline{\beta}) = \cup_{\substack{\overline{\alpha}\in\Lambda_{k+1} \\ \overline{\alpha}>\overline{\beta}}} S(\overline{\alpha})$.

5) *If* $0 \leq m \leq k$, $\overline{\gamma} \in \Lambda_{k+1-m}$ *and* $\overline{\gamma} \leq \overline{\alpha}$ *then*

$d(\overline{\alpha},m) = y(\overline{\gamma},m) \cap J(S(\overline{\alpha}))$.

6) *If* $\overline{\gamma} \in \Lambda_q$, $1 \leq q \leq k$ *and* $0 \leq m \leq q - 1$ *then* $d(\overline{\gamma},m) \subseteq U(\overline{\gamma},m,k-1)$.

7) *If* $\overline{\gamma} \in \Lambda_q$, $1 \leq q \leq k$, $0 \leq m \leq q - 1$ *and* $d(\overline{\gamma},m) = \emptyset$ *then* $U(\overline{\gamma},m,k-1) = \emptyset$.

8) *If* $\overline{\gamma} \in \Lambda_q$, $\overline{\gamma}_1 \in \Lambda_{q'}$, $1 \leq q \leq k$, $1 \leq q' \leq k_1$, $0 \leq m \leq q - 1$, $0 \leq m_1 \leq q' - 1$, $k_1 < k$ *and* $d(\overline{\gamma},m) \subseteq U(\overline{\gamma}_1,m_1,k_1-1)$ *then* $U(\overline{\gamma},m,k-1) \subseteq U(\overline{\gamma}_1,m_1,k_1-1)$.

9) *If* $\overline{\gamma}_1 \in \Lambda_{q'}$, $\overline{\gamma}_2 \in \Lambda_{q''}$, $1 \leq q' \leq k$, $1 \leq q'' \leq k$, $0 \leq m_1 \leq q' - 1$, $0 \leq m_2 \leq q'' - 1$ *and* $d(\overline{\gamma}_1,m_1) \neq d(\overline{\gamma}_2,m_2)$ *then* $U(\overline{\gamma}_1,m_1,k-1) \cap U(\overline{\gamma}_2,m_2,k-1) = \emptyset$.

10) *If* $\overline{\gamma} \in \Lambda_q$, $1 \leq q \leq k - 1$, $0 \leq m \leq q - 1$, $S \in S(\overline{\gamma})$, $d \in D(S)$ *and* $(d \times \{S\}) \cap U(\overline{\gamma},m,k-1) \neq \emptyset$ *then* $d \times \{S\} \subseteq U(\overline{\gamma},m,k-2)$.

11) *If* $\overline{\gamma} \in \Lambda_q$, $1 \leq q \leq k$, $0 \leq m \leq q - 1$, $\overline{\gamma} \leq \overline{\alpha}$, $S \in S(\overline{\alpha})$, $d \in D_{q-m-1}(S)$, $d \times \{S\} = d(\overline{\gamma},m) \cap$

$(C \times \{S\})$, $d' \in D(S)(1)$, $d' \neq d$ *and* $d' \subseteq st(d,n(\overline{\alpha}))$ *then, either* $\deg(d') > k$ *or* $\deg(d') = q - m - 1$ *and* $\text{type}(d', D_{q-m-1}(S)) < \text{type}(d, D_{q-m-1}(S))$.

12) *If* $\overline{\gamma} \in \Lambda_q$, $1 \leq q \leq k$, $0 \leq m \leq q - 1$, $\overline{\gamma} \leq \overline{\alpha}$, $S \in S(\overline{\alpha})$, $d \in D(S)$ *and* $d \times \{S\} = d(\overline{\gamma},m) \cap (C \times \{S\})$ *then* $st(d,n(\overline{\alpha})) = C_{s(\overline{\gamma},m,\overline{\alpha})}$.

13) *If* $\overline{\gamma} \in \Lambda_q$, $1 \leq q \leq k$ *and* $0 \leq m \leq q - 1$ *then* $U(\overline{\gamma},m,k-1) = \bigcup_{\overline{\alpha} \geq \overline{\gamma}, \overline{\alpha} \in \Lambda_{k+1}} ((C_{s(\overline{\gamma},m,\overline{\alpha})} \times S(\overline{\alpha})) \cap J(S(\overline{\alpha})))$.

14) *If* $\overline{\gamma} \in \Lambda_{k+1-m}$, $0 < m \leq k$, $\overline{\gamma} \leq \overline{\alpha}$, $S_1 \in S(\overline{\alpha})$, $S_2 \in S(\overline{\alpha})$, $d_1 \in D(S_1)$, $d_2 \in D(S_2)$, $d_1 \times \{S_1\} = y(\overline{\gamma},m) \cap (C \times \{S_1\})$, $d_2 \times \{S_2\} = y(\overline{\gamma},m) \cap (C \times \{S_2\})$ *then* $st(d_1,n(\overline{\alpha})) = st(d_2,n(\overline{\alpha}))$.

Proof. We prove the lemma by induction on k.

Let $k = 0$. Set $n(\overline{\alpha}) = 1$ for every $\overline{\alpha} \in \Lambda_1$. There exists a one-to-one correspondence between the set $S(0,1)$ and a subset of Λ_1. If $Q \in S(0,1)$ corresponds to $\overline{\alpha} \in \Lambda_1$, then we set $S(\overline{\alpha}) = Q$. If there is no element of $S(0,1)$ which corresponds to $\overline{\alpha} \in \Lambda_1$ then we set $S(\overline{\alpha}) = \emptyset$.

For every $\overline{\alpha} \in \Lambda_1$, let $y(\overline{\alpha},0), y(\overline{\alpha},1), \cdots$ be an ordering of the set of all D-sets of $J(S(\overline{\alpha}))$ with respect to $(0,1)$-equivalence. Set $d(\overline{\alpha},0) = y(\overline{\alpha},0)$, $\overline{\alpha} \in \Lambda_1$.

Obviously, properties 1), 2), 3), 5) and 14) are true for $k = 0$.

Suppose that the lemma is true for all integers $k < p$, $p > 0$. We prove the lemma for $k = p$.

For every $S \in S(\overline{\gamma})$, $\overline{\gamma} \in \Lambda_q$, $q \leq p$ and for every $d(\overline{\gamma},m)$, $0 \leq m \leq q - 1$, by $d(S,\overline{\gamma},m)$ we denote the element of $D(S)$ for which $d(S,\overline{\gamma},m) \times \{S\} = d(\overline{\gamma},m) \cap (C \times \{S\})$. For every $U(\overline{\gamma},m,k-1)$, where $\overline{\gamma} \in \Lambda_q$, $1 \leq q \leq k$, $0 \leq m \leq q - 1$ and $k \leq p - 1$ by $U(S,\overline{\gamma},m,k-1)$ where $S \in S(\overline{\gamma})$ we denote the subset of S for which $U(S,\overline{\gamma},m,k-1) \times \{S\} = U(\overline{\gamma},m,k-1) \cap (C \times \{S\})$. Obviously, if $d(\overline{\gamma},m) = \emptyset$ or $U(\overline{\gamma},m,k-1) = \emptyset$ then $d(S,\overline{\gamma},m) = \emptyset$ or $U(S,\overline{\gamma},m,k-1) = \emptyset$, respectively. We observe that by property 13) it follows that the set $U(S,\overline{\gamma},m,k-1)$ is an open subset of S. Also, for every $\overline{\gamma} \in \Lambda_{p+1-m}$, $0 < m \leq p$, $S \in S(\overline{\gamma})$ we denote by $y(S,\overline{\gamma},m)$ the element d of $D(S)$ for which $d \times \{S\} = y(\overline{\gamma},m) \cap (C \times \{S\})$.

Let $\overline{\alpha} \in \Lambda_{p+1}$. There exists a uniquely determined element $\overline{\beta} \in \Lambda_p$ such that $\overline{\beta} \leq \overline{\alpha}$. Let $S \in S(\overline{\beta})$.

There exists an integer $t_1(S,p) \geq 0$ such that if $\overline{\gamma} \in \Lambda_q$, $\overline{\gamma}_1 \in \Lambda_{q'}$, $1 \leq q \leq p$, $0 \leq m \leq q - 1$, $1 \leq q' \leq p$, $0 \leq m_1 \leq q' - 1$, $k_1 < p$, $S \in S(\overline{\gamma}) \cap S(\overline{\gamma}_1)$ and $d(\overline{\gamma},m) \subseteq U(\overline{\gamma}_1,m_1,k_1-1)$, then $st(d(S,\overline{\gamma},m),t_1(S,p)) \subseteq U(S,\overline{\gamma}_1,m_1,k_1-1)$.

Also, there exists an integer $t_2(S,p) \geq 0$ such that if $\overline{\gamma}_1 \in \Lambda_{q'}$, $\overline{\gamma}_2 \in \Lambda_{q''}$, $1 \leq q' \leq p$, $1 \leq q'' \leq p$, $0 \leq m_1 \leq q' - 1$, $0 \leq m_2 \leq q'' - 1$, $S \in S(\overline{\gamma}_1) \cap S(\overline{\gamma}_2)$ and $d(\overline{\gamma}_1,m_1) \neq d(\overline{\gamma}_2,m_2)$, then $st(d(S,\overline{\gamma}_1,m_1), t_2(S,p)) \cap st(d(S,\overline{\gamma}_2,m_2), t_2(S,p)) = \emptyset$.

Since $D(S)$ is an upper semi-continuous partition there exists an integer $t_3(S,p) \geq 0$ such that if $\overline{\gamma} \leq \overline{\beta}$, $\overline{\gamma} \in \Lambda_q$, $1 \leq q \leq p - 1$, $0 \leq m \leq q - 1$, $d \in D(S)$ and $d \cap st(d(S,\overline{\gamma},m), t_3(S,p)) \neq \emptyset$, then $d \subseteq U(S,\overline{\gamma},m,p-2)$.

Finally, since 1) $D(S)$ is an upper semi-continuous partition, 2) $D_k(S)$, $k = 0,1,2,\cdots$, is a closed subset of

$D(S)$ having type $\leq \alpha$ and 3) $D_{k_1}(S) \cap D_{k_2}(S) = \emptyset$, $k_1 \neq k_2$, there exists an integer $t_4(S,p) \geq 0$ such that if $\bar{\gamma} \in \Lambda_q$, $1 \leq q \leq p$, $0 \leq m \leq q - 1$, $\bar{\gamma} \leq \bar{\beta}$, $d' \in D(S)(1)$, $d' \neq d(S,\bar{\gamma},m)$ and $d' \subseteq st(d(S,\bar{\gamma},m),t_4(S,p))$, then, either $\deg(d') > p$, or $\deg(d') = q - m - 1$ (we observe that $\deg(d(\bar{\gamma},m)) = q - m - 1$) and $type(d',D_{q-m-1}(S)) < type(d(S,\bar{\gamma},m),D_{q-m-1}(S))$.

Set $t(S,p) = \max\{n(\bar{\beta}),t_1(S,p),t_2(S,p),t_3(S,p),t_4(S,p), p+1\}$.

Now, in the set $S(\bar{\beta})$ we define an equivalence relation. Let $S_1,S_2 \in S(\bar{\beta})$. We say that $S_1 \sim S_2$ if and only if 1) $t(S_1,p) = t(S_2,p)$, 2) S_1 and S_2 are $(p,t(S_1,p))$-equivalent, 3) $st(d(S_1,\bar{\gamma},m),t(S_1,p)) = st(d(S_2,\bar{\gamma},m),t(S_2,p))$ for every $\bar{\gamma} \leq \bar{\beta}$, $\bar{\gamma} \in \Lambda_q$, $1 \leq q \leq p$ and $0 \leq m \leq q - 1$ and 4) $st(y(S_1,\bar{\gamma},m),t(S_1,p)) = st(y(S_2,\bar{\gamma},m),t(S_2,p))$ for every $\bar{\gamma} \leq \bar{\beta}$, $\bar{\gamma} \in \Lambda_{p+1-m}$, $0 < m \leq p$.

It is clear that the set of all equivalence classes is countable. Hence, there exists a one-to-one correspondence between this set and a subset of the set of all elements $\bar{\delta}$ of Λ_{p+1} for which $\bar{\beta} \leq \bar{\delta}$. If there is no equivalence class which corresponds to the element $\bar{\alpha}$, then we set $S(\bar{\alpha}) = \emptyset$. In the opposite case, we denote by $S(\bar{\alpha})$ the equivalence class which corresponds to the element $\bar{\alpha}$.

Set $n(\bar{\alpha}) = t(S,p)$ where $S \in S(\bar{\alpha})$. Obviously, the number $n(\bar{\alpha})$ is independent of the element S of $S(\bar{\alpha})$.

Let $y(\bar{\alpha},0),y(\bar{\alpha},1),\cdots$ be an arbitrary ordering of the set of all D-sets of $J(S(\bar{\alpha}))$ with respect to $(p,n(\bar{\alpha}))$-equivalence.

Set $d(\overline{\alpha},m) = y(\overline{\gamma},m) \cap J(S(\overline{\alpha}))$ where $\overline{\gamma} \in \Lambda_{p+1-m}$, $\overline{\gamma} \leq \overline{\alpha}$ and $m = 0,1,\cdots,p$.

Let $\overline{\gamma} \in \Lambda_q$, $1 \leq q \leq p$, $\overline{\gamma} \leq \overline{\alpha}$, $0 \leq m \leq q-1$ and $S_0 \in S(\overline{\alpha})$. We define the sets $s(\overline{\gamma},m,\overline{\alpha})$ and $U(\overline{\gamma},m,p-1)$ setting $s(\overline{\gamma},m,\overline{\alpha}) = \{\overline{i} \in L_{n(\overline{\alpha})}: C_{\overline{i}} \subseteq st(d(S_0,\overline{\gamma},m),n(\overline{\alpha}))\}$ and $U(\overline{\gamma},m,p-1) = \cup_{\overline{\alpha}\geq\overline{\gamma},\overline{\alpha}\in\Lambda_{p+1},S\in S(\overline{\alpha})}((st(d(S,\overline{\gamma},m),n(\overline{\alpha})) \times \{S\}) \cap J(S(A)))$ (*)

By condition 3) of the definition of the equivalence relation in the set $S(\overline{\beta})$ it follows that the set $s(\overline{\gamma},m,\overline{\alpha})$ is independent of $S_0 \in S(\overline{\alpha})$. Obviously, if $d(\overline{\gamma},m) = \emptyset$, then $s(\overline{\gamma},m,\overline{\alpha}) = \emptyset$ and $U(\overline{\gamma},m,p-1) = \emptyset$.

Now, we prove that the properties of the lemma are true for the $k = p$.

Property 1) follows by the definition of the number $n(\overline{\alpha})$. Property 2) is independent of p. Properties 3) and 4) follow by the construction of the sets $S(\overline{\alpha})$, $\overline{\alpha} \in \Lambda_{p+1}$. Property 5) follows by the determination of the set $d(\overline{\alpha},m)$. Properties 6) and 7) follow by the construction of the sets $U(\overline{\gamma},m,p-1)$.

It is easy to see that properties 8), 9), 10) and 11) follow by the definition of the numbers $t_1(S,p)$, $t_2(S,p)$, $t_3(S,p)$ and $t_4(S,p)$, respectively, and by the construction of the sets $U(\overline{\gamma},m,p-1)$. Property 12) follows by the definition of the sets $s(\overline{\gamma},m,\overline{\alpha})$.

In the type (*) using the property 12) we have $\cup_{S\in S(\alpha)}((st(d(S,\overline{\gamma},m),n(\overline{\alpha})) \times \{S\}) \cap J(S(A))) = (C_{s(\overline{\gamma},m,\overline{\alpha})} \times S(\overline{\alpha})) \cap J(S(\overline{\alpha}))$. From this follows property 13).

Property 14) follows by condition 4) of the definition of the equivalence relation in the set $S(\bar{\beta})$. The proof of the lemma is completed.

4. By $T(A)$ we denote the set of all non-empty sets of the form $d(\bar{\alpha},m)$, $\bar{\alpha} \in \Lambda_{k+1}$, $0 \leq m \leq k$ and $k = 0,1,2,\cdots$ and all singletons $\{x\}$ where x belongs to $J(A)$ and does not belong to any set $d(\bar{\alpha},m)$. The elements $d(\bar{\alpha},m)$ (resp. $\{x\}$) of the set $T(A)$ are called elements of the *first kind* (resp. the *second kind*).

By $U(A)$ we denote the set of all sets $U(\bar{\gamma},m,k-1)$, $\bar{\gamma} \in \Lambda_q$, $1 \leq q \leq k$, $0 \leq m \leq q-1$ and $k = 0,1,2,\cdots$ and all sets of the form $(C_t \times S(\bar{\alpha})) \cap J(A)$ where t is a subset of L_k, $k = 0,1,2,\cdots$ and $\bar{\alpha} \in \Lambda$.

If $U \in U(A)$, then by $O(U)$ we denote the set of all elements of $T(A)$ which are contained in the set U. If $U = U(\bar{\gamma},m,k-1)$ or $U = (C_t \times S(\bar{\alpha})) \cap J(A)$ then we set $O(\bar{\gamma},m,k-1) = O(U)$ or $O(C_t,S(\bar{\alpha})) = O(U)$, respectively.

By $O(A)$ we denote the set of all sets $O(U)$, $U \in U(A)$. Obviously, the set $O(A)$ is countable.

In the future, the sets $U(\bar{\gamma},m,k-1)$ and $O(\bar{\gamma},m,k-1)$ (resp. $(C_t \times S(A)) \cap J(A)$ and $O(C_t,S(\bar{\alpha})))$ are called elements of the *first kind* (resp. the *second kind*) of the sets $U(A)$ and $O(A)$, respectively.

Also, for every $S \in S(\bar{\gamma})$, $\bar{\gamma} \in \Lambda_q$, $1 \leq q$, $0 \leq m \leq q-1$, by $d(S,\bar{\gamma},m)$ (resp. $y(S,\bar{\gamma},m)$) we denote the element d of $D(S)$ for which $d \times \{S\} = d(\bar{\gamma},m) \cap (C \times \{S\})$ (resp. $d \times \{S\} = y(\bar{\gamma},m) \cap (C \times \{S\})$). Obviously, $d \in D_{q-m-1}(S)$ (resp. $d \in D_{q-1}(S)$).

Lemma 6. *The set* $O(A)$ *is a basis of open sets for a topology on* $T(A)$.

Proof. It is sufficient to prove that: 1) for every $d \in T(A)$ there exists $O \in O(A)$ such that $d \in O$ and 2) if $d \in O_1 \cap O_2$ where $O_1, O_2 \in O(A)$, then there exists $O \in O(A)$ such that $d \in O \subseteq O_1 \cap O_2$.

Let $d \in T(A)$. If $d = d(\overline{\alpha}, m)$, $\overline{\alpha} \in \Lambda_{k+1}$ and $0 \leq m \leq k$ or $d = \{(\alpha, S)\}$ where $S \in S(\overline{\alpha})$ then, obviously, $d \in O(C, S(\overline{\alpha})) = O \in O(A)$.

Let $d \in O_1 \cap O_2$ where $O_1, O_2 \in O(A)$. First, we suppose that $d = \{x\}$, where $x = (a, S)$. Then, there exist $\overline{i}_1, \overline{i}_2 \in L$ and $\overline{\alpha}_1, \overline{\alpha}_2 \in \Lambda$ such that $d \in O(C_{\overline{i}_1}, S(\overline{\alpha}_1)) \subseteq O_1$ and $d \in O(C_{\overline{i}_2}, S(\overline{\alpha}_2)) \subseteq O_2$. If O_1 or O_2 is an element of the first kind, then the existence of $\overline{i}_1$ or $\overline{i}_2$ of L follows by the structure of the elements of $U(A)$ of the first kind (see property 13) of Lemma 5). Let $\overline{i} \in L$ and $\overline{\alpha} \in \Lambda$ such that $a \in C_{\overline{i}} \subseteq C_{\overline{i}_1} \cap C_{\overline{i}_2}$ and $S \in S(\overline{\alpha}) \subseteq S(\overline{\alpha}_1) \cap S(\overline{\alpha}_2)$. Obviously, $d \in O(C_{\overline{i}}, S(\overline{\alpha})) = O \subseteq O_1 \cap O_2$.

Now, let $d = d(\overline{\gamma}, m)$, $\overline{\gamma} \in \Lambda_q$, $0 \leq m \leq q - 1$.

First, we prove that if $d(\overline{\gamma}, m) \subseteq O(C_t, S(\overline{\beta})) = O' \in O(A)$, where $t \subseteq L_n$ and $\overline{\beta} \in \Lambda_{q'}$, then $d(\overline{\gamma}, m) \subseteq O(\overline{\gamma}, m, k-1) \subseteq O'$ for $k \geq \max\{n, q+1\}$. Indeed, let $k \geq \max\{n, q+1\}$, $\overline{\alpha} \in \Lambda_{k+1}$ and $\overline{\gamma} \leq \overline{\alpha}$. Since $d(\overline{\gamma}, m) \neq \emptyset$ and $d(\overline{\gamma}, m) \subseteq O(C_t, S(\overline{\beta}))$ we have that $S(\overline{\gamma}) \subseteq S(\overline{\beta})$. Hence, $S(\overline{\alpha}) \subseteq S(\overline{\beta})$. Let $S \in S(\overline{\alpha})$. Since $d(S, \overline{\gamma}, m) \times \{S\} \subseteq d(\overline{\gamma}, m) \subseteq C_t \times S(\overline{\beta})$ we have $d(S, \overline{\gamma}, m) \subseteq C_t$. By property 12) of Lemma 5 we have $C_{s(\overline{\gamma}, m, \overline{\alpha})} = st(d(S, \overline{\gamma}, m), n(\overline{\alpha}))$. Since $n(\overline{\alpha}) \geq k + 1 > n$, $C_{s(\overline{\gamma}, m, \overline{\alpha})} \subseteq C_t$. Hence, $C_{s(\overline{\gamma}, m, \overline{\alpha})} \times S(\overline{\alpha}) \subseteq C_t \times S(\overline{\beta})$ for every $\overline{\alpha} \geq \overline{\gamma}$,

$\overline{\alpha} \in \Lambda_{k+1}$. From the structure of the set $U(\overline{\gamma},m,k-1)$ (see property 13) of Lemma 5) we have $U(\overline{\gamma},m,k-1) \subseteq C_t \times S(\overline{\beta})$. Thus, $d(\overline{\gamma},m) \subseteq O(\overline{\gamma},m,k-1) \subseteq O'$.

Using the above proposition (and properties 6) and 8) of Lemma 5) we may find integers $k_1 \geq 0$ and $k_2 \geq 0$ such that $d \in O(\overline{\gamma},m,k_1-1) \subseteq O_1$ and $d \in O(\overline{\gamma},m,k_2-1) \subseteq O_2$. Set $O = O(U)$, where $U = U(\overline{\gamma},m,k-1)$, $k \geq \max\{k_1,k_2\}$. Then $d \in O \subseteq O_1 \cap O_2$. The proof of the lemma is completed.

5. In the future, in the set $T(A)$ we consider the topology which has the set $O(A)$ as a basis of open sets.

Lemma 7. *The space* $T(A)$ *is a Hausdorff space.*

Proof. Let $d_1,d_2 \in T(A)$, $d_1 \neq d_2$. Consider the cases: 1) $d_1 = \{(a_1,S_1)\}$, $d_2 = \{(a_2,S_2)\}$, 2) $d_1 = \{(a,S)\}$, $d_2 = d(\overline{\gamma},m)$, $\overline{\gamma} \in \Lambda_q$, $0 \leq m \leq q-1$ and 3) $d_1 = d(\overline{\gamma}_1,m_1)$, $d_2 = d(\overline{\gamma}_2,m_2)$, $\overline{\gamma}_1 \in \Lambda_{q'}$, $\overline{\gamma}_2 \in \Lambda_{q''}$, $q' \geq q''$, $0 \leq m_1 \leq q'-1$, $0 \leq m_2 \leq q''-1$.

In the first case, either $a_1 \neq a_2$, or $S_1 \neq S_2$. By the definition of the (k,n)-equivalence, if $a_1 \neq a_2$ (resp. $S_1 \neq S_2$), then there exist an integer $n \geq 1$ and $\overline{i},\overline{j} \in L_n$, $\overline{i} \neq \overline{j}$ (resp. $\overline{\alpha}_1,\overline{\alpha}_2 \in \Lambda_n$, $\overline{\alpha}_1 \neq \overline{\alpha}_2$) such that $a_1 \in C_{\overline{i}}$ and $a_2 \in C_{\overline{j}}$ (resp. $S_1 \in S(\overline{\alpha}_1)$ and $S_2 \in S(\overline{\alpha}_2)$) (we observe that $S_1 \neq S_2$ means that the elements S_1 and S_2 of $S(A)$ are different though it is possible that $S_1 \equiv S_2$ as subset of C). Set $O_1 = O(C_{\overline{i}}, S(\overline{\alpha}'))$ and $O_2 = O(C_{\overline{j}}, S(\overline{\alpha}''))$, where $\overline{\alpha}',\overline{\alpha}'' \in \Lambda_1$, $S_1 \in S(\overline{\alpha}')$ and $S_2 \in S(\overline{\alpha}'')$ (resp. $O_1 = O(C,S(\overline{\alpha}_1))$ and $O_2 = O(C,S(\overline{\alpha}_2))$). Then $d_1 \in O_1$, $d_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$.

In the second case, either $S \notin S(\overline{\gamma})$, or $S \in S(\overline{\gamma})$ and $a \notin d(S,\overline{\gamma},m)$. Let $S \notin S(\overline{\gamma})$, $\overline{\gamma}' \in \Lambda_q$ and $S \in S(\overline{\gamma}')$. Setting $O_1 = O(C,S(\overline{\gamma}'))$ and $O_2 = O(C,S(\overline{\gamma}))$ we have $d_1 \in O_1$, $d_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$.

Let $S \in S(\overline{\gamma})$ and $a \notin d(S,\overline{\gamma},m)$. There exists an integer $k \geq q + 1$ such that $st(a,k) \cap st(d(S,\overline{\gamma},m),k) = \emptyset$. Let $\overline{\alpha} \in \Lambda_{k+1}$, $S \in S(\overline{\alpha})$, $O_1 = O(st(a,k),S(\overline{\alpha}))$ and $O_2 = O(\overline{\gamma},m,k-1)$. Obviously, $d_1 \in O_1$ and $d_2 \in O_2$. By property 13) of Lemma 5 we have that $U(\overline{\gamma},m,k-1) \cap (st(a,k) \times S(\overline{\alpha})) = (C_{s(\overline{\gamma},m,\overline{\alpha})} \times S(\overline{\alpha})) \cap (st(a,k) \times S(\overline{\alpha}))$.

Since $n(\overline{\alpha}) \geq k + 1$, by property 12) of Lemma 5, $st(d(S,\overline{\gamma},m),k) \supseteq C_{s(\overline{\gamma},m,\overline{\alpha})}$. Since $st(a,k) \cap st(d(S,\overline{\gamma},m),k) = \emptyset$ we have $U(\overline{\gamma},m,k-1) \cap (st(a,k) \times S(\overline{\alpha})) = \emptyset$. Hence, $O_1 \cap O_2 = \emptyset$.

In the third case, setting $O_1 = O(\overline{\gamma}_1,m_1,k-1)$ and $O_2 = O(\overline{\gamma}_2,m_2,k-1)$, where $k \geq q' + 1$, by properties 6) and 9) of Lemma 5, we have $d_1 \in O_1$, $d_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. The proof of the lemma is completed.

6. *Lemma* 8. *If* $U \in U(A)$ *then* $Bd(O(U)) \subseteq \{d \in T(A): d \cap U \neq \emptyset, d \cap (J(A) \setminus U) \neq \emptyset\}$.

Proof. Let $d \in Bd(O(U)) = cl(O(U)) \setminus O(U)$. Since $d \not\subseteq U$, $d \cap (J(A) \setminus U) \neq \emptyset$. We prove that $d \cap U \neq \emptyset$. It is sufficient to prove that if $d \cap U = \emptyset$, then $d \notin cl(O(U))$.

Consider the following cases: 1) $d = \{(a,S)\}$, $U = (C_t \times S(\overline{\alpha})) \cap J(A)$, $t \subseteq L_n$ and $\overline{\alpha} \in \Lambda_q$, 2) $d = \{(a,S)\}$, $U = U(\overline{\gamma},m,k-1)$, $\overline{\gamma} \in \Lambda_q$, $0 \leq m \leq q - 1$ and $k \geq q$, 3) $d = d(\overline{\gamma},m)$, $\overline{\gamma} \in \Lambda_q$, $0 \leq m \leq q - 1$, $U = (C_t \times S(\overline{\alpha})) \cap J(A)$, $t \subseteq L_n$ and $\overline{\alpha} \in \Lambda_q$, and 4) $d = d(\overline{\gamma},m)$, $\overline{\gamma} \in \Lambda_q$, $0 \leq m \leq q - 1$,

$U = U(\overline{\gamma}_1, m_1, k_1 - 1)$, $\overline{\gamma}_1 \in \Lambda_{q'}$, $0 \leq m \leq q' - 1$ and $q' \leq k_1$.

In the first case, since $d \cap U = \emptyset$, we have, either $a \notin C_t$, or $S \notin S(\overline{\alpha})$. If $a \notin C_t$, then $st(a,n) \cap C_t = \emptyset$. Let $\overline{\beta} \in \Lambda_q$ and $S \in S(\overline{\beta})$ then $(st(a,n) \times S(\overline{\beta})) \cap (C_t \times S(\overline{\alpha})) = \emptyset$. Setting $O = O(st(a,n), S(\overline{\beta}))$ we have that $d \in O$ and $O \cap O(U) = \emptyset$. This means that $d \notin cl(O(U))$.

If $S \notin S(\overline{\alpha})$ then setting $O = O(C, S(\overline{\beta}))$ where $\overline{\beta} \in \Lambda_q$ and $S \in S(\overline{\beta})$, we have $d \in O$ and $O \cap O(U) = \emptyset$, that is $d \notin cl(O(U))$.

In the second case, either $S \notin S(\overline{\gamma})$ or there exists an element $\overline{\alpha} \in \Lambda_{k+1}$ such that $\overline{\alpha} \geq \overline{\gamma}$, $S \in \overline{\alpha}$ and $a \notin C_{s(\overline{\gamma},m,\overline{\alpha})}$. This follows by the structure of the set $U(\overline{\gamma}, m, k-1)$ (see property 13) of Lemma 5). If $S \notin \overline{\gamma}$, then setting $O = O(C, S(\overline{\gamma}_1))$, where $\overline{\gamma}_1 \in \Lambda_q$ and $S \in S(\overline{\gamma}_1)$ we have $d \in O$ and $O \cap O(\overline{\gamma}, m, k-1) = \emptyset$, that is, $d \notin cl(O(U))$.

If $a \notin C_{s(\overline{\gamma},m,\overline{\alpha})}$, where $\overline{\alpha} \in \Lambda_{k+1}$, $\overline{\alpha} \geq \overline{\gamma}$ and $S \in \overline{\alpha}$, then $st(a, n(\overline{\alpha})) \cap C_{s(\overline{\gamma},m,\overline{\alpha})} = \emptyset$ and hence, $(st(a,n(\overline{\alpha})) \times S(\overline{\alpha})) \cap ((C_{s(\overline{\gamma},m,\overline{\alpha})} \times S(\overline{\alpha})) \cap J(A)) = (st(a,n(\overline{\alpha})) \times S(\overline{\alpha})) \cap U(\overline{\gamma}, m, k-1) = \emptyset$. Since $d \in O = O(st(a,n(\overline{\alpha})), S(\overline{\alpha}))$, $d \notin cl(O(U))$.

In the third case, if $S(\overline{\gamma}) \cap S(\overline{\alpha}) = \emptyset$, then $U(\overline{\gamma}, m, q-1) \cap U = \emptyset$, that is, $d \notin cl(O(U))$. Let $S(\overline{\gamma}) \cap S(\overline{\alpha}) \neq \emptyset$. Then for every $S \in S(\overline{\gamma}) \cap S(\overline{\alpha})$ we have $d(S, \overline{\gamma}, m) \cap C_t = \emptyset$ and, hence, $st(d(S,\overline{\gamma},m), n) \cap C_t = \emptyset$, that is, $(st(d(S,\overline{\gamma},m),n) \times \{S\}) \cap U = \emptyset$. From this and by properties 12) and 13) of Lemma 5, it follows that $U(\overline{\gamma}, m, k-1) \cap U = \emptyset$ if $k = \max\{q,n\}$, that is, $d \notin cl(O(U))$.

In the fourth case, obviously, $d(\overline{\gamma},m) \neq d(\overline{\gamma}_1,m_1)$. If $S(\overline{\gamma}) \cap S(\overline{\gamma}_1) = \emptyset$, then $U(\overline{\gamma},m,k-1) \cap U(\overline{\gamma}_1,m_1,k_1-1) = \emptyset$ for every $k \geq q$, that is, $d \notin cl(O(U))$.

Let $S(\overline{\gamma}) \cap S(\overline{\gamma}_1) \neq \emptyset$. If $q \leq k_1$, then by property 9) of Lemma 5, $U(\overline{\gamma},m,k_1-1) \cap U(\overline{\gamma}_1,m_1,k_1-1) = \emptyset$, that is, $d \notin cl(O(U))$.

If $q > k_1$, then $\overline{\gamma}_1 \leq \overline{\gamma}$. By the structure of the set $U(\overline{\gamma}_1,m_1,k_1-1)$ (see property 13) of Lemma 5) there exists an element $\overline{\alpha} \in \Lambda_{k_1+1}$ such that $\overline{\gamma}_1 \leq \overline{\alpha}$, $\overline{\alpha} \leq \overline{\gamma}$, $d(\overline{\gamma},m) \subseteq C \times S(\overline{\alpha})$ and $d(\overline{\gamma},m) \cap (C_{s(\overline{\gamma}_1,m_1,\overline{\alpha})} \times S(\overline{\alpha})) = \emptyset$. Hence, as in the third case, there exists an integer $k \geq q$ such that $U(\overline{\gamma},m,k-1) \cap (C_{s(\overline{\gamma}_1,m_1,\overline{\alpha})} \times S(\overline{\alpha})) = U(\overline{\gamma},m,k-1) \cap U(\overline{\gamma}_1,m_1,k_1-1) = \emptyset$, that is $d \notin cl(O(U))$. The proof of the lemma is completed.

7. *Lemma* 9. *The space* T(A) *is a regular space.*

Proof. Let $d \in O(U)$, $U \in U(A)$. We must prove that there exist $U_0 \in U(A)$ such that $d \in O(U_0) \subseteq cl(O(U_0)) \subseteq O(U)$.

Consider the cases: 1) $d = \{(a,S)\}$, $U = (C_t \times S(\overline{\alpha})) \cap J(A)$, $t \subseteq L_{n+1}$ and $\overline{\alpha} \in \Lambda_{k+1}$, 2) $d = \{(a,S)\}$, $U = U(\overline{\gamma},m,k-1)$, $\overline{\gamma} \in \Lambda_q$, $0 \leq m \leq q-1$ and $q \leq k$, 3) $d = d(\overline{\gamma},m)$, $U = U(\overline{\gamma}_1,m_1,k_1-1)$, $\overline{\gamma} \in \Lambda_q$, $0 \leq m \leq q-1$, $\overline{\gamma}_1 \in \Lambda_{q'}$, $0 \leq m_1 \leq q'-1$ and $q' \leq k_1$ and 4) $d = d(\overline{\gamma},m)$, $U = (C_t \times S(\overline{\alpha})) \cap J(A)$, $\overline{\gamma} \in \Lambda_q$, $0 \leq m \leq q-1$, $t \subseteq L_n$ and $\overline{\alpha} \in \Lambda_{k+1}$.

In the first case, since D(S) is an upper semi-continuous partition, there exists an integer $q \geq \max\{k+1,n+1\}$ such that if $d' \in D(S)$ and $d' \cap st(a,q) \neq \emptyset$, then $d' \subseteq C_t$.

Let $\overline{\delta} \in \Lambda_{q+1}$ and $S \in S(\overline{\delta})$. Set $U_1 = (st(a,q) \times S(\overline{\delta})) \cap J(A)$. By the choice of the integer q, $d \subseteq O(U_1) \subseteq O(U)$. We prove that $cl(O(U_1)) \subseteq O(U) \cup F$ where F is a finite set.

Let $d_1 \in Bd(O(U_1))$. By Lemma 8 d_1 is an element of the first kind. Let $d_1 = d(\overline{\delta}_1,m_1)$ where $\overline{\delta}_1 \in \Lambda_{q'}$, and $0 \leq m_1 \leq q' - 1$. By F we denote the set of all such elements d_1 for $q' \leq q + 1$. It is easy to see that the set F is finite.

Now, suppose that $q' \geq q + 2$. We prove that $d_1 \in O(U)$. Indeed, in the opposite case, by Lemma 8 we have that $d_1 \cap U_1 \neq \emptyset$ and $d_1 \cap (J(A) \setminus U) \neq \emptyset$. Let $(a_1,S_1) \in d_1 \cap U_1$ and $(a_2,S_2) \in d_1 \cap (J(A) \setminus U)$.

By property 14) of Lemma 5 (setting $k + 1 = q'$, $m = m_1$ and $\overline{\alpha} = \overline{\delta}_1$) if $m_1 > 0$ we have that $st(d(S_1,\overline{\delta}_1,m_1), n(\overline{\delta}_1)) = st(d(S_2,\overline{\delta}_1,m_1), n(\overline{\delta}_1))$. If $m_1 = 0$, then $d(\overline{\delta}_1,0) = y(\overline{\delta}_1,0)$. Since S_1 and S_2 are $(q'-1,n(\overline{\delta}_1))$-equivalent there exist a subset $t(d_1)$ of $L_{n(\overline{\delta}_1)}$ and an element $z(d_1)$ of $M(S(\overline{\delta}_1), q' - 1, t(d_1))$ such that $h(D(S_1), q' - 1, t(d_1))(z(d_1)) = d(S_1,\overline{\delta}_1,0)$ and $h(D(S_2), q' - 1, t(d_1))(z(d_1)) = d(S_2,\overline{\delta}_1,0)$. This means that $d(S_1,\overline{\delta}_1,0) \in D(S_1)(q' - 1),t(d_1))$ and $d(S_2,\overline{\delta}_1,0) \in D(S_2)(q' - 1,t(d_1))$, that is, $st(d(S_1,\overline{\delta}_1,0), n(\overline{\delta}_1)) = st(d(S_2,\overline{\delta}_1,0),n(\overline{\delta}_1))$. Thus, $st(d(S_1,\overline{\delta}_1,m_1), n(\overline{\delta}_1)) = st(d(S_2,\overline{\delta}_1,m_1),n(\overline{\delta}_1))$ for $0 \leq m \leq q' - 1$. Since $n(\overline{\delta}_1) \geq q' \geq q + 2$ from the above it follows that $d(S_1,\overline{\delta}_1,m_1) \cap st(a,q) \neq \emptyset$ and $d(S_1,\overline{\delta}_1,m_1) \cap (C \setminus C_t) \neq \emptyset$.

Hence, since $t \subseteq L_{n+1}$ we have that $d(S_1,\overline{\delta}_1,m_1) \in D_{k_1}(S_1)$, where $k_1 \leq n$. Since S_1 and S are $(q,n(\overline{\delta}))$-equivalent, there is an element $d' \in D_{k_1}(S)$ such that $st(d',n(\overline{\delta})) =$

$st(d(S_1,\overline{\delta}_1,m_1),n(\overline{\delta}))$. From the above and since $n(\overline{\delta}) \geq q + 1 > n + 1$ it follows that $d' \cap st(a,q) \neq \emptyset$ and $d' \cap (C \setminus C_t) \neq \emptyset$.

By the choice of the set $st(a,q)$ this is impossible. Hence, $d_1 \cap (J(A) \setminus U) = \emptyset$, that is, $d_1 \in O(U)$.

Since the space $T(A)$ is Hausdorff, there is an open neighbourhood $O(U_0)$, $U_0 \in U(A)$ of d_1 such that $O(U_0) \cap F = \emptyset$ and $O(U_0) \subseteq O(U_1)$. Then $d \in O(U_0) \subseteq cl(O(U_0)) \subseteq O(U)$.

In the second case, by the structure of the set $U(\overline{\gamma},m,k-1)$ (see property 13) of Lemma 5) there exists an element $\overline{\alpha} \in \Lambda_{k+1}$ such that $d \subseteq U'$ where $U' = C_{s(\overline{\gamma},m,\overline{\alpha})} \times S(\overline{\alpha}) \subseteq U$. By the first case there exists $U_0 \in U(A)$ such that $d \in O(U_0) \subseteq cl(O(U_0)) \subseteq O(U') \subseteq O(U)$.

Consider the third case. Let k be an integer such that $k > \max\{k_1,q\}$. Then, by property 8) of Lemma 5 we have that $d \subseteq U(\overline{\gamma},m,k) \subseteq U(\overline{\gamma}_1,m_1,k_1-1)$. Set $U_0 = U(\overline{\gamma},m,k)$. We prove that $d \in O(U_0) \subseteq cl(O(U_0)) \subseteq O(\overline{\gamma},m,k-1) \subseteq O(U)$.

Indeed, let $d_1 \in cl(O(U_0)) \setminus O(U_0)$. Then, by Lemma 8, $d_1 \cap U_0 \neq \emptyset$. Obviously, d_1 is an element of the first kind. Let $d_1 = d(\overline{\delta},m')$ where $\overline{\delta} \in \Lambda_{q''}$ and $0 \leq m' \leq q'' - 1$.

If $q'' \leq k + 1$, then by property 9) of Lemma 5, we must have that $U(\overline{\gamma},m,k) \cap U(\overline{\delta},m',k) = \emptyset$ which is impossible.

Hence, $q'' > k + 1$. This means that there exists an element $\overline{\alpha} \in \Lambda_{k+2}$ such that $\overline{\gamma} \leq \overline{\alpha} \leq \overline{\delta}$, that is, $S(\overline{\delta}) \subseteq S(\overline{\alpha}) \subseteq S(\overline{\gamma})$. By the structure of the set $U(\overline{\gamma},m,k)$ (see property 13) of Lemma 5) we have $d_1 \cap U(\overline{\gamma},m,k) = d_1 \cap ((C_{s(\overline{\gamma},m,\overline{\alpha})} \times S(\overline{\alpha})) \cap J(A)) \neq \emptyset$. This means that there is an element S_1 of $S(\overline{\delta})$ such that $d(S_1,\overline{\delta},m') \cap C_{s(\overline{\gamma},m,\overline{\alpha})} \neq \emptyset$. We prove that this is true for every $S \in S(\overline{\delta})$.

Indeed, let $S \in S(\overline{\delta})$. By property 5) of Lemma 5 we have $d(\overline{\delta},m') = y(\overline{\delta}',m') \cap J(S(\overline{\delta}))$, where $\overline{\delta}' \in \Lambda_{q''-m'}$, and $\overline{\delta}' \leq \overline{\delta}$. Obviously, $y(S,\overline{\delta}',m') = d(S,\overline{\delta},m')$. If $m' \neq 0$, then by property 14) of Lemma 5 $st(y(S_1\overline{\delta}',m'),n(\overline{\delta})) = st(y(S,\overline{\delta}',m'),n(\overline{\delta}))$, that is, $st(d(S_1,\overline{\delta},m'),n(\overline{\delta})) = st(d(S,\overline{\delta},m'),n(\overline{\delta}))$. Since $n(\overline{\delta}) \geq n(\overline{\alpha})$ it follows that $d(S,\overline{\delta},m') \cap C_{s(\overline{\gamma},m,\overline{\alpha})} \neq \emptyset$.

Suppose now that $m' = 0$. Then $d(\overline{\delta},0) = y(\overline{\delta},0)$, that is, $d(\overline{\delta},0)$ is a D-set of $S(\overline{\delta})$ with respect to $(q''-1,n(\overline{\delta}))$-equivalence (see property 5) of Lemma 5). Hence, there exist a subset $t(d_1)$ of $L_{n(\overline{\delta})}$ and an element $z(d_1)$ of $M(S(\overline{\delta}),q'' - 1,t(d_1))$ such that $h(D(S_1),q'' - 1,t(d_1))(z(d_1)) = d(S_1,\overline{\delta},0)$ and $h(D(S),q'' - 1,t(d_1))(z(d_1)) = d(S,\overline{\delta},0)$. This means that $d(S_1,\overline{\delta},0) \in D(S_1)(q'' - 1,t(d_1))$ and $d(S,\overline{\delta},0) \in D(S)(q'' - 1,t(d_1))$, that is, $st(d(S_1,\overline{\delta},m'), n(\overline{\delta})) = st(d(S,\overline{\delta},m),n(\overline{\delta})) = C_{t(d_1)}$.

As above, since $n(\overline{\delta}) \geq n(\overline{\alpha})$ and $d(S_1,\overline{\delta},0) \cap C_{s(\overline{\gamma},m,\overline{\alpha})} \neq \emptyset$ we have that $d(S,\overline{\delta},0) \cap C_{s(\overline{\gamma},m,\overline{\alpha})} \neq \emptyset$.

Thus, for all cases, for every $S \in S(\overline{\delta})$ we have $d(S,\overline{\delta},m') \cap C_{s(\overline{\gamma},m,\overline{\alpha})} \neq \emptyset$. By property 10) of Lemma 5, we have that $d(S,\overline{\delta},m') \times \{S\} \subseteq U(\overline{\gamma},m,k-1)$ for every $S \in S(\overline{\delta})$ and, hence, $d(\overline{\delta},m') \subseteq U(\overline{\gamma},m,k-1)$, that is, $d_1 = d(\overline{\delta},m') \in O(\overline{\gamma},m,k-1) \subseteq O(U)$. Hence, $d \in O(U_0) \subseteq cl(O(U_0)) \subseteq O(U)$.

Finally, consider the fourth case. There exists an integer $k_1 \geq q$ such that $d \subseteq U(\overline{\gamma},m,k_1) \subseteq (C_t \times S(\overline{\alpha})) \cap J(A)$ (see the proof of Lemma 6). By the third case, there exists $U_0 \in U(A)$ such that $d \in O(U_0) \subseteq cl(O(U_0)) \subseteq O(U)$. The proof of the lemma is completed.

8. *Lemma* 10. *For every* $O \in \mathcal{O}(A)$, type(Bd(O)) $\leq \alpha$.

Proof. For every element $d(\bar{\alpha},m)$, where $\bar{\alpha} \in \Lambda_{k+1}$, $0 \leq m \leq k$, by type$(d(\bar{\alpha},m))$ *(resp.* deg$(d(\bar{\alpha},m))$*)* we denote the ordinal number type$(y(\bar{\gamma},m))$ (resp. the integer deg$(y(\bar{\alpha},m))$, where $\bar{\gamma} \in \Lambda_{k+1-m}$, $\bar{\gamma} \leq \bar{\alpha}$ and $d(\bar{\alpha},m) = y(\bar{\gamma},m) \cap J(S(\bar{\alpha}))$).

Since D(S) is an $M(\alpha)$-partition, for every $S \in S(A)$ we have that type$(d(\bar{\alpha},m) \leq \alpha$ for every $d(\bar{\alpha},m)$.

Let $O \in \mathcal{O}(A)$. Consider two cases: 1) $O = O(U)$, where $U = (C_t \times S(\bar{\beta})) \cap J(A)$, $t \subseteq L_n$, $\bar{\beta} \in \Lambda_q$, $n = 0,1,\cdots$, $q = 1,2,\cdots$ and 2) $O = O(U)$, where $U = U(\bar{\gamma},m,k-1)$, $\bar{\gamma} \in \Lambda_q$, $0 \leq m \leq q - 1$, $q \leq k$.

Consider the first case. Let $d(\bar{\alpha},m) \in$ Bd(O), where $\bar{\alpha} \in \Lambda_{q'}$, $0 \leq m \leq q' - 1$. Let $k \geq \max\{q',n,q\}$. We prove that if $d_1 = d(\bar{\alpha}_1,m_1) \in O(\bar{\alpha},m,k-1) \cap$ Bd(O), where $\bar{\alpha}_1 \in \Lambda_{q''}$, $0 \leq m_1 \leq q'' - 1$ and $d(\bar{\alpha}_1,m_1) \neq d(\bar{\alpha},m)$, then type$(d(\bar{\alpha}_1,m_1))$ $<$ type$(d(\bar{\alpha},m))$.

Indeed, let $\bar{\gamma} \in \Lambda_{q'-m}$, $\bar{\gamma} \leq \bar{\alpha}$, $\bar{\gamma}_1 \in \Lambda_{q''-m_1}$, $\bar{\gamma}_1 \leq \bar{\alpha}_1$, $d(\bar{\alpha},m) = y(\bar{\gamma},m) \cap J(S(\bar{\alpha}))$, $d(\bar{\alpha}_1,m_1) = y(\bar{\gamma}_1,m_1) \cap J(S(\bar{\alpha}_1))$ and $S \in S(\bar{\alpha}_1) \cap S(\bar{\alpha})$. By the above definition of the type of the elements of T(A) of the first kind we have that type$(d(\bar{\alpha}_1,m_1))$ = type$(y(\bar{\gamma}_1,m_1))$ and type$(d(\bar{\alpha},m))$ = type$(y(\bar{\gamma},m))$.

On the other hand, type$(y(\bar{\gamma}_1,m_1)$ = type$(d(S,\bar{\alpha}_1,m_1)$, $D_{q''-m_1-1}(S))$ and type$(y(\bar{\gamma},m))$ = type$(d(S,\bar{\alpha},m)$, $D_{q'-m-1}(S))$. Since $d(\bar{\alpha}_1,m_1) \subseteq U(\bar{\alpha},m,k-1)$ we have that $q'' \geq k + 1$, because in the opposite case $U(\bar{\alpha}_1,m_1,k-1) \cap U(\bar{\alpha},m,k-1) = \emptyset$ which is impossible. Hence, there exists an element $\bar{\delta} \in \Lambda_{k+1}$ such that $\bar{\alpha} \leq \bar{\delta} \leq \bar{\alpha}_1$. Let $S \in S(\bar{\alpha}_1)$. Then, by the structure of the set $U(\overline{\alpha},m,k-1)$ (see property 13) of Lemma 5) we have that $d(S,\overline{\alpha}_1,m_1) \subseteq st(d(S,\overline{\alpha},m),n(\overline{\delta}))$. By property 11) of Lemma 5, either $deg(d(S,\overline{\alpha}_1,m_1)) > k$, or $deg(d(S,\overline{\alpha}_1,m_1) = q' - m - 1$ and $type(d(S,\overline{\alpha}_1,m_1),D_{q'-m-1}(S)) < type(d(S,\overline{\alpha},m), D_{q'-m-1}(S))$.

Since $d(\overline{\alpha}_1,m_1) \in Bd(O)$, $S(\overline{\alpha}_1) \cap S(\overline{\beta}) \neq \emptyset$. Since $q'' \geq k + 1 > q$, $S(\overline{\alpha}_1) \subseteq S(\overline{\beta})$. Let $(a_1,S_1) \in d(\overline{\alpha}_1,m_1) \cap U$. Then $d(S_1,\overline{\alpha}_1,m_1) \cap C_t \neq \emptyset$.

We prove that if $d(S_1,\overline{\alpha}_1,m_1) \subseteq C_t$, then for every $S \in S(\overline{\alpha}_1)$ we have that $d(S,\overline{\alpha}_1,m_1) \subseteq C_t$.

Indeed, let $d(S_1,\overline{\alpha}_1,m_1) \subseteq C_t$ and $S \in S(\overline{\alpha}_1)$. If $m_1 > 0$, then by property 14) of Lemma 5 $st(d(S_1,\overline{\alpha}_1,m_1),n(\overline{\alpha}_1)) = st(d(S,\overline{\alpha}_1,m_1),n(\overline{\alpha}_1))$. Since $n(\overline{\alpha}_1) \geq q'' + 1 \geq k + 2 \geq n + 2$ we have that $st(d(S,\overline{\alpha}_1,m_1),n(\overline{\alpha}_1)) \subseteq C_t$ and, hence, $d(S,\overline{\alpha}_1,m_1) \subseteq C_t$.

If $m_1 = 0$, then the set $d(\overline{\alpha}_1,0) = y(\overline{\alpha}_1,0)$ is a D-set of $S(\overline{\alpha}_1)$ with respect to $(q''-1,n(\overline{\alpha}_1))$-equivalence. This means that there is a subset $t(d_1)$ of $L_{n(\overline{\alpha}_1)}$ and an element $z(d_1)$ of $M(S(\overline{\alpha}_1),q'' - 1,t(d_1))$ such that $h(D(S_1),q'' - 1, t(d_1))(z(d_1)) \in D(q'' - 1,t(d_1))$ and $h(D(S),q'' - 1,t(d_1))(z(d_1)) \in D(q'' - 1,t(d_1))$, that is, $st(d(S_1,\overline{\alpha}_1,0),n(\overline{\alpha}_1)) = st(d(S,\overline{\alpha}_1,0),n(\overline{\alpha}_1)) = C_{t(d_1)}$. Since $n(\overline{\alpha}_1) \geq n + 2$ and $d(S_1,\overline{\alpha}_1,0) \subseteq C_t$ we have that $d(S,\overline{\alpha}_1,0) \subseteq C_t$.

Hence, for all $S \in S(\overline{\alpha}_1)$ we have that $d(S,\overline{\alpha}_1,m_1) \subseteq C_t$, that is, $d(\overline{\alpha}_1,m_1) \subseteq (C_t \times S(\overline{\beta})) \cap J(A)$. This means that $d(\overline{\alpha}_1,m_1) \not\in Bd(O)$ which is impossible.

Thus, $d(S_1,\overline{\alpha}_1,m_1) \cap C_t \neq \emptyset$ and $d(S_1,\overline{\alpha}_1,m_1) \not\subseteq C_t$. This means that $deg(d(S_1,\overline{\alpha}_1,m_1)) = deg(d(\overline{\alpha}_1,m_1)) \leq n - 1 < k$.

Hence, $\deg(d(S_1,\bar{\alpha}_1,m_1)) = q' - m - 1 = q'' - m_1 - 1$ and $\text{type}(d(\bar{\alpha}_1,m_1)) < \text{type}(d(\bar{\alpha},m))$.

From the above it follows that if $\text{type}(d(\bar{\alpha},m)) = 1$, then $O(\bar{\alpha},m,k-1) \cap \text{Bd}(O) = \{d(\bar{\alpha},m)\}$, that is, the point $d(\bar{\alpha},m)$ is an isolated point of $\text{Bd}(O)$. Hence, $\text{type}(d(\bar{\alpha},m), \text{Bd}(O)) = 1$, that is, $\text{type}(d(\bar{\alpha},m),\text{Bd}(O)) \leq \text{type}(d(\bar{\alpha},m))$. By induction, it follows that for every point $d(\alpha,m)$ of $\text{Bd}(O)$ we have $\text{type}(d(\bar{\alpha},m),\text{Bd}(O)) \leq \text{type}(d(\bar{\alpha},m))$. Hence, $\text{type}(\text{Bd}(O)) \leq \alpha$.

We now consider the second case. We prove that the set $\text{Bd}(O)$ is the free union of the sets $\text{Bd}(O(C_{s(\bar{\gamma},m,\bar{\alpha})}, S(\bar{\alpha})))$, $\bar{\alpha} \geq \bar{\gamma}$, $\bar{\alpha} \in \Lambda_{k+1}$.

Indeed, let $d(\bar{\beta},m_1) \in \text{Bd}(O)$ where $\bar{\beta} \in \Lambda_{q'}$, and $0 \leq m_1 \leq q' - 1$. If $q' \leq k$, then by property 9) of Lemma 5, $O(\bar{\beta},m_1,k-1) \cap O(\bar{\gamma},m,k-1) = \emptyset$, that is, $d(\bar{\beta},m_1) \notin \text{Bd}(O)$. Hence, $q' > k$. Since $q' > q$ and $d(\bar{\beta},m_1) \cap U(\bar{\gamma},m,k-1) \neq \emptyset$ it follows that there exists an element $\bar{\alpha}$ of Λ_{k+1} such that $\bar{\alpha} \leq \bar{\beta}$, that is, $S(\bar{\beta}) \subseteq S(\bar{\alpha})$. Hence, $d(\bar{\beta},m_1) \cap U(\bar{\gamma},m,k-1) = d(\bar{\beta},m_1) \cap ((C_{s(\bar{\gamma},m,\bar{\alpha})} \times S(\bar{\alpha})) \cap J(A))$ and $d(\bar{\beta},m_1) \cap (J(A)\setminus U(\bar{\gamma},m,k-1)) \subseteq d(\bar{\beta},m_1) \cap (J(A)\setminus(C_{s(\bar{\gamma},m,\bar{\alpha})} \times S(\bar{\alpha})))$. This means that $d(\bar{\beta},m_1) \in \text{Bd}(O(C_{s(\bar{\gamma},m,\bar{\alpha})},S(\bar{\alpha}))$.

Conversely, let $d(\bar{\beta},m_1) \in \text{Bd}(O(C_{s(\bar{\gamma},m,\bar{\alpha})},S(\bar{\alpha})))$, where $\bar{\beta} \in \Lambda_{q'}$, $0 \leq m_1 \leq q' - 1$, $\bar{\alpha} \in \Lambda_{k+1}$ and $\bar{\gamma} \leq \bar{\alpha}$. Then $d(\bar{\beta},m_1) \cap ((C_{s(\bar{\gamma},m,\bar{\alpha})} \times S(\bar{\alpha})) \cap J(A)) \neq \emptyset$ (see Lemma 8), that is, $d(\bar{\beta},m_1) \cap U(\bar{\gamma},m,k-1) \neq \emptyset$. If $q' \leq k$, then $U(\bar{\beta},m_1,k-1) \cap U(\bar{\gamma},m,k-1) = \emptyset$ which is impossible. Hence, $q' > k$. There exists an element $\bar{\alpha} \in \Lambda_{k+1}$ such that $\bar{\beta} \geq \bar{\alpha}$, that is, $S(\bar{\beta}) \subseteq S(\bar{\alpha})$. From this it follows that

$d(\overline{\beta},m_1) \cap (J(A)\smallsetminus(C_{s(\overline{\gamma},m,\overline{\alpha})} \times S(\overline{\alpha}))) \not\subseteq U(\overline{\gamma},m,k-1)$ and, hence, $d(\overline{\beta},m_1) \cap (J(A)\smallsetminus U(\overline{\gamma},m,k-1)) \neq \emptyset$. This means that $d(\overline{\beta},m_1) \in Bd(O)$. Moreover, if $\overline{\alpha}_1 \in \Lambda_{k+1}$ and $\overline{\alpha}_1 \neq \overline{\alpha}$, then $S(\overline{\beta}) \cap S(\overline{\alpha}_1) = \emptyset$. Hence, $d(\overline{\beta},m_1) \cap (C_{s(\overline{\gamma},m,\overline{\alpha}_1)} \times S(\overline{\alpha}_1)) = \emptyset$ that is, $d(\overline{\beta},m_1) \notin Bd(O(C_{s(\overline{\gamma},m,\overline{\alpha}_1)},S(\overline{\alpha}_1)))$.

Hence, $Bd(O(C_{s(\overline{\gamma},m,\overline{\alpha}_1)},S(\overline{\alpha}_1))) \cap Bd(O(C_{s(\overline{\gamma},m,\overline{\alpha}_2)}, S(\overline{\alpha}_2))) = \emptyset$ for every $\overline{\alpha}_1,\overline{\alpha}_2 \in \Lambda_{k+1}$, $\overline{\alpha}_1 \neq \overline{\alpha}_2$.

Also, the sets $O(C,S(\overline{\alpha}_1))$ and $O(C,S(\overline{\alpha}_2))$ are open neighbourhoods of the sets $Bd(O(C_{s(\overline{\gamma},m,\overline{\alpha}_1)},S(\overline{\alpha}_1)))$ and $Bd(O(C_{s(\overline{\gamma},m,\overline{\alpha}_2)},S(\overline{\alpha}_2)))$, respectively, with empty intersection.

Thus, $Bd(O)$ is the free union of the closed sets $Bd(O(C_{s(\overline{\gamma},m,\overline{\alpha})},S(\overline{\alpha})))$, $\overline{\alpha} \geq \overline{\gamma}$, $\overline{\alpha} \in \Lambda_{k+1}$. By the first case, for every $\overline{\alpha} \geq \overline{\gamma}$, $\overline{\alpha} \in \Lambda_{k+1}$, we have that $\text{type}(Bd(O(C_{s(\overline{\gamma},m,\overline{\alpha})}, S(\overline{\alpha})))) \leq \alpha$.

Hence, $\text{type}(Bd(O)) \leq \alpha$. The proof of the lemma is completed.

9. *Corollary. The space* T(A) *is metrizable with a countable basis space having type* $\leq \alpha$.

III. 1. By $R(\leq\alpha)$ we denote the set of all spaces having rim-type $\leq \alpha$.

A pair (S,D) where S is a subset of the Cantor ternary set C and D is a partition of S is called a *representation* of a space X iff the quotient space D is homeomorphic to the space X.

Let $\alpha = \beta + n$ where β is a limit ordinal number and n is a non-negative integer.

Let $M(\alpha)$ be a countable set of spaces having the property: a space X is homeomorphic to an element of $M(\alpha)$ iff, either X is a compact space and type(X) < β, or X is an element of the set $G(\beta,m)$, where $0 \leq m \leq n$. We consider that two different elements of $M(\alpha)$ are not homeomorphic. The existence of such set follows by Lemma 2.

A representation (S,D) of a space X is called a $M(\alpha)$-*representation* iff the partition D of S is a $M(\alpha)$-partition.

Lemma 11. *Let* $X \in R(\leq\alpha)$. *There exists an element* $\hat{X}$ *of* $R(\leq\alpha)$ *containing* X *topologically and having an* $M(\alpha)$-*representation*.

Proof. Let $X \in R(\leq\alpha)$. Let Z be an extension of X and $B(Z) = \{T_1, T_2, \cdots\}$ be a basis of open sets of Z having all the properties of Lemma 4.

As in section I.5 of $[I_4]$ we set $A_0^i = cl(T_i)$ and $A_1^i = Z \setminus T_i$, $i = 1,2,\cdots$. For every $i_1 \cdots i_m \in L_m$, $m = 1,2,\cdots$ we set $Z_{i_1 \cdots i_m} = A_{i_1}^1 \cap \cdots \cap A_{i_m}^m$. We define a subset S(Z) of C and a map q(Z) of S(Z) into Z. The point a of C belongs to S(Z) if and only if $Z_{\bar{I}(a,1)} \cap Z_{\bar{I}(a,2)} \cap \cdots \neq \emptyset$. Obviously, for every point $a \in S(Z)$, the set $Z_{\bar{I}(a,1)} \cap Z_{\bar{I}(a,2)} \cap \cdots$ is a singleton. If $\{x\} = Z_{\bar{I}(a,1)} \cap Z_{\bar{I}(a,2)} \cap \cdots$, then we set $q(Z)(a) = x$.

Set $D(Z) = \{(q(Z))^{-1}(x) : x \in Z\}$. Obviously, D(Z) is a partition of S(Z). Also, set $Bd(B(Z)) = \cup_{i=1}^{\infty} Bd(T_i)$.

The set S(Z), the map q(Z) and the partition D(Z) have the following properties (see section I.5 and Lemma 2 of $[I_4]$):

1) $q(Z)(C_{\bar{i}} \cap S(Z)) = Z_{\bar{i}}$, $\bar{i} \in L$,

2) if $x \notin Bd(B(Z))$ then $(q(Z))^{-1}(x)$ consists of one point only,

3) if $x \in Bd(B(Z))$ then $(q(Z))^{-1}(x)$ consists of exactly two points,

4) the map $q(Z)$ is continuous,

5) the map $q(Z)$ is closed,

6) the partition $D(Z)$ is an upper semi-continuous partition of $S(Z)$.

Let $p(Z)$ be the natural projection of $S(Z)$ onto $D(Z)$. Since $D(Z)$ is upper semi-continuous, the projection $p(Z)$ is closed.

Let $i(Z)$ be the map of Z onto $D(Z)$ for which $p(Z) = i(Z) \circ q(Z)$. Obviously, the map $i(Z)$ is one-to-one. Since both maps $p(Z)$ and $q(Z)$ are continuous and closed, the map $i(Z)$ is also continuous and closed. Hence $i(Z)$ is a homeomorphism of Z onto $D(Z)$.

Set $\hat{X} = X \cup (\cup_{i=1}^{\infty}(Bd(T_i) \setminus (Bd(T_i))^{(\beta)}))$. By Lemma 4, $\hat{X}$ is an element of $R(\leq\alpha)$.

Also, set $S(\hat{X}) = (q(Z))^{-1}(\hat{X})$, $D(\hat{X}) = i(Z)(\hat{X})$. Obviously, $(p(Z))^{-1}(D(\hat{X})) = S(\hat{X})$. Let $i(\hat{X}) = i(Z)|_{\hat{X}}$, $q(\hat{X}) = q(Z)|_{S(\hat{X})}$ and $p(\hat{X}) = p(Z)|_{S(\hat{X})}$. Obviously, $p(\hat{X}) = i(\hat{X}) \circ q(\hat{X})$. It is clear that the map $p(\hat{X})$ is continuous and closed. Hence, $D(\hat{X})$ is an upper semi-continuous partition and the space $D(\hat{X})$ has the quotient topology. Since $i(\hat{X})$ is a homeomorphism of $\hat{X}$ onto $D(\hat{X})$, the pair $(S(\hat{X}),D(\hat{X}))$ is a representation of $\hat{X}$.

We prove that $(S(\hat{X}),D(\hat{X}))$ is an $M(\alpha)$-representation of $\hat{X}$. By the above properties 2) and 3) of the map $q(Z)$

it follows that every element of $D(\hat{X})$ is a singleton, or it consists of exactly two points.

Hence, in order to prove that $(S(\hat{X}),D(\hat{X}))$ is a $M(\alpha)$-representation it is sufficient to prove that for every integer $k \geq 0$ and for every subset t of L_m, $m = 0,1,2,\cdots$ the subset $D(\hat{X})(k,t)$ is homeomorphic to an element of the set $M(\alpha)$.

Let $k \geq 0$ be an integer and let t be a subset of the set L_m, $m \geq 0$. Consider the set $D(\hat{X})(k,t)$. Obviously, $D(\hat{X})(k,t) = D(Z)(k,t) \cap D(\hat{X})$.

Let $d \in D(Z)(k,t)$. Then, $d = (q(Z))^{-1}(x)$ for some $x \in Z$. By the definition of the map $i(Z)$, $(i(Z))^{-1}(d) = x$.

By the construction of the set $S(Z)$ it follows that $d \in D(Z)_{k-1}$ if and only if $x \in Bd(T_k)$. On the other hand, $d \cap C_{\bar{i}} \neq \emptyset$ if and only if $x \in Z_{\bar{i}}$ (see property 1) of the set $S(Z)$ and the map $q(Z)$). Hence, $(i(Z))^{-1}(D(Z)(k-1,t)) = (\cap_{\bar{i}\in t} Z_{\bar{i}}) \cap Bd(T_k)$. Set $Bd_t(T_k) = (\cap_{\bar{i}\in t} Z_{\bar{i}}) \cap Bd(T_k)$.

Since $Bd(T_i) \cap Bd(T_j) = \emptyset$ if $i \neq j$ we have that the set $Bd_t(T_k)$ is an open and closed subset of the set $Bd(T_k)$.

We have $(i(\hat{X}))^{-1}(D(\hat{X})(k-1,t)) = Bd_t(T_k) \cap \hat{X}$. Hence, if $\text{type}(Bd_t(T_k)) < \beta$, then $Bd_t(T_k) \subseteq \hat{X}$. This means that $D(\hat{X})(k-1,t)$ is homeomorphic to $Bd_t(T_k)$ and, hence, it is homeomorphic to an element of $M(\alpha)$, because $Bd_t(T_k)$, as a compact space having type $< \beta$, is homeomorphic to an element of $M(\alpha)$.

Suppose that $\text{type}(Bd_t(T_k)) \geq \beta$. Then, $Bd_t(T_k) \setminus (Bd_t(T_k))^{(\beta)} \subseteq \hat{X}$ and since $Bd_t(T_k) \cap \hat{X} \subseteq Bd(T_k) \cap \hat{X} = (X \cap (Bd(T_k))) \cup (Bd(T_k) \setminus (Bd(T_k)^{(\beta)}))$ (we have) $\text{type}(Bd_t(T_k) \cap \hat{X}) \leq \beta + n$. The above means that the set $Bd_t(T_k) \cap \hat{X}$ and, hence, $D(\hat{X})(k-1,t)$ is homeomorphic to an element of $M(\alpha)$. The proof of the lemma is completed.

2. A representation $(S(X),D(X))$ of a space X is called *complete* if the set $C\setminus(\cup_{k=0}^{\infty}cl(D_k^*(\hat{X})))$ is a subset of the set $S(X)$.

Lemma 12. *Let* $X \in R(\leq\alpha)$. *There exists an element* X^* *of* $R(\leq\alpha)$ *containing topologically the space* X *and having a complete* $M(\alpha)$*-representation.*

Proof. Let X be a space and $(S(\hat{X}),D(\hat{X}))$ be the $M(\alpha)$-representation of $\hat{X}$ constructed in Lemma 11.

Set $S_c(\hat{X}) = S(\hat{X}) \cup (C\setminus(\cup_{k=0}^{\infty}cl(D_k^*(\hat{X}))))$. By $D_c(\hat{X})$ we denote a partition of the set $S_c(\hat{X})$ which is defined as follows: 1) every element of the set $D(\hat{X})(1)$ is an element of the partition $D_c(\hat{X})$ and 2) if $x \in S_c(\hat{X})$ and x does not belong to any element of $D(\hat{X})(1)$ then the singleton $\{x\}$ is an element of $D_c(\hat{X})$.

Since $D_c(\hat{X})(1) = D(\hat{X})(1)$ the partition $D_c(\hat{X})$ is an $M(\alpha)$-partition of the set $S_c(\hat{X})$. Hence, by the construction of the set $S_c(\hat{X})$, the pair $(S_c(\hat{X}),D_c(\hat{X}))$ is a complete $M(\alpha)$-representation of the quotient space $D_c(\hat{X})$. Obviously, the space $D(\hat{X})$ and, hence, the spaces $\hat{X}$ and X are homeomorphic to a subset of the space $D_c(\hat{X})$. Setting $X^* = D_c(\hat{X})$ we complete the proof of the lemma.

3. *Theorem. In the family* $R(\leq\alpha)$ *there exists a universal element* T *having the property of finite intersection with respect to a given subfamily* R_1 *of* $R(\leq\alpha)$*, where the power of* R_1 *is less than or equal to the continuum.*

Proof. Let $X \in R(\leq\alpha)$. By Lemma 12 there exists a space $X^* \in R(\leq\alpha)$ containing topologically the space X and having a complete $M(\alpha)$-representation $(S(X^*),D(X^*))$.

By A' we denote the set of all pairs $(S(X^*),D(X^*))$ where $X \in R(\leq\alpha)\setminus R_1$. We consider that if $(S(X_1^*),D(X_1^*))$ and $(S(X_2^*),D(X_2^*))$ are different elements of A' then, either $S(X_1^*) \neq S(X_2^*)$, or $D(X_1^*) \neq D(X_2^*)$. We observe that for every element $(S(X^*),D(X^*))$ the set $D(X^*)(1)$ uniquely determines the set $S(X^*)$. Since the set $D(X^*)(1)$ is uniquely determined by a sequence of countable subsets of $C \times C$ we have that the power of A' is less than or equal to the continuum.

By A_1 we denote the set of all pairs $(S(X^*),D(X^*))$ where $X \in R_1$. We consider that if X_1 and X_2 are different elements of R_1 then $(S(X_1^*),D(X_1^*))$ and $(S(X_2^*),D(X_2^*))$ are different elements of A_1.

Let A be the free union of the sets A' and A_1. Then, the power of A is less than or equal to the continuum.

Set $T = T(A)$ where $T(A)$ is the space constructed in section II for the set A. By the corollary of section II.9, T is a metrizable space with a countable basis having rim-type $\leq \alpha$, that is, T is an element of $R(\leq\alpha)$.

Let $(S(X^*),D(X^*))$ be an element of A. Consider the subset $T(S(X^*))$ of $T(A)$ which consists of the points d of $T(A)$ for which $d \cap (C \times \{S(X^*)\}) \neq \emptyset$.

We prove that the subset $T(S(X^*))$ of $T(A)$ is homeomorphic to the quotient space $D(X^*)$.

We observe that by the construction of the elements of $T(A)$ (see Lemma 5) it follows that if $d \in T(S(X^*))$, then the subset d' of S(X*) for which $d' \times \{S(X^*)\} = d \cap (C \times \{S(X^*)\})$ is an element of D(X*).

Conversely, if d' is an element of D(X*), then there exists an element $d \in T(S(X^*))$ such that $d' \times \{S(X^*)\} = d \cap (C \times \{S(X^*)\})$.

We define a map h(X*) of T(S(X*)) into D(X*) setting $h(X^*)(d) = d'$. Obviously, the map h(X*) is one-to-one and "onto."

Now, we prove that H(X*) is a homeomorphism of T(S(X*)) onto D(X*). Indeed, let $h(X^*)(d) = d'$ and let V be an open neighbourhood of d' in D(X*). Since D(X*) is an upper semi-continuous partition of S(X*) we may assume that there is an open subset V' of S(X*) such that the set V is the set of all elements of D(X*) which have a non-empty intersection with V'. There exists an integer $m \geq 0$ such that $st(d',m) \subseteq V'$. Obviously, there is a subset t of L_m for which $C_t = st(d',m)$. Consider the set $O(C_t,S(A))$. This is an open subset of T(A) which contains the element d. Set $W = T(S(X^*)) \cap O(C_t,S(A))$. The set W is an open neighbourhood of d in the space T(S(X*)).

We prove that $h(X^*)(W) \subseteq V$. Indeed, let $d_1 \in W$. Then, $d_1 \subseteq C_t \times S(A)$. Hence, if $h(X^*)(d_1) = d_1'$, then $d_1' \subseteq C_t$ and, hence, $d_1' \subseteq V'$, that is, $d_1' \in V$. Thus, $h(X^*)(W) \subseteq V$ and, hence, h(X*) is a continuous map.

Conversely, let W be an open neighbourhood of d in T(S(X*)) and $d = (h(X^*))^{-1}(d')$. We may assume that $W = T(S(X^*)) \cap O$, where O is an element of O(A) (see section II.4). There is a neighbourhood $O_1 \in O(A)$ of d in T such that $d \in O_1 \subseteq cl(O_1) \subseteq O$ (see Lemma 9). Let $O_1 = O(U)$, where $U \in U(A)$. By V' we denote the open subset of C for which $V' \times \{S(X^*)\} = U \cup (C \times \{S(X^*)\}$. Obviously, $d' \subseteq V'$. By V we denote the open subset of $D(X^*)$ which consists of elements d_1' of $D(X^*)$ such that $d_1' \subseteq V'$. Obviously, V is an open neighbourhood of d in $D(X^*)$.

We prove that $(h(X^*))^{-1}(V) \subseteq W$. Indeed, let $d_1' \in V$. Then, $d_1' \subseteq V'$. Hence, $d_1' \times \{S(X^*)\} \subseteq U$. This means that if $d_1 = (h(X^*))^{-1}(d_1')$ then $d_1 \cap U \neq \emptyset$. By Lemma 8, $d_1 \in cl(O(U))$ and, hence, $d_1 \in O$. Since $d_1 \in T(S(X^*))$, $d_1 \in W$. Thus, $(h(X^*))^{-1}(V) \subseteq W$ and the map $(h(X^*))^{-1}$ is continuous. Hence, the map $h(X^*)$ of $T(S(X^*))$ onto $D(X^*)$ is a homeomorphism.

Let $(S(X_1^*),D(X_1^*))$ and $(S(X_2^*),D(X_2^*))$ be two different elements of A. We prove that the set $T(S(X_1^*)) \cap T(S(X_2^*))$ is finite.

Indeed, the elements $S(X_1^*)$ and $S(X_2^*)$ of $S(A)$ are different. Hence, there exist an integer $k \geq 0$ and two different elements $\overline{\alpha}_1$ and $\overline{\alpha}_2$ of the set Λ_{k+1} such that $S(X_1^*) \in S(\overline{\alpha}_1)$ and $S(X_2^*) \in S(\overline{\alpha}_2)$.

Suppose that $d \in T(S(X_1^*)) \cap T(S(X_2^*))$. Obviously, d is an element of the first kind. Let $d = d(\overline{\alpha},m)$, $\overline{\alpha} \in \Lambda_q$ and $0 \leq m \leq q - 1$. If $q \geq k + 1$ then, either $\overline{\alpha} \not\geq \overline{\alpha}_1$ or $\overline{\alpha} \not\geq \overline{\alpha}_2$. This means that, either $d \cap (C \times S(X_1^*)) = \emptyset$, or $d \cap (C \times S(X_2^*)) = \emptyset$. In both cases, $d \notin T(S(X_1^*)) \cap T(S(X_2^*))$.

Hence, $q \leq k + 1$. Obviously, the set of all points $d(\overline{\alpha},m)$ of T for which $\overline{\alpha} \in \Lambda_q$, $0 \leq m \leq q - 1$ and $q \leq k + 1$ is finite. Hence, the set $T(S(X_1^*)) \cap T(S(X_2^*))$ is finite, too.

Now, let X be an element of $R(\leq\alpha)$. By $g(X)$ we denote a homeomorphism of X into $D(X^*)$. Set $i_X = (h(X^*))^{-1} \circ g(X)$. Obviously, i_X is a homeomorphism of X into $T(S(X_1^*)) \subseteq T$. Hence, the space T is a universal element of the family $R(\leq\alpha)$.

Let X_1 and X_2 be two different elements of R_1. Then, $S(X_1^*)$ and $S(X_2^*)$ are different elements of $S(A)$. Hence, the set $T(S(X_1^*)) \cap T(S(X_2^*))$ is finite. Then, the set $i_{X_1}(X_1) \cap i_{X_2}(X_2)$ is finite, too. Thus, the space T has the property of finite intersection with respect to subfamily R_1 of $R(\leq\alpha)$. The proof of the theorem is completed.

Corollary 1. *In the family* $R(\leq\alpha)$ *there exists a universal element having the property of finite intersection with respect to the subfamily* $R^{com}(\leq\alpha)$ *of all compact spaces having rim-type* $\leq \alpha$.

Using Theorem 8 of [I-T] and Theorem 3 of [I_4] we have

Corollary 2. *Let* $\alpha = \beta + n$ *where* β *is a limit ordinal number (or* 0*) and* n *is a non-negative integer. There exists a continuum of rim-type* $\leq \beta + 2n + \min\{\alpha, 1\}$ *having the property of finite intersection with respect to the family of all compact spaces of rim-type* $\leq \alpha$.

In particular,

Corollary 3. *There exists a continuum of rim-type* 2 *having the property of finite intersection with respect to the family of all rim-finite compact spaces.*

Corollary 4. *For a given space* X *of rim-type* $\leq \alpha$ *there exists a space of rim-type* $\leq \alpha$ *having the property of finite intersection with respect to the family of all closed subsets of* X.

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University of Patras

Patras, Greece