TOPOLOGY PROCEEDINGS
Volume 11, 1986
Pages 291–308
http://topology.auburn.edu/tp/

# $\kappa$-METRIC SPACES AND FUNCTION SPACES

by

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**Topology Proceedings**

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| **E-mail:** | topolog@auburn.edu |
| **ISSN:** | 0146-4124 |

# $\kappa$-METRIC SPACES AND FUNCTION SPACES

**Takesi Isiwata**

Ščepin [13,14] introduced the notions of $\kappa$-metrizability and capacity as a generalization of metric spaces and locally compact groups, and proved the k-metrizability is productive [15]. Bennett, Lewis and Luksic [2] showed that $\kappa$-metrizability is equivalent to faithful capacity, and that $\kappa$-metrizability is not closed-hereditary. Dranišnikov [6] defined the additivity of k-metric space and asserted that a compact additive $\kappa$-metric space is metrizable. In our previous papers [10,11], we studied metrization problems of additive $\kappa$-metric spaces, and compact (or realcompact) $\kappa$-metrizable extension of $\kappa$-metric spaces respectively. After the previous paper [11] was printed, through G. D. Dimov, the author knew the following result due to Chigogidze [3] (see Th. 3.3); $\beta X$ is $\kappa$-metrizable iff X is pseudocompact and $\kappa$-metrizable. Thus, in particular, any of $\beta R$, $\beta Q$ and $\beta N$ is not $\kappa$-metrizable.

In this paper, we prove, in Section 3, that if X is locally compact and $\beta X - X$ is $\kappa$-metric space, then X is pseudocompact, and that any of $\beta R - R$, $\beta Q - Q$ and $\beta N - N$ is not $\kappa$-metrizable. In Section 4, we introduce the notion of $X(\aleph_0)$-points as a generalization of strictly $\aleph_0$-continuous functions in the sense of Arhangel'skiǐ [1], and show that any dense subspace X of the product of realcompact $\kappa$-metric space $Y_\alpha$, $\alpha \in \Gamma$, $\upsilon X$ precisely consists of $X(\aleph_0)$-points if any point of $Y_\alpha$ is $G_\delta$ for each $\alpha \in \Gamma$ (see Th. 4.3). As corollaries, there follows Corson's result [4; Th. 2] as well as Uspenškii's one [18, Th. 1]. In Section 5, we give a characterization of a space X such that $R^X$ precisely consists of strictly $\aleph_0$-continuous (or $\aleph_0$-continuous) functions, and related examples. The author wishes to thank Y. Tanaka for his helpful advice.

## 1. Definitions and Preliminaries

All spaces in this paper are Tychonoff. We denote by N the set of positive integers, by R the set of real numbers, by Q the set of rational numbers, by Rc(X) the family of regular closed subsets of X, by C(X) the set of continuous real-valued functions on X, and by $\beta X(\upsilon X)$ the Stone-Čech compactification (the Hewitt realcompactification) of X. We assume familiarity with [7], whose terminology will be used throughout, and use the following abbreviations: pc = pseudocompact, clopen = closed and open, rc = realcompact, $X^* = \beta X - X$ and c-$\kappa$ extension = compact $\kappa$-metrizable extension.

(X,d) is a $\kappa$-metric space if d is a mapping X from $X \times Rc(X)$ to $[0,\infty)$ satisfying the following conditions: (K1) $d(x,C) = 0 \Leftrightarrow x \in C$. (K2) if $C \subset D$, then $d(x,C) \geq d(x,D)$ for every $x \in X$. (K3) $d(x,C)$ is continuous in x for every C. (K4) $d(x,cl(\cup C_\alpha)) = \inf d(x,C_\alpha)$ for every increasing transfinite sequence $\{C_\alpha\}$. d is called a $\kappa$-metric on X. We can assume that any $\kappa$-metric d satisfies $d(x,\emptyset) \leq 1$ for every $x \in X$ [14; p. 179]. Let us put $\mathcal{U}(\Gamma) = \{U_\alpha; \alpha \in \Gamma\}$ where $U_\alpha$ is a subset of X, and $\cup\mathcal{U}(\Gamma) = \cup\{U_\alpha; \alpha \in \Gamma\}$. By $\{N_1,N_2;\Gamma\}$ we mean the fact that $N_1$ and $N_2$ are disjoint countable subsets of $\Gamma$ and $|\Gamma| \geq \aleph_0$. The following Ščepin's results will be useful.

1.1. *Proposition* ([14], pp. 179-180). *Let* $(X,d)$ *be a* $\kappa$*-metric space.*

(1) *A subspace* $Y$ *of* $X$ *is* $\kappa$*-metrizable if* $Y$ *satisfies one of the following cases* (i) $Y$ *is dense,* (ii) $Y$ *is open, and* (iii) $Y \in Rc(X)$.

(2) *Let* $\mathcal{U}(\Gamma) = \{U_\alpha \in Rc(X)\}$. *Then for any* $\varepsilon > 0$ *and any* $x \in X$*, there exists a finite subset* $\Gamma_1$ *of* $\Gamma$ *such that* $d(x, \cup\mathcal{U}(\Gamma_1)) < d(x, cl(\cup\mathcal{U}(\Gamma)) + \varepsilon$.

1.2. *Lemma. Let* $(X,d)$ *be a* $\kappa$*-metric space and* $\mathcal{U}(\Gamma) = \{U_\alpha \in Rc(X)\}$, $|\Gamma| \geq \aleph_0$. *Then we have*

(1) *If* $x \in cl(\cup\mathcal{U}(\Gamma)) - \cup\mathcal{U}(\Gamma)$, *then there exists* $\{N_1, N_2; \Gamma\}$ *with* $x \in cl(\cup\mathcal{U}(N_i))$, $i = 1,2$.

(2) *If* $cl(\cup\mathcal{U}(N_1)) \cap cl(\cup\mathcal{U}(N_2)) = \emptyset$ *for any* $\{N_1, N_2, \Gamma\}$, *then* $\mathcal{U}(\Gamma)$ *is discrete.*

(3) *If* $|\Gamma| > \aleph_0$ *and* $\mathcal{U}(\Gamma) = \{U_\alpha \in Rc(X)\}$ *is pairwise disjoint, then there exist a subset* $\Gamma_1 \subset \Gamma$ *with* $|\Gamma| = |\Gamma_1|$ *and a discrete family* $\mathcal{V}(\Gamma_1) = \{V_\alpha \in Rc(X)\}$ *with* $V_\alpha \subset U_\alpha$ *for each* $\alpha \in \Gamma_1$.

*Proof.* (1) For $\varepsilon = 1$, put $A(1) = \Gamma_1$ in 1.1(2). Since $x \in cl(\cup\mathcal{U}(\Gamma - A(1)))$, again for $\varepsilon = 1/2$, apply 1.1(2) to $\mathcal{U}(\Gamma - A(1))$, and take a finite subset $A(2)$ of $\Gamma - A(1)$ with $d(x, \cup\mathcal{U}(A(2)) < 1/2$. Repeating this process, we have a finite subset $A(n)$ of $\Gamma - \cup\{A(i); 1 \leq i < n\}$ with $d(x, \cup\mathcal{U}(A(n)) < d(x, cl(\cup\mathcal{U}(\Gamma - \cup\{A(i); 1 \leq i < n\})) + 1/n$. It is easily seen, by (K1) and (K2), that $N_1 = \cup\{A(2n-1); n \in N\}$ and $N_2 = \cup\{A(2n); n \in N\}$ are desired subsets of $\Gamma$.

(2) follows from (1).

(3) To prove (3), we use Šč̌epin's method used in the proof of [14, Th. 12]. For any $\alpha \in \Gamma$, we fix a point $x_\alpha \in U_\alpha$. Then for any given $\varepsilon > 0$, and $C_\alpha = \mathrm{cl}(X - U_\alpha) \in Rc(X)$, $D(\varepsilon) = \{x_\alpha; d(x,C_\alpha) \geq \varepsilon\}$ is a discrete closed subspace of X (cf. [14, Th. 12]). Let us put $A(\varepsilon) = \{\alpha; x_\alpha \in D(\varepsilon)\}$ and $W_\alpha = \{x; d(x,C_\alpha) > \varepsilon/2\}$ for $\alpha \in A(\varepsilon)$. Then $V_\alpha = \mathrm{cl}W_\alpha \subset U_\alpha$. We claim that $\mathcal{V}(A(\varepsilon)) = \{V_\alpha\}$ is discrete. Suppose that $p \in \mathrm{cl}(\cup\mathcal{V}(A(\varepsilon)) - \mathcal{V}(A(\varepsilon))$. By 1.1(1), there exists a finite subset A of $A(\varepsilon)$ with $d(p,\cup\mathcal{V}(A)) < \varepsilon/3$. Since d is continuous in x, there exists an open set $0 \ni p$ such that $d(y,\cup\mathcal{V}(A)) < \varepsilon/3$ for each $y \in 0$. If $0 \cap W_\alpha \not\ni y$ for $\alpha \notin A$, then $y \in W_\alpha$ implies $d(y,C_\alpha) > \varepsilon/2$, and $\cup\mathcal{V}(A) \subset X - U_\alpha$ implies $d(y,C_\alpha) \leq d(y,\cup\mathcal{V}(A))$. These inequalities imply $\varepsilon/2 < \varepsilon/3$, a contradiction. Thus $\mathcal{V}(A(\varepsilon))$ is discrete. Let $\varepsilon = 1/n$, $n \in N$. Then $\Gamma = \cup A(n)$. Since $|\Gamma| > \aleph_0$, we have $|A(m)| = |\Gamma|$ for some $m \in N$.

## 2. Properties of $\kappa$-Metric Spaces

We recall that X satisfies the *countable chain condition* (=(CCC)) if every family of pairwise disjoint non-empty open subsets of X has cardinality $\leq \aleph_0$, and that X is an SL-*space* if every open cover of X contains a countable subfamily whose union is dense in X [12]. If X is either Lindelöf or satisfies (CCC), then X is SL [15]. A point x of X is a P-*point* if every $G_\delta$-set containing x is a neighborhood of x. A space X is a P-*space* if every point of X is a P-point, equivalently every cozero set is C-embedded [7]. X is an F-*space* if every cozero set is C*-embedded, equivalently for any $f \in C(X)$, cozero sets $\{x; f(x) > 0\}$ and $\{x; f(x) < 0\}$ are completely separated [5]. A P-space is an F-space. A compact space Y containing a $\kappa$-metric space $(X,d)$ as a dense subspace is said to be a *compact $\kappa$-metrizable* (= c-$\kappa$) *extension* of $(X,d)$ if there exists a $\kappa$-metric $d^*$ on Y such that $d^*(x,C) = d(x,C \cap X)$ for $C \in Rc(Y)$ and $x \in X$ [11]. We note that $cl_Y(C \cap X) = C$ for $C \in Rc(Y)$.

2.1. *Proposition. Let* $(X,d)$ *be a $\kappa$-metric space. Then we have*

(1) *Any* P-*point is isolated.*

(2) *If* X *is* rc, *then the cardinal of every family of pairwise disjoint open subsets is nonmeasurable.*

(3) *If* X *is* pc, *then* X *is* SL.

(4) *If* X *is* SL, *then* X *satisfies* (CCC).

(5) *If there exists a* c-$\kappa$ *extension* Y *of* X, *then* X *satisfies* (CCC).

(6) *If any two disjoint cozero sets of* X *have disjoint closures, especially* X *is an* F-*space, then* X *is discrete.*

*Proof.* (1) Let p be a non-isolated P-point. Let $\mathcal{U} = \{C_\alpha \in Rc(X); p \notin C_\alpha\}$. Since $p \in cl(\cup\mathcal{U})$, there exists a subfamily $\mathcal{U}(N) = \{C_n\}$ with $p \in cl(\cup\mathcal{U}(N))$ by 1.2(1), but $p \in int(\cap(X - C_n))$, a contradiction.

(2) From 1.2(3) and the fact that every closed discrete subset of a rc space has a nonmeasurable cardinal.

(3) If X is not SL, there exists a discrete family of open subsets of X by 1.2(3). This is impossible because X is pc.

(4) Let $W(\Gamma) = \{W_\alpha\}$ be a disjoint family of open subsets with $|\Gamma| > \aleph_0$. For each $\alpha$, take $U_\alpha \in Rc(X)$ with $U_\alpha \subset W_\alpha$. Apply 1.2(3) to $\mathcal{U}(\Gamma) = \{U_\alpha\}$. Then an open covering $\{W_\alpha, X - cl(\cup \mathcal{V}(\Gamma_1)); \alpha \in \Gamma_1\}$ has no countable subfamily whose union is dense in X.

(5) From [14, Th. 12] or 1.2(3), Y satisfies (CCC), so does X.

(6) Let x be a non-isolated point. As in the proof of [14, Th. 11], there exists a decreasing sequence $\{V_n\}$ of regular open neighborhoods of x such that $U_1 = \cup\{V_{2k} - clV_{2k+1}; k \in N\}$ and $U_2 = \cup\{V_{2k-1} - clV_{2k}; k \in N\}$ are disjoint, but $x \in clU_1 \cap clU_2$. Since $V_{2k} - clV_{2k+1}$ and $V_{2k-1} - clV_{2k}$ are regular open sets of a $\kappa$-metric space X, these are cozero sets, so are $U_1$ and $U_2$, a contradiction. (2.1(6) (in case X is an F-space) is due to [17].)

### 3. $\kappa$-Metrizability of $\beta X$, $\nu X$ and $X^*$

3.1. *Lemma. Let* $\mathcal{U}(N) = \{F_n \in Rc(X)\}$ *be a discrete family of compact subsets of a space* X. *Then there exists a disjoint family* $\mathcal{V}(\Gamma) = \{L_\alpha \in Rc(X^*)\}$ *with* $|\Gamma| > \aleph_0$.

*Proof*. Let $\Gamma$ be an index set with $|\Gamma| > \aleph_0$ such that for each $\alpha \in \Gamma$, there exists $N_\alpha \subset N$ with $|N_\alpha| = \aleph_0$, $N = \cup\{N_\alpha; \alpha \in \Gamma\}$ and $|N_\alpha \cap N_\beta| < \aleph_0$ for $\alpha \cdot \beta \in \Gamma$. Let us put $E_\alpha = \cup\{F_i; i \in N_\alpha\}$ and $K_\alpha = cl_{\beta X}E_\alpha - E_\alpha$. Then $E_\alpha \cap E_\beta$ is a compact subset of X. We claim that $int_{X^*}K_\alpha \neq \emptyset$. We fix a point $x_n \in intF_n$. Let $f_i \in C(X)$, $i \in N_\alpha$ such that $f_i(x_i) = 1$, $f_i = 0$ on $X - F_i$ and $0 \leq f_i \leq 1$. Since $\mathcal{U}(N_\alpha)$ is discrete, $f_\alpha = \Sigma f_i \in C(X)$. We take $g_\alpha \in C(\beta X)$ with

$g_\alpha | X = f_\alpha$ and $p_\alpha \in cl_{\beta X} B_\alpha - B_\alpha$ where $B_\alpha = \{x_i; i \in N_\alpha\}$. Then $g_\alpha(p_\alpha) = 1$ and $g_\alpha = 0$ on $\beta X - cl_{\beta X} E_\alpha$, so $p_\alpha \in int_{X*} K_\alpha$. We take $L_\alpha \in Rc(X^*)$ such that $p_\alpha \in int_{X*} L_\alpha \subset L_\alpha \subset K_\alpha \cap W_\alpha$ where $W_\alpha = \{y \in X^*; g_\alpha(y) > 1/2\}$. For $\alpha \neq \beta$, $f_\alpha = 0$ on $E_\beta - (E_\alpha \cap E_\beta)$ and hence $g_\alpha = 0$ on $K_\beta$, so $L_\alpha \cap L_\beta = \emptyset$. Thus $\mathcal{V}(\Gamma) = \{L_\alpha\}$ is pairwise disjoint in $X^*$.

3.2. *Lemma. Let* $\mathcal{U}(\Gamma) = \{U_\alpha \in Rc(X)\}$, $|\Gamma| \geq \aleph_0$ *be a family of pairwise disjoint subsets of a space* X. *Then we have*

(1) *If* $\mathcal{U}(\Gamma)$ *satisfies the following* (a) *and* (b), *then* X *is not* $\kappa$*-metrizable.*

(a) *For any* $\{N_1, N_2; \Gamma\}$, $cl(\cup\mathcal{U}(N_1)) \cap cl(\cup\mathcal{U}(N_2)) = \emptyset$.

(b) $\cup\mathcal{U}(\Gamma)$ *is not closed.*

(2) *If* $\mathcal{U}(\Gamma)$ *satisfies the following* (c), *then* $\beta X$ *is not* $\kappa$*-metrizable.*

(c) *For any* $\{N_1, N_2; \Gamma\}$, $\cup\mathcal{U}(N_1)$ *and* $\cup\mathcal{U}(N_2)$ *are completely separated.*

(3) *If* $\mathcal{U}(\Gamma)$ *satisfies* (c) *and the following* (d), *then* $\cup X$ *is not* $\kappa$*-metrizable.*

(d) $\cup\{cl_{\cup X} U; \alpha \in \Gamma\}$ *is not closed in* $\cup X$.

(4) *If each* $U_\alpha$ *is non-compact, clopen and* $\mathcal{U}(\Gamma)$ *is discrete, then* $X^*$ *is not* $\kappa$*-metrizable.*

*Proof.* (1) If X is $\kappa$-metrizable, then 1.2(1) holds, a contradiction.

(2) $\mathcal{V}(\Gamma) = \{cl_{\beta X} U_\alpha\}$ is a family of pairwise disjoint regular closed subsets of $\beta X$. Replacing X by $\beta X$ in (1), it is easy to see that $\mathcal{V}(\Gamma)$ satisfies (a) and (b) in $\beta X$, so $\beta X$ is not $\kappa$-metrizable.

(3) Replacing $\beta X$ by $\upsilon X$ in (2) and apply the method used in the proof of (2).

(4) $Cl_{\beta X}U_\alpha$ being clopen in $\beta X$, $F_\alpha = cl_{\beta X}U_\alpha - U_\alpha \in Rc(X^*)$ and $E_\alpha = cl_{\beta X}F_\alpha \subset cl_{\beta X}U_\alpha$. Thus $\mathcal{V}(\Gamma) = \{E_\alpha\}$ is pairwise disjoint in $\beta X$, and hence there exists a point $p \in cl_{\beta X}(\cup\mathcal{V}(\Gamma)) - \cup\mathcal{V}(\Gamma)$. The discreteness of $\mathcal{U}(\Gamma)$ implies $p \in X^*$. Since $\cup F_\alpha$ is dense in $\cup E_\alpha$, we have $p \in cl_{\beta X}(\cup\mathcal{W}(\Gamma))$ where $\mathcal{W}(\Gamma) = \{F_\alpha\}$. $\mathcal{W}(\Gamma)$ is pairwise disjoint in $X^*$, and satisfies (a) and (b) in (1) replacing X by $X^*$. Thus $X^*$ is not $\kappa$-metrizable.

The following theorem 3.3 is due to Chigogidze and Volov. The proof of this theorem is not given in [3,5], however, for example, we can give the proofs of the "only if" part of (1), (2), (3) and (4) of theorem 3.3 by the method used in this paper, that is, by 3.4(1), 2.1(3,4), 4.5(2) and 3.4(1.5), respectively.

3.3. *Theorem.* (1) *$\beta X$ is $\kappa$-metrizable iff X is* pc *and $\kappa$-metrizable* [3, Th. 2].

(2) *If X is* pc *$\kappa$-metrizable, X satifies* (CCC) [3, Corollary 2].

(3) *The product of $\kappa$-metrizable* pc *spaces is* pc [3, Corollary 3].

(4) *If X is* pc *and Y is a* c-$\kappa$ *extension of X, $\beta X = Y$* (see [5, p. 1259]).

3.4. *Theorem.* (1) *If $\beta X$ is $\kappa$-metrizable, X is* pc.

(2) *If X is locally compact and $X^*$ is $\kappa$-metrizable, X is* pc.

(3) *Let* X *be a topological sum* $\Sigma X_\alpha$ *of spaces* $X_\alpha$, $\alpha \in \Gamma$. *If* $\Gamma$ *is measurable,* $\cup X$ *is not* $\kappa$*-metrizable.*

(4) *If* X *has a* c-$\kappa$ *extension* Y, $\cup X$ *is* $\kappa$*-metrizable and* $\cup X \subset Y$.

(5) *If* X *is* pc *and* Y *is a* c-$\kappa$ *extension of* X, $\beta X = Y$.

(6) *If a locally compact space* X *is topologically complete and* $X^*$ *is* $\kappa$*-metrizable, then* $\beta X = X$.

(7) *Any of* $\beta R - R$, $\beta Q - Q$ *and* $\beta N - N$ *is not* $\kappa$*-metrizable.*

*Proof.* (1) If X is not pc, there exists a discrete family $\mathcal{U}(N) = \{U_n \in Rc(X)\}$. The discreteness of $\mathcal{U}(N)$ implies $\cup\mathcal{U}(M) \in Rc(X)$ for any $M \subset N$. On the other hand, X is $\kappa$-metrizable by 1.1(1), and hence $\cup\mathcal{U}(M)$ is a zero set. Thus $\cup\mathcal{U}(N_1)$ and $\cup\mathcal{U}(N_2)$ are completely separated for any $\{N_1, N_2; N\}$. Thus $\beta X$ is not $\kappa$-metrizable by 3.2(2), a contradiction.

(2) If X is not pc, then there exists a discrete family $\mathcal{U}(N) = \{F_n \in Rc(X)\}$ of compact subsets of X. But $X^*$ is compact $\kappa$-metrizable. Thus, by 3.1 and 2.1(4), $X^*$ is not $\kappa$-metrizable, a contradiction.

(3) X is not rc [7], and $\cup X - X \subset cl_{\cup X}(\cup\mathcal{U}(\Gamma))$ where $\mathcal{U}(\Gamma) = \{X_\alpha\}$. Thus $\cup X$ is not $\kappa$-metrizable by 3.2(3) (3.4(3) (in case each $X_\alpha$ consists of a single point) is due to [17]).

(4) Follows from [11, Lemma 2.3].

(5) X being pc, we have $\beta X = \cup X$, and hence $\beta X = Y$ by (4).

(6) By (2) X is pc. Since a topologically complete pc space is compact, we have $X = \beta X$ (3.4(6) (in case X is locally compact and rc) is due to [17]).

(7) Any of $R^*$ and $N^*$ are not $\kappa$-metrizable by (6). $Q^*$ is not $\kappa$-metrizable by 3.2(4).

We recall that $(X,d)$ is an *additive $\kappa$-metric space* if $d$ satisfies the following (SK4) instead of (K4).

(SK4) For $\mathcal{U}(\Gamma) = \{U_\alpha \in Rc(X)\}$ and any $x \in X$,

$$d(x, cl(\cup\mathcal{U}(\Gamma))) = \inf\{d(x,U_\alpha);\ \alpha \in \Gamma\}.$$

We note that any metric space is an additive $\kappa$-metric space, but the converse does not hold; indeed, the Sorgenfrey line is a desired space [16]. In [10], we proved that a pc additive $\kappa$-metric space is metrizable. From this and 3.4(1,2) we have

3.5. *Theorem. For an additive $\kappa$-metric space $(X,d)$, we have*

(1) *If $\beta X$ is $\kappa$-metrizable, then $X$ is compact and metrizable.*

(2) *If $X$ is locally compact and $X^*$ is $\kappa$-metrizable, then $X$ is compact and metrizable.*

3.6. *Example.* There exists a $\kappa$-metric space $X$ which has no c-$\kappa$ extensions. Indeed, let $X$ be a topological sum of $\aleph_1$ many copies of $R$. The $X$ is a desired space by 2.1(5).

## 4. Function Spaces

We recall terminologies used in [1] in which we restrict cardinality to $\aleph_0$. Let $C_p(X)$ be the space of real-valued continuous functions defined on $X$ with the topology of pointwise convergence. Note that $C_p(X)$ is considered as a dense subspace of $R^X$. A function $f \in R^X$ is called $\aleph_0$-*continuous* if $f|A$ is continuous on A for each $A \subset X$ with $|A| = \aleph_0$. We recall that a function $f \in R^X$ is *strictly* $\aleph_0$-*continuous* if for each $A \subset X$ with $|A| = \aleph_0$, there exists $g \in C_p(X)$ with $f|A = g|A$. A subset A of X is $\aleph_0$-*embedded in* X if for every $x \in X - A$, there exists a $G_\delta$-set G of X with $x \in G \subset X - A$. A space X is called a *moscow space* if every regular closed subset is a union of $G_\delta$-subsets of X. A $\kappa$-metric space is moscow. We denote by $q(X) = \aleph_0$ the fact that X is $\aleph_0$-embedded in $\beta X$, equivalently, X is rc.

We define the notion of $X(\aleph_0)$-point as a generalization of strictly $\aleph_0$-continuous functions. Let $Y = \Pi Y_\alpha$, $\alpha \in \Gamma$ and $X \subset Y$. We say that a point $p = (p_\alpha) \in Y$ is an $X(\aleph_0)$-*point* if for any countable subset $\Gamma_1 \subset \Gamma$ there exists a point $x = (x_\alpha) \in X$ such that $x_\alpha = p_\alpha$ for each $\alpha \in \Gamma_1$. Let $S(X(\aleph_0))$ be the set of $X(\aleph_0)$-points. Obviously $S(X(\aleph_0)) \supset X$. For each $p = (p_\alpha) \in Y$, let us put $\Sigma_p = \{y;\ |\{\alpha: y_\alpha \neq p_\alpha\}| \leq \aleph_0\}$, which is a $\Sigma$-*product* of $Y_\alpha$, introduced by Corson [4]. We show that for any dense subspace X of the product or rc $\kappa$-metric spaces $Y_\alpha$, $\upsilon X$ precisely consists of $X(\aleph_0)$-points if any point of $Y_\alpha$ is $G_\delta$. As corollaries, there follows Corson's result [4, Th. 2] as well as Uspenskiĭ's one [18, Th. 1]. It is easily verified that 4.1(2) below holds.

4.1. *Proposition.* (1) *If* X *is dense in a moscow space* Y *and* $q(X) = \aleph_0$, *then* X *is* $\aleph_0$-*embedded in* Y[1].

(2) *Let* $p \in Y = \Pi Y_\alpha$, $\alpha \in \Gamma$. *Then* $\Sigma_p$ *is dense in* $Y$ *and every point* $y$ *of* $Y$ *is a* $\Sigma_p(\aleph_0)$*-point.*

4.2. *Theorem.* *Let* $X$ *be dense in* $Y = \Pi Y_\alpha$, $\alpha \in \Gamma$. *Suppose that* $Y$ *is moscow and any point of* $Y_\alpha$ *is* $G_\delta$ *for each* $\alpha \in \Gamma$. *Then we have*

(1) *If* $Y \supset \upsilon X$, *then* $\upsilon X = S(X(\aleph_0))$.

(2) *If* $p \in Y$ *and* $Y \supset \upsilon\Sigma_p$, *then* $\upsilon\Sigma_p = Y$.

*Proof.* We claim that $S(X(\aleph_0)) \subset \upsilon X$. Let $p = (p_\alpha) \in S(X(\aleph_0)) - \upsilon X$ and $\Gamma_1$ any countable subset of $\Gamma$. Since $p \in S(X(\aleph_0))$, there exists some point $x = (x_\alpha) \in X$ with $x_\alpha = p_\alpha$ for each $\alpha \in \Gamma_1$. On the other hand, it is easily verified by 4.1(1) that for $p$ and $\Gamma_1$, there exists a $G_\delta$-set $G(\Gamma_1,\Gamma)$ of $Y$ such that $p \in G(\Gamma_1,\Gamma) = \Pi_{\Gamma_1}\{p_\alpha\} \times \Pi_{\Gamma-\Gamma_1} Y_\alpha \subset Y - \upsilon X$. This implies $x \in Y - \upsilon X$, a contradiction. Next we claim that if $p \in \upsilon X - X$, then $p \in S(X(\aleph_0))$. For any countable set $\Gamma_1 \subset \Gamma$ and for any $G_\delta$-set $G(\Gamma_1,\Gamma) \ni p$, there exists $f \in C(Y)$ such that $p \in Z(f) \subset G(\Gamma_1,\Gamma)$ and $Z(f) \cap X \neq \emptyset$. Let $x = (x_\alpha) \in Z(f) \cap X$. Since $x \in G(\Gamma_1,\Gamma)$, $x_\alpha = p_\alpha$ for each $\alpha \in \Gamma_1$. Hence $p \in S(X(\aleph_0))$. Thus we have $\upsilon X = S(X(\aleph_0))$. (2) follows from (1) and 4.1(2).

4.3. *Theorem.* *For each* $\alpha \in \Gamma$, *let* $Y_\alpha$ *be a* rc $\kappa$*-metric space, and every point of* $Y_\alpha$ *be* $G_\delta$.

(1) *If* $X$ *is dense in* $Y = \Pi Y_\alpha$, *then* $\upsilon X = S(X(\aleph_0)) \subset Y$.

(2) *For any* $p \in Y$, $\upsilon\Sigma_p = S(\Sigma_p(\aleph_0)) = Y$.

*Proof.* Since $Y$ is $\kappa$-metrizable by [15, Th. 2], $X$ is $\kappa$-metrizable by 1.1. Then, since $Y$ is rc, $\upsilon X \subset Y$ by [11, Th. 2.5(1)]. Thus (1) and (2) follow from 4.2(1) and 4.2(2) respectively.

4.4. *Corollary. Let* $Y$ *be the product space of* $Y_\alpha$, $\alpha \in \Gamma$.

(1) *If* $Y_\alpha = R$ *for each* $\alpha \in \Gamma$, *then we have*

(i) $UC_p(X)$ *precisely consists of strictly* $\aleph_0$*-continuous functions.*

(ii) $C_p(X)$ *is* rc *iff every strictly* $\aleph_0$*-continuous function is continuous* (Uspenškii [18, Th. 1]).

(2) *If each* $Y_\alpha$ *is separable and metrizable, then* $U\Sigma_p = Y$ *for any* $p \in Y$ (Corson [4, Th. 2]).

(3) *If each* $Y_\alpha$ *is compact and metrizable, then* $\beta X = Y$ *for any* pc *dense subspace* $X$ *of* $Y$ (from 3.4(5)).

4.5. *Theorem. Let* $X_\alpha$ *be a* $\kappa$*-metric space for each* $\alpha \in \Gamma$.

(1) *If* $UX_\alpha$ *is* $\kappa$*-metrizable for each* $\alpha \in \Gamma$, *then* $U(\Pi X) = \Pi UX_\alpha$.

(2) *If* $\beta X_\alpha$ *is* $\kappa$*-metrizable for each* $\alpha \in \Gamma$, *then* $\Pi X_\alpha$ *is* pc *and* $\beta(\Pi X_\alpha) = \Pi\beta X_\alpha$.

*Proof.* (1) Let us put $UX_\alpha = Y_\alpha$. Since $\kappa$-metrizability is productive, $Y = \Pi Y_\alpha$ is $\kappa$-metrizable. While $X = \Pi X_\alpha$ is dense in $Y$. Then, by [11, Th. 2.5(1)]. $UX \subset Y$. Let $p \in L = Y - X$. By 4.1, there exists a $G_\delta$-set $G$ of $Y$ such that $p \in G = \Pi_{\Gamma_1} G(p_\alpha) \times \Pi_{\Gamma - \Gamma_1} Y_\alpha \subset L$ for some countable subset $\Gamma_1 \subset \Gamma$ where $G(p_\alpha)$ is a $G_\delta$-set in $Y_\alpha$. On the other hand, $G(p_\alpha) \cap X_\alpha \neq \emptyset$. Thus $G \cap \Pi X_\alpha \neq \emptyset$, and hence $G \cap UX \neq \emptyset$, a contradiction, so $UX = Y$.

(2) By 3.3(1), each $X_\alpha$ is pc, so $\beta X_\alpha = UX_\alpha$. Thus $\beta(\Pi X_\alpha) = \Pi\beta X_\alpha$ by (1), and hence $\Pi X_\alpha$ is pc by [8, Th. 1].

4.6. *Theorem.* *Suppose that* $X_\alpha$ *has a* c-$\kappa$ *extension* Y *for each* $\alpha \in \Gamma$,

(1) $X = \Pi_\alpha X_\alpha$ *satisfies* (CCC).

(2) *Any* $f \in C(X)$ *depends on countably many coordinates, i.e.,* f *is represented as the composition of a projection onto some countable facet of the product and of a continuous function on this facet.*

(3) *For any* $p \in Y = \Pi Y_\alpha$, *any* $f \in C(\Sigma_p)$ *depends on countably many coordinates, and* f *is continuously extendable over* Y.

*Proof.* (1) $Y = \Pi Y_\alpha$ is a c-$\kappa$ extension of X, and hence (1) follows from 2.1(5).

(2) Follows from (1) and [15, Th. 2].

(3) Since $\Sigma_p$ is dense in Y, $\Sigma_p$ satisfies (CCC), and hence it is an SL-space. It is easily verified that by the method used in [15, p. 19], $\Sigma_p$ is an S-space (for the definition of S-spaces, see [15]). The first part follows from [15, Th. 4]. The latter part follows from the fact that a projection of Y onto a countable facet is onto.

## 5. Special Function Spaces

In this section, we characterize a space X such that every $f \in R^X$ is $\aleph_0$-continuous (strictly $\aleph_0$-continuous) and give related examples. X is said to be *well separated* [9] if any countable discrete closed subset of X is C-embedded. An F-space satisfies 5.1(8) below (see 14.25 and 14M(5) in [7]).

5.1. *Theorem. We have the following implications:* (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Leftrightarrow$ (7) *and* (5) $\Rightarrow$ (8).

(1) *X is a* P-*space.*

(2) *Any countable subset of* X *is* C-*embedded.*

(3) *Any countable subset of* X *is closed and* C-*embedded.*

(4) *Any* $f \in R^X$ *is strictly* $\aleph_0$-*continuous.*

(5) *Any countable subset of* X *is closed and* C*-*embedded.*

(6) *Any countable subset of* X *is closed.*

(7) *Any* $f \in R^X$ *is* $\aleph_0$-*continuous.*

(8) *Any countable subset of* X *is* C*-*embedded.*

*Proof.* (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follow from 3B and 4K(2) in [7] respectively. To prove 5.1, it suffices to show that (3) $\Rightarrow$ (4) and (7) $\Rightarrow$ (6).

(3) $\Rightarrow$ (4) Let $f \in R^X$ and A any countable subset of X. Since any countable subset of X is closed, it is discrete. Thus $f|A$ is continuous, and hence $f|A$ is continuously extended over X, so f is strictly $\aleph_0$-continuous.

(7) $\Rightarrow$ (6) Let A be a countable subset and $p \in \text{cl}A - A$. Then $A \cup \{p\}$ is countable. Define f as follows: $f(p) = 1$ and $f = 0$ on $X - \{p\}$. Then f is not $\aleph_0$-continuous. Thus $\text{cl}A = A$.

5.2. *Corollary.* (1) $R^X = UC_p(X)$ *iff any* $f \in R^X$ *is strictly* $\aleph_0$-*continuous iff any* $f \in R^X$ *is* $\aleph_0$-*continuous and* X *is well-separated.*

(2) *If* X *is countably compact and any* $f \in R^X$ *is* $\aleph_0$-*continuous, then* X *is a finite set.*

(3) *If* X *is* pc *and any* $f \in R^X$ *is strictly* $\aleph_0$-*continuous, then* X *is a finite set.*

5.3. *Examples.* (1) *There exists a non* F-*space* X *satisfying* 5.1(3). Let A be the set of isolated point in $W(\omega_1+1)$, $Z = W(\omega_1+1) \times W(\omega_1+1)$, $A_n = A \times \{n\}$, $p = (\omega_1, \omega)$ and $X = (\cup\{A_n; n \in N\}) \cup \{p\}$. Then the subspace X of Z is a desired space. Indeed, $\cup\{A_n; n \in N\}$ is a cozero set but not C*-embedded, i.e., is not an F-space. Since every point x of $X - \{p\}$ is isolated and p is not contained in the closure of any countable subset, X satisfies 5.1(3).

(2) *There exists a space* X *satisfying* 5.1(6) *but neither* 5.1(4) *nor* 5.1(8). Let A be the set of point of $W(\omega_2+1)$ having an uncountable base of neighborhoods, and B the set of isolated points of $W(\omega_2+1)$. Let us put $Z = W(\omega_2+1) \times W(\omega+1)$, $B_n = B \times \{n\}$, $A^* = A \times \{\omega\}$, $D = \{\omega_2\} \times N$ and $X = (\cup\{B_n; n \in N\}) \cup A^* \cup D$. Then the subspace X of Z is a desired space. Indeed, it is routinely proved that any countable subset is closed, and that D is not C*-embedded. Thus X does not satisfy 5.1(8), hence, X is not an F-space. A function f defined as follows: $f = (-1)^n$ on $B_n \cup (\omega_2, n)$, $n \in N$, and $f = 0$ on $A^*$, is $\aleph_0$-continuous but not strictly $\aleph_0$-continuous. Thus X does not satisfy 5.1(4).

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