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EVERY STRICT ρ-SPACE IS θ-REFINABLE

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EVERY STRICT p-SPACE IS θ-REFINABLE

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1. Introduction

Is every strict p-space θ-refinable? This question has been asked many times and there are many partial results in the literature ([G], [CJ], [D_2], [W]). We prove here that the answer is yes. (Note that perhaps the more modern term for θ-refinable is submetacompact.)

For a nice discussion of the history of the question see the paper of S. Davis ([D_1]). Davis points out that our yes answer to this question solves other unsolved questions:

1. Does θ-refinable p-space characterize strict p-space? (Yes.)

2. Is every strict p-space with a G_δ-diagonal developable? (Yes.)

3. Is every perfect image of a strict p-space also a strict-p space? (Yes.)

2. Main Result

Every strict p-space is θ-refinable.

We use the characterization of ($T_{3\frac{1}{2}}$) strict p-spaces by D. Burke ([B]):

We assume we have a space X and a countable family $\{\mathcal{G}_n | n \in \omega\}$ of open covers of X such that if, for each $x \in X$, $P_x = \bigcap_{n\in\omega}(\mathcal{G}_n^*(x))$, then

(a) P_x is compact, and

(b) if U is an open set with $P_x \subset U$, there is an $n \in \omega$ with $\mathcal{G}_n^*(x) \subset U$.

Our space X is then said to be θ-refinable if for every open cover $\mathcal{U}$ of X there is a countable family $\mathbb{H}$ of open covers of X each refining $\mathcal{U}$ such that, for every $x \in X$, there is a $\mathcal{H} \in \mathbb{H}$ such that x belongs to at most finitely many members of $\mathcal{H}$.

So we assume that $\mathcal{U}$ is an open cover of X. We then use the powerful theorem of H. Junnila ([J]) which says, we may assume that there is an ordinal κ such that $\mathcal{U} = \{U_\alpha | \alpha < \kappa\}$ and that for $\alpha < \beta < \kappa$, $U_\alpha \subset U_\beta$.

We assume, without loss of generality, that for all $n \in \omega$, $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$. Then by our assumption on the family $\{\mathcal{G}_n | n \in \omega\}$:

Lemma 1. *If* $x \in X$, J *is an infinite subset of* ω *and for each* $j \in J$ *there is a point* x_j *and a term* G_j *of* $\mathcal{G}_j$ *with* x *and* x_j *both in* G_j, *then* $\overline{\{x_j | j \in J\}} \cap P_x \neq \emptyset$.

Proof that $\mathcal{U}$ has a θ-refinement.

For each $x \in X$, define

$$\alpha(x) = \min\{\alpha | x \in U_\alpha\}$$

$$\beta(x) = \min\{\beta | P_x \subset U_\beta\}$$

$$e(x) = \min\{n | st(x,\mathcal{G}_n) \subset U_{\beta(x)}\}.$$

If $y \in X$ and $j \in \omega$, choose a finite subset $\mathcal{G}_{jy}$ from $\{G \in \mathcal{G}_j | y \in G\}$ with $P_y \subset \cup\mathcal{G}_{jy}$ and define $G_{jy} = (\cap\mathcal{G}_{jy}) \cap U_{\alpha(y)}$.

For each $m \in \omega$ we define by induction a countable family $\mathbb{H}_m$ of open covers of X refining $\mathcal{U}$. Our θ-refinement will be $\mathbb{H} = \cup_{m\in\omega}\mathbb{H}_m$.

Define $\mathbb{H}_0 = \{\mathcal{U}\}$.

Assume $m \in \omega$ and that $\mathbb{H}_m$ has been defined; we proceed to construct $\mathbb{H}_{m+1}$.

Let $F = \cup F_n$, where $F_n = \{f | f: n + 1 \to \omega\}$.

Fix $\mathcal{H} \in \mathbb{H}_m$, if $n \in \omega$, define $\mathcal{H}_n = \{\mathcal{V} \subset \mathcal{H} | |\mathcal{V}| = n + 1\}$.

By induction, for each $f \in F_n$, we define a family $\mathcal{G}_{\mathcal{H}f}$ of open sets.

Define $\mathcal{G}_{\mathcal{H}\phi} = \phi$.

Suppose we are given $f: n + 1 \to \omega$ from F_n and that $\mathcal{G}_{\mathcal{H}(f \restriction n)}$ has been defined. If $\mathcal{V} \in \mathcal{H}_n$, define

$$G_{\mathcal{V}f} = \cup\{G_{f(n)y} | \mathcal{V} = \{V \in \mathcal{H} | y \in V\} \text{ and } y \notin \cup\mathcal{G}_{\mathcal{H}(f \restriction n)}\}.$$

Define $\mathcal{G}_{\mathcal{H}f} = \mathcal{G}_{\mathcal{H}(f \restriction n)} \cup \{G_{\mathcal{V}f} | \mathcal{V} \in \mathcal{H}_n\}$.

Let $\mathcal{A}_m = \{A | A \subset \mathbb{H}_m \times F, |A| < \omega\}$ and $\mathcal{B}_m = \{\langle A, i, j\rangle | A \in \mathcal{A}_m \text{ and } i, j \in \omega\}$.

For $A \in \mathcal{A}_m$, define $\mathcal{H}_A = \cup\{\mathcal{G}_{\mathcal{H}f} | \langle \mathcal{H}, f\rangle \in A\}$.

For $B = \langle A, i, j\rangle \in \mathcal{B}_m$ and $\beta < \kappa$, define $H_{B\beta} = \cup\{G_{jy} | y \notin \cup\mathcal{H}_A, e(y) \leq i \text{ and } \beta(y) = \beta\}$, then let $\mathcal{H}_B = \{H_{B\beta} | \beta < \kappa\}$.

If $\mathcal{C}$ is a finite subset of $\mathcal{A}_m \cup \mathcal{B}_m$, let $\mathcal{H}_\mathcal{C} = \cup\{\mathcal{H}_C | C \in \mathcal{C}\}$.

If $k \in \omega$, define

$$\mathcal{H}_{k\mathcal{C}} = \mathcal{H}_\mathcal{C} \cup \{G \cap U_\alpha | G \in \mathcal{G}_k, \alpha < \kappa, G \setminus \cup\mathcal{H}_\mathcal{C} \neq \phi\}.$$

Finally $\mathbb{H}_{m+1} = \mathbb{H}_m \cup \{\mathcal{H}_{k\mathcal{C}} | k \in \omega, \mathcal{C} \text{ is a finite subset of } \mathcal{A}_m \cup \mathcal{B}_m\}$.

We want to prove that $\mathbb{H} = \cup_{m\in\omega}\mathbb{H}_m$ is a θ-refinement of $\mathcal{U}$.

Certainly by the definition of $\mathcal{H}_{k\mathcal{C}}$, $\mathbb{H}$ is a family of open covers of X.

Since $\mathbb{H}_0 = \{\mathcal{U}\}$, we can assume inductively that $\mathbb{H}_m$ is countable. Then since F is countable, $\mathcal{A}_m$ is countable, $\mathcal{B}_m$ is countable and $\mathbb{H}_{m+1}$ is countable. Hence $\mathbb{H}$ is a countable family.

Again since $\mathcal{H}_0 = \{\mathcal{U}\}$, we can assume inductively that each $\mathcal{H} \in \mathcal{H}_m$ is a refinement of $\mathcal{U}$. Thus for a fixed $\mathcal{H} \in \mathcal{H}_m$, if $\mathcal{V} \in \mathcal{H}_n$, $(\cap\mathcal{V}) \subset U_\alpha$ for some $\alpha < \kappa$. But $G_{\mathcal{V}f}$ is the union of sets of form $G_{f(n)y}$, $G_{f(n)y} \subset U_{\alpha(y)}$ for some y in $\cap\mathcal{V}$, for all these y's, $\alpha(y) < \alpha$, hence $U_{\alpha(y)} \subset U_\alpha$, so $G_{\mathcal{V}f} \subset U_\alpha$. Thus for $A \in \mathcal{A}_m$, $\mathcal{G}_A$ refines $\mathcal{U}$.

For $B \in \mathcal{B}_m$, since $\alpha(y) \leq \beta(y)$ and $G_{jy} \subset U_{\alpha(y)}$ for all $j \in \omega$, $H_{B\beta} \subset U_\beta$. Thus each term of $\mathcal{H}_{m+1}$ refines $\mathcal{U}$.

This shows that each term of $\mathcal{H}$ refines $\mathcal{U}$.

Now define $Y = \{y \in X \mid$ there is a $\mathcal{H} \in \mathcal{H}$ such that y is in at most finitely many members of $\mathcal{H}\}$.

If X = Y we have now shown that $\mathcal{H}$ is a θ-refinement of $\mathcal{U}$.

Otherwise there is an $x \in X\setminus Y$ such that $\beta(x) = \min\{\beta(y) \mid y \in X\setminus Y\}$. Fix one such x, we prove that the existence of such an x leads to a contradiction.

First we prove a lemma.

Lemma 2. *Suppose* $\mathcal{H} \in \mathcal{H}$. *There is an* $f: \omega \to \omega$ *such that for all* $n \in \omega$, x *belongs to only finitely many members of* $\mathcal{G}_{\mathcal{H}(f \restriction n)}$.

Proof. We define f inductively.

Let $f \restriction 0 = \phi$.

Assume $f \restriction n$ has been defined. For $i \in \omega$ define $f_i \in F_n$ by $f_i \restriction n = f \restriction n$ and $f_i(n) = i$.

Suppose that for every i, there exists $\mathcal{V}_i \in \mathcal{H}_n$ with $x \in G_{\mathcal{V}_i f_i}$ and the elements of $\{\mathcal{V}_i \mid i \in \omega\}$ distinct.

By the Δ-system lemma, there are an infinite subset J of ω and an $\mathcal{R} \subset \mathcal{H}$ with $\mathcal{V}_i \cap \mathcal{V}_j = \mathcal{R}$ for all $i \neq j$ in J. Observe that for $i \in J$, $\mathcal{V}_i \not\subset \mathcal{R}$, since $\{\mathcal{V}_i | i \in \omega\}$ are distinct.

For each i, since $x \in G_{\mathcal{V}_i f_i}$, we can choose an x_i with $\mathcal{V}_i = \{V \in \mathcal{H} | x_i \in V\}$, $x_i \notin \cup\mathcal{G}_{\mathcal{H}(f\restriction n)}$, and $x \in G_{ix_i}$. There is a limit point p of $\{x_i | i \in J\}$ in P_x, since $\overline{\{x_i | i > h, i \in J\}} \cap P_x \neq \phi$ for any $h \in \omega$ by Lemma 1.

Let $\mathcal{V} = \{V \in \mathcal{H} | p \in V\}$. Observe that $\mathcal{V}$ is finite, actually $|\mathcal{V}| \leq n + 1$. If $|\mathcal{V}| > n + 1$ there is a subset $\mathcal{Y}$ of $\mathcal{V}$ with $|\mathcal{Y}| = n + 2$. There is an $i \in J$ with $x_i \in \cap\mathcal{Y}$. But then $|\mathcal{V}_i| \geq n + 2$, contradicting $\mathcal{V}_i \in \mathcal{H}_n$.

Thus $\cap\mathcal{V}$ is an open neighborhood of p. Choose $i \neq j$ in J with x_i, x_j in $\cap\mathcal{V}$. Then $\mathcal{V} \subset \mathcal{V}_i \cap \mathcal{V}_j \subset \mathcal{R}$, so $\mathcal{V} \in \mathcal{H}_{\zeta}$ for some $\zeta < n$. But then $p \in \cup\mathcal{G}_{\mathcal{H}(f\restriction n)}$, since $\mathcal{G}_{\mathcal{H}(f\restriction n)}$ covers all points belonging to only $\leq n$ members of $\mathcal{H}$. Thus there is an $x_h \in \cup\mathcal{G}_{\mathcal{H}(f\restriction n)}$ contradicting our definition of the x_h's.

Since the proof of Lemma 2 is complete, we return to the proof that there is no $x \in X \setminus Y$ with $\beta(x)$ minimal.

For $i \in \omega$, define $X_i = \{p \in X | e(p) \leq i\}$. Let $S = \{p \in X | \alpha(x) \leq \alpha(p) \leq \beta(p) < \beta(x)\}$.

Lemma 3. $\overline{S \cap X_i} \subset S$.

Proof. Suppose $z \in \overline{S \cap X_i}$. Since $\alpha(y) \geq \alpha(x)$ for all $y \in S$, $\alpha(z) \geq \alpha(x)$.

Suppose $\beta(z) \geq \beta(x)$. Since $z \in G_{iz}$, there is a $y \in G_{iz} \cap S \cap X_i$. Thus $\beta(y) < \beta(x)$ and $e(y) \leq i$, so $st(y, \mathcal{G}_i) \subset U_{\beta(y)} \subset U_{\beta(x)}$. But there is a point $w \in P_z$ with $\alpha(w) = \beta(z)$ and a $G \in \mathcal{G}_{iz}$ with $w \in G$. Since $G_{iz} = (\cap\mathcal{G}_{iz}) \cap U_{\alpha(z)}$, $w \in G \in \mathcal{G}_i$ and $y \in G$.

Since $\alpha(w) > \beta(y)$, $G \not\subset U_{\beta(y)}$, but $st(y, \mathcal{G}_i) \subset U_{\beta(y)}$ which is a contradiction. This proves the Lemma 3.

By the minimality of $\beta(x)$, if $q \in S$, there are $n \in \omega$ and $\mathcal{H} \in \mathbb{H}$ such that q belongs to exactly n members of $\mathcal{H}$. If f satisfies the conditions of Lemma 2 for this $\mathcal{H}$, then $q \in \cup \mathcal{G}_{\mathcal{H}(f \upharpoonright n)}$ and x belongs to only finitely many members of $\mathcal{G}_{\mathcal{H}(f \upharpoonright n)}$.

Since for a fixed $i \in \omega$ $P_x \cap \overline{(S \cap X_i)}$ is a closed subset of the compact set P_x, it is compact.

So we can choose finitely many $\mathcal{H}_0, \cdots, \mathcal{H}_\ell \in \mathbb{H}$ and $f_0, \cdots, f_\ell \in F$ such that

$$P_x \cap \overline{(S \cap X_i)} \subset \cup_{n=0}^{\ell} \cup \mathcal{G}_{\mathcal{H}_h f_h},$$

and for each $h \leq \ell$, x belongs to only finitely many members of $\mathcal{G}_{\mathcal{H}_h f_h}$.

Find m_i so that $\mathcal{H}_h \in \mathbb{H}_{m_i}$, $h = 0, \cdots, \ell$.

Define $A_i = \{\langle \mathcal{H}_0, f_0 \rangle, \cdots, \langle \mathcal{H}_\ell f_\ell \rangle\}$. Then $P_x \cap \overline{(S \cap X_i)} \subset \cup \mathcal{H}_{A_i}$ and x belongs to only finitely many members of $\mathcal{H}_{A_i}$.

Lemma 4. *For all* $i \in \omega$, *there is* $j \in \omega$ *such that* x *belongs to at most finitely many* $H_{B_j \beta}$ *where* $B_j = \langle A_i, i, j \rangle$.

Proof. Suppose that for each $j \in \omega$, $x \in H_{B_j \beta}$ for infinitely many $\beta < \kappa$. Then there are $\beta_0 < \beta_1 < \cdots$ and that $\beta(x_j) = \beta_j$ and $x \in G_{jx_j}$.

Observe each $x_j \in S$. Certainly $\alpha(x_j) \geq \alpha(x)$, since $x \in G_{jx_j}$ and $G_{jx_j} \subset U_{\alpha(x_j)}$. Also $\beta(x_j) < \beta(x)$. For if $\beta(x_j) \geq \beta(x)$, choose $h > j + e(x)$, then $\beta(x_h) > \beta(x_j) \geq \beta(x)$

so there is a $G \in \mathcal{G}_{hx_h}$ with $G \not\subset U_{\beta(x)}$. However this contradicts $e(x) < h$ and $x \in G \in \mathcal{G}_h$ and the assumption $x \in G_{hx_h} \subset G$.

Since for each $j \in \omega$, G_{jx_j} is contained in a member of $\mathcal{G}_j$ and both x and x_j are in G_{jx_j}, there is a $p \in P_x \cap \overline{\{x_j | j \in \omega\}}$ by Lemma 1. Since $\overline{\{x_j | j \in \omega\}} \subset S$ by Lemma 3, we have $p \in S$. Since $p \in P_x \cap \overline{(S \cap X_i)}$, by our choice of A_i, $p \in \cup\mathcal{H}_{A_i}$, which is open. Thus there is an $x_j \in \cup\mathcal{H}_{A_i}$. But this contradicts our choice of x_j's and proves Lemma 4.

If $i > e(y)$ for some $y \in X$, then $y \in \cup(\mathcal{H}_{\langle A,i,j \rangle} \cup \mathcal{H}_A)$ for all A and j. Thus since P_x is compact, by Lemma 4 there is a finite set I of i's and for each $i \in I$ a j_i such that P_x is covered by $\cup_{i \in I}(\mathcal{H}_{\langle A_i,i,j_i \rangle} \cup \mathcal{H}_{A_i})$ and x belongs to only finitely many terms of $\cup(\mathcal{H}_{\langle A_i,i,j_i \rangle} \cup \mathcal{H}_{A_i})$.

Let $m = \max\{m_i | i \in I\}$. Then $A_i \in \mathcal{A}_m$ for all $i \in I$ and $B_i = \langle A_i,i,j_i \rangle \in \mathcal{B}_m$. If $\mathcal{C} = \{A_i | i \in I\} \cup \{B_i | i \in I\}$, then $P_x \subset \cup\mathcal{H}_{\mathcal{C}}$ and x belongs to only finitely many members of $\mathcal{H}_{\mathcal{C}}$. There is a $k \in \omega$ with $st(x,\mathcal{G}_k) \subset \cup\mathcal{H}_{\mathcal{C}}$. Thus $\mathcal{H}_{k\mathcal{C}} \in \mathbb{H}_{m+1}$ and x belongs to at most finitely many members of $\mathcal{H}_{k\mathcal{C}}$ contradicting the assumption that there is no term of $\mathbb{H}$ having x in only finitely many of its members.

This proves that $\mathbb{H}$ is a θ-refinement of $\mathcal{U}$.

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References

[B] D. K. Burke, *On p-spaces and wΔ-spaces*, Pacific J. Math. 35 (1970), 285-296.

[BS] ________ and R. A. Stoltenberg, *A note on p-spaces and Moore spaces*, Pacific J. Math. 30 (1969), 601-608.

[CJ] J. Chaber and H. J. K. Junnila, *On θ-refinability of strict p-spaces*, Gen. Top. Appl. 10 (1979), 233-236.

[D_1] S. W. Davis, *The strict p-space problem*, preprint.

[D_2] ________, *Covering properties of strict p-spaces*, Abstracts, Amer. Math. Soc. 1 (1980), 615.

[G] R. F. Gittings, *Subclasses of p-spaces and strict p-spaces*, Top. Proc. 3 (1978), 335-346.

[Gr] G. Gruenhage, *Generalized metric spaces*, Handbook of Set-Theoretic Topology, 423-501. North-Holland.

[J] H. J. K. Junnila, *On submetacompactness*, Top. Proc. 3 (1978), 375-405.

[Wa] K. A. Wagner, *θ-refinability and strict p-spaces*, Ph.D. Thesis, University of Pittsburgh, 1985.

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