TOPOLOGY PROCEEDINGS

Volume 11, 1986

Pages 281–290

http://topology.auburn.edu/tp/

THE FUNDAMENTAL GROUP AND WEAKLY CONFLUENT MAPPINGS OF *ANR*s, II

by

J. Grispolakis and E. D. Tymchatyn

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

THE FUNDAMENTAL GROUP AND WEAKLY CONFLUENT MAPPINGS OF ANRs, II

J. Grispolakis and E. D. Tymchatyn[1]

1. Introduction

In [7] Smale proved the following:

1.1 *Theorem* (Smale). *Let* $f: (X,x_0) \to (Y,y_0)$ *be a proper open mapping of a locally arcwise connected Hausdorff space* X *onto an* LC^1 *metric space* Y. *Then* $f_{\#}\pi(X,x_0)$ *has finite index in* $\pi(Y,y_0)$.

In [2] Grispolakis extended Smale's result to confluent mappings and in [3] he obtained some partial results for weakly confluent mappings.

Since the mapping $f: I \to S^1$ defined by $f(x) = e^{4\pi ix}$ is weakly confluent, it is clear that some additional hypotheses are required for Smale's theorem to be true for weakly confluent mappings. In Section 3, we give our extension of Smale's theorem, which is the main result of this paper. We also give some examples to show that the structure of the fundamental group alone is not sufficient to determine whether a space is the weakly confluent image of a simply connected manifold.

[1]This research was done while the first author was a Visiting Professor at the University of Saskatchewan. The first author was supported in part by the National Research Council of Greece and by NSERC grant No. A7854, and the second author was partially supported by NSERC grant No. A5616.

2. Preliminaries

By a *mapping* we always mean a continuous function. A mapping $f: X \to Y$ from a compact Hausdorff space X onto a Hausdorff space Y is said to be *confluent* (respectively, *weakly confluent*) provided that for every compact connected subset K of Y and for each (resp., for some) component C of $f^{-1}(K)$ we have that $f(C) = K$. It is known that monotone mappings and open mappings are confluent (see [11, p. 148]). Confluent mappings are weakly confluent. The composition of weakly confluent maps is weakly confluent.

A *continuum* is a compact, connected, metric space. By a n-*manifold* we mean a topological n-manifold with or without boundary. By an ANR we mean a metric absolute neighbourhood retract. A space is said to be LC^1 provided that it is locally connected and locally simply connected.

We say that a subset A *does not locally separate* the space Y provided that for any open connected subset U of Y the set $U \setminus A$ is connected. By I we denote the unit interval [0,1] and by $\dot{I}$ we denote the set of endpoints of I. A point y is said to be a *separating point* of Y if y disconnects Y.

A metric space X is said to have the *disjoint arcs property* provided each pair of mappings $f,g: I \to X$ can be approximated arbitrarily closely by mappings $f',g': I \to X$ such that $f'(I) \cap g'(I) = \emptyset$. It is easy to see that a locally compact, locally connected metric space with the disjoint arcs property has no local separating points and has no non-empty, planar, open set. It can be shown that the converse is also true.

2.1 *Lemma. Let* Y *be a locally compact, locally connected metric space with the disjoint arcs property. Let* $\mathcal{C}$ *denote the space of mappings* $f\colon (I,\dot{I}) \to (Y,y_0)$ *with the supremum metric. Then* $\mathcal{C}$ *contains a dense* G_δ *subset consisting of mappings* $f\colon (I,\dot{I}) \to (Y,y_0)$ *such that* f *embeds* $I \setminus \dot{I}$ *into* $Y \setminus \{y_0\}$.

Proof. By the disjoint arcs property applied to a single mapping of $(I,\dot{I})$ into (Y,y_0) viewed as a pair of mappings, it follows that every mapping of an interval into Y can be approximated arbitrarily closely by a mapping into $Y \setminus \{y_0\}$.

Let $S = \{[a,b]\colon a,b$ are rationals, $0 < a < b < 1\}$. For $[a,b] \in S$ let $U_{[a,b]} = \{f \in \mathcal{C}\colon f([a,b]) \subset Y \setminus \{y_0\}\}$. Then $U_{[a,b]}$ is clearly open in $\mathcal{C}$. We show that $U_{[a,b]}$ is also dense in $\mathcal{C}$. Let $g'\colon I \to Y$ be a mapping with g' near g so that $g'([a,b]) \subset Y \setminus \{y_0\}$. Let A be an arc of small diameter which contains $g'(\dot{I}) \cup \{y_0\}$ so that y_0 is not an endpoint of A. Define $f \in \mathcal{C}$ so that f is close to g and $f(I) \subset A \cup g'(I)$. For $[a,b],[c,d] \in S$ with $[a,b] \cap [c,d] = \emptyset$, let $V_{[a,b],[c,d]} = \{f \in \mathcal{C}\colon f([a,b]) \cap f([c,d]) = \emptyset\}$. As above $V_{[a,b],[c,d]}$ is a dense open set in $\mathcal{C}$. Hence, $\cap U_{[a,b]} \cap \cap V_{[a,b],[c,d]}$ is the required dense G_δ subset of $\mathcal{C}$ (where the intersection is taken over all members of S).

A pair (Y,y_0) is said to have the *avoidable arcs property* if Y is a space, $y_0 \in Y$ and if for each $\varepsilon > 0$ and each mapping $f\colon (I,\dot{I}) \to (Y,y_0)$ there exists a mapping $g\colon (I,\dot{I}) \to (Y,y_0)$ such that $g|[0,1-\varepsilon]$ is an embedding, diameter $(g[1-\varepsilon,1]) < \varepsilon$ and $[f] = [g]$.

It follows from Lemma 2.1 that if Y is an LC^1 metric continuum with the disjoint arcs property, then (Y,y_0) has the avoidable arcs property for each $y_0 \in Y$.

It is easy to see that if Y is a locally connected metric continuum such that (Y,y_0) has the avoidable arcs property, then Y has no local separating points that are not separating points of Y.

It is fairly easy to see from Lemma 2.2 that if Y is a 2-manifold, then (Y,y_0) has the avoidable arcs property for each interior point $y_0 \in Y$.

2.2 *Lemma. Let* $f: (I,\dot{I}) \to (Y,y_0)$ *be a mapping into a locally connected continuum. Then there exists for each* $\varepsilon > 0$, *a graph* $G \subset Y$ *and a mapping* $g: (I,\dot{I}) \to (G,y_0)$ *such that* $d(f,g) < \varepsilon$. *If* y_0 *is not a local separating point of* Y, *then* G *can be taken so that* y_0 *is a point of order two in* G *and* $g^{-1}(y_0) = \{0,1\}$.

Proof. Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of I so that $f([x_i,x_{i+1}])$ has diameter less than $\varepsilon/2$ for each $i \in \{0,\cdots,n-1\}$. For each $i \in \{0,\cdots,n-1\}$ let A_i be an arc in $f([x_i,x_{i+1}])$ irreducible from $f(x_i)$ to $f(x_{i+1})$. Define $h: (I,\dot{I}) \to (\cup_{i=0}^{n-1}A_i,y_0)$ so that $h(x_i) = f(x_i)$ and h carries $[x_i,x_{i+1}]$ linearly onto A_i. Then h is continuous and h is $\varepsilon/2$ close to f. Then $\cup_{i=0}^{n-1}A_i$ is a one-dimensional locally connected continuum.

Let $\mathcal{U}$ be an order two partition of $\cup_{i=0}^{n-1}A_i$ by closed sets of diameter less than $\varepsilon/2$, with connected interiors and zero-dimensional boundaries (see R. H. Bing, *Partitioning continuous curves*, Bull. Amer. Math. Soc. 58 (1952), 536-556). Let G be a graph in $\cup_{i=0}^{n-1} A_i$ such that $y_0 \in G$, $G \cap U$ is a tree for each $U \in \mathcal{U}$. Then there is a retraction $r: \cup_{i=0}^{n-1} A_i \to G$ such that $r(U) \subset U$ for each $U \in \mathcal{U}$. Let $g = r \circ h$.

If y_0 is not a local separating point of Y, then y_0 is locally avoidable by arcs in Y, and so G can be taken so that y_0 is a point of order two in G. We may suppose y_0 is in the interior of an element $U \in \mathcal{U}$. By adding a small arc to G in $U \setminus \{y_0\}$, such that this arc meets both components of $G \cap (U \setminus \{y_0\})$, g may be perturbed by a small motion so that $g^{-1}(y_0) = \{0,1\}$.

2.3 *Question.* If Y is an LC^1 continuum with no local separating points does (Y, y_0) have the avoidable arcs property for some $y_0 \in Y$?

3. Weakly confluent mappings on ANRs

We are grateful to Professor J. Tits for some very helpful discussions concerning the following lemma.

3.1 *Lemma. Let* G *be a group and let* $H_1, \cdots, H_n$ *be finitely many subgroups of* G. *If* G *is covered by finitely many cosets of these subgroups, then there exists some* i, $1 \leq i \leq n$ *such that* H_i *has finite index in* G.

Proof. Let m_i be a positive integer and $x_{i,j} \in G$ such that

$$G = \cup_{i=1}^{n} \cup_{j=1}^{m_i} H_i x_{i,j}.$$

If H_1 has infinite index in G, then there exists some x in G such that for each j, $1 \leq j \leq m_1$, we have that $H_1 x \neq H_1 x_{1,j}$. Then

$$H_1 x \subset \cup_{i=2}^{n} \cup_{j=1}^{m_i} H_i x_{i,j}.$$

This implies that

$$H_1 \subset \cup_{i=2}^{n} \cup_{j=1}^{m_i} H_i x_{i,j} x^{-1},$$

and, hence, we have that

$$G = \cup_{i=2}^{n} [\cup_{j=1}^{m_i} (H_i x_{i,j} \cup \cup_{\ell=1}^{m_1} H_i x_{i,j} x^{-1} x_{1,\ell})].$$

Thus, G is covered by finitely many cosets of the subgroups $H_2, \cdots, H_n$. By induction, the lemma is proved.

3.2 *Proposition. Let* G *be a subgroup, and* $\alpha_1, \cdots, \alpha_n$, $\beta_1, \cdots, \beta_n$ *elements of* G *such that*

$$G = \cup_{i=1}^{n} \alpha_i H \beta_i.$$

Then H *has finite index in* G.

Proof. For each i, $1 \leq i \leq n$, let $H_i = \alpha_i H \alpha_i^{-1}$. Then

$$G = \cup_{i=1}^{n} H_i \alpha_i \beta_i.$$

By Lemma 3.1, there exists some i such that H_i has finite index in G. Since H_i is a conjugate subgroup of H, we infer that H has finite index in G.

3.3 *Theorem. Let* $f\colon X \to Y$ *be a weakly confluent mapping from a locally connected continuum* X *onto an* LC^1 *space* Y, *where* (Y, y_0) *has the avoidable arcs property. Then* $f_\# \pi(X)$ *has finite index in* $\pi(Y)$.

Proof. Let x_0 be an arbitrary but fixed point of $f^{-1}(y_0)$. Let $(\tilde{Y}, p)$ be a covering space of Y with

(1) $$p_\# \pi(\tilde{Y}, z_1) = f_\# \pi(X, x_0),$$

where z_1 is a point of $p^{-1}(y_0)$. Since Y is a LC^1 compact metric space, $\pi(Y, y_0)$ is a countable group, and hence $P^{-1}(y_0)$ is a countable set. Let $p^{-1}(y_0) = \{z_1, z_2, \cdots\}$.

Condition (1) implies that there exists a lifting $\tilde{f}\colon (X, x_0) \to (\tilde{Y}, z_1)$ of the mapping $f\colon (X, x_0) \to (Y, y_0)$. Then

$\tilde{f}(X)$ is a subcontinuum of $\tilde{Y}$, and the fact that $f = p \circ \tilde{f}$ is weakly confluent implies that $p|\tilde{f}(X)$ is a weakly confluent mapping of $\tilde{f}(X)$ onto Y. By the compactness of $\tilde{f}(X)$, we infer that $\tilde{f}(X) \cap p^{-1}(y_0)$ is a finite set. Without loss of generality, we may assume that

(2) $\tilde{f}(X) \cap p^{-1}(y_0) = \{z_1, \cdots, z_n\}$.

Consider mappings $f_1, f_2, \cdots$ with $f_i\colon (I, \dot{I}) \to (Y, y_0)$, which represent the elements of the group $\pi(Y, y_0)$.

Let U be a connected open set in Y so that $y_0 \in U$ and $p^{-1}(U) = U_1 \cup U_2 \cup \cdots$, where the U_i are pairwise disjoint open sets such that for each i $z_i \in U_i$ and $U = p(U_i)$. Since (Y, y_0) has the avoidable arcs property we may assume that for each i there exists $\varepsilon_i > 0$ so that $f_i|[0, 1-\varepsilon_i]$ is an embedding and $f_i([1-\varepsilon_i, 1]) \subset U$.

For each i let A_i be an arc in $\tilde{f}(X)$ such that $p(A_i) = f_i([0, 1-\varepsilon_i])$. Then p carries A_i homeomorphically onto the arc $f_i([0, 1-\varepsilon_i])$. Let $z_{m_i} \in A_i \cap p^{-1}(y_0)$, let n_i be the unique integer so that $(p|A_i)^{-1}(f_i(1-\varepsilon_i)) \subset U_{n_i}$. Define a lifting g_i of f_i by

$$g_i(x) = \begin{cases} (p|A_i)^{-1}(f_i(x)), & \text{if } x \in [0, 1-\varepsilon_i] \\ (p|U_{n_i})^{-1}(f_i(x)), & \text{if } x \in [1-\varepsilon_i, 1]. \end{cases}$$

For $m \in \{1, \cdots, n\}$ let $h_m\colon I \to \tilde{Y}$ be a path such that $h_m(0) = z_1$ and $h_m(1) = z_m$. Let $\bar{h}_m$ be the inverse of the path h_m. Let $\alpha_m = [p \circ h_m]$ and $\beta_m = [p \circ h_m]^{-1}$.

For $i \in \{1, 2, \cdots\}$ consider the loop $\bar{h}_{n_i} * g_i * h_{m_i}$ in $\tilde{Y}$. Then $[\bar{h}_{n_i} * g_i * h_{m_i}] \in \pi(\tilde{Y}, z_1)$, and hence

$$p_{\#}([\bar{h}_{n_i} * g_i * h_{m_i}]) \in p_{\#}(\pi(\tilde{Y}, z_1))$$

or, equivalently,

$$[p \circ h_{n_i}]^{-1} \quad [p \circ g_i] \quad [p \circ h_{m_i}] \in p_\#(\pi(\tilde{Y},z_1)).$$

Since $f_i = p \circ g_i$ we have

$$\alpha_{n_i}^{-1} \cdot [f_i] \cdot \beta_{m_i}^{-1} \in p_\#(\pi(\tilde{Y},z_1))$$

or

$$[f_i] \in \alpha_{n_i} \cdot p_\#(\pi(\tilde{Y},z_1)) \cdot \beta_{m_i}.$$

Hence

$$\pi(Y,y_0) = \cup_{k,m=1}^{n} \alpha_k f_\#(\pi(X,x_0))\beta_m,$$

and by Proposition 3.2 $f_\#(\pi(X,x_0))$ has finite index in $\pi(Y,y_0)$.

3.4 *Corollary. Let* f: X → Y *be a weakly confluent mapping from a locally connected continuum* X *onto an* LC^1 *space* Y, *where* (Y,y_0) *has the avoidable arcs property. If* $f_\#(\pi(X))$ *is finite, then* $\pi(Y)$ *is a finite group. In particular, if the fundamental group of* X *is finite, then the fundamental group of* Y *is also finite.*

Walsh [10] has obtained a converse to the Smale theorem.

3.5 *Theorem* (Walsh). *A mapping* f: M → Y *from a compact, connected, piecewise linear,* n-*manifold* M, $n \geq 3$, *into a connected* ANR Y *is homotopic to an open mapping of* M *onto* Y *if and only if* $f_\#(\pi(M,x))$ *has finite index in* $\pi(Y,f(x))$.

Theorem 3.3 and Theorem 3.5 give rise to the following result, since open mappings onto continua are weakly confluent.

3.6 *Corollary. Let* f *be a mapping from a compact, connected, piecewise linear,* n*-manifold,* $n \geq 3$*, onto an* ANR Y *with the avoidable arcs property at some point* $y_0 \in Y$. *Then the following are equivalent:*

(i) f *is homotopic to a weakly confluent mapping,*

(ii) f *is homotopic to an open mapping,*

(iii) $f_{\#}(\pi(X))$ *has finite index in* $\pi(Y)$.

We close with two examples to show that the hypotheses of Theorem 3.3 are necessary. We construct a continuum Y which is the union of two real projective planes Y_1 and Y_2 such that $Y_1 \cap Y_2$ is an arc. Each of Y_1 and Y_2 has fundamental group $\mathbb{Z}_2$ and, hence by Theorem 3.5 is the open image of I^3 under a mapping such that the preimage of each point has finitely many components. It follows from the Van Kampen Theorem that the fundamental group of Y is $\mathbb{Z}_2 * \mathbb{Z}_2$, i.e. the free product of two copies of the integers modulo two. By Theorem 3.3 Y is not the weakly confluent image of I^3. This example answers in the negative problems 1 and 2 of [1].

We remark that Theorem 3.3 fails if Y contains local separating points. To see this, let Y be the wedge of two projective planes. By [3, Theorem 4.1], Y is the weakly confluent image of I^3. However, the fundamental group of Y is the free product $\mathbb{Z}_2 * \mathbb{Z}_2$.

References

1. J. Grispolakis, *On Čech cohomology and weakly confluent mappings into* ANRs, Fund. Math. 111 (1981), 235-245.

2. ________, *Confluent mappings and the fundamental group*, Bull. Pol. Acad. Sci. 28 (1980), 643-646.

3. ________, *The fundamental group and weakly confluent mappings on ANRs, I*, Top. Proc. 11 (1986), 267-279.
4. S. T. Hu, *Theory of retracts*, Wayne State Univ. Press, Detroit, 1965.
5. ________, *Homotopy theory*, Academic Press, New York, 1959.
6. A. G. Kurosh, *Theory of groups, I*, Chelsea, London, 1955.
7. S. Smale, *A note on open mappings*, Proc. Amer. Math. Soc. 8 (1957), 391-394.
8. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.
9. J. J. Walsh, *Monotone and open mappings on manifolds, I*, Trans. Amer. Math. Soc. 209 (1975), 419-432.
10. ________, *Light open and open mappings on manifolds, II*, Trans. Amer. Math. Soc. 217 (1976), 271-284.
11. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc., Providence, 1942.

Tech. University of Crete

Chania 73100, Greece

and

University of Saskatchewan

Saskatoon, Saskatchewan, Canada S7N 0W0