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by

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A PROPERTY RELATED TO THE STRICT p-SPACE PROBLEM

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0. Introductions and Definitions

In this article, we study a property which arose from work on the so-called "strict p-space problem." That problem has recently been solved by S. Jiang [J], but the behavior of the property seems sufficiently interesting that we study it here "for its own sake."

0.1. *Definition* [D]. We say that a space X satisfies (*) if and only if for every open cover $\mathcal{U}$ of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of open covers of X, each refining $\mathcal{U}$, such that for each $x \in X$ there exists $n_x \in \omega$ with $\overline{st(x, \mathcal{V}_{n_x})} \subseteq st(x, \mathcal{U})$. If this can be done so that $\mathcal{V}_n = \mathcal{V}_0$ for all $n \in \omega$, i.e. with one refinement, then we say X satisfies strong (*).

This property is useful for getting from G_δ-diagonal to G_δ^*-diagonal (both defined below). That is, it was shown in [D] that if a space X has G_δ-diagonal and satisfies (*), then X has G_δ^*-diagonal. In particular, if strict p-spaces always satisfied (*), then strict p-spaces with G_δ-diagonal would be developable. This is, of course, made obsolete by Jiang's result. It also follows from an old result of Hodel [Ho] that if X is a wΔ-space with G_δ-diagonal which satisfies (*), then X is developable. So this property may be of help on the following.

0.2. *Open Question* [B_1].[1] Is every $w\Delta$-space with G_δ-diagonal developable?

To begin this study of (*), we make a few remarks about preservation to subspaces and easy consequences of these remarks.

0.3. *Remark.* Closed subspaces inherit (*).

The proof is routine and is omitted. One consequence of this is that neither irreducible nor σ-minimal base can imply (*). It is shown in [DS] that every space can be embedded as a closed subspace of an irreducible space, and it is shown in [B_2] every space can be embedded as a closed subspace of a space with σ-minimal base.

0.4. *Remark.* Open subspaces need not inherit (*).

In section 2, we will describe an example, due to Burke, which is locally compact T_2 and does not satisfy (*). Such a space will be an open subspace of any of its compactifications.

We close this section with a list of definitions. The references given contain the definition but may not be the original source.

0.5. *Definitions.*

0.5.0 [Bo]. A space X is a *$w\Delta$-space* if and only if there is a sequence $\langle \mathcal{G}_n : n \in \omega \rangle$ of open covers of X such

[1]This question has recently been shown to have consistent negative answers under CH by K. Alster, D. Burke and S. Davis and under $\underline{b} = \underline{c}$ by Z. Balogh.

that if $x \in X$ and $x_n \in st(x,\mathcal{G}_n)$ for each $n \in \omega$, then $\langle x_n: n \in \omega \rangle$ has a cluster point.

0.5.1 [C]. A space X has G_δ-*diagonal* if and only if there is a sequence $\langle \mathcal{G}_n: n \in \omega \rangle$ of open covers of X such that if $x \in X$, then $\{x\} = \bigcap_{n\in\omega} st(x,\mathcal{G}_n)$.

0.5.2 [Ho]. A space X has G_δ^*-*diagonal* if and only if there is a sequence $\langle \mathcal{G}_n: n \in \omega \rangle$ of open covers of X such that if $x \in X$, then $\{x\} = \bigcap_{n\in\omega} \overline{st(x,\mathcal{G}_n)}$.

0.5.3 [WW]. A space X is θ-*refinable* if and only if for every open cover $\mathcal{U}$ of X there exists a sequence $\langle \mathcal{V}_n: n \in \omega \rangle$ of open covers of X, each refining $\mathcal{U}$, such that for each $x \in X$ there exists $n_x \in \omega$ such that $\{V: x \in V \in \mathcal{V}_{n_x}\}$ is finite.

0.5.4 [SW]. A space X is *orthocompact* if and only if for every open cover $\mathcal{U}$ of X there exists an open cover $\mathcal{V}$ of X, refining $\mathcal{U}$, such that $\cap\mathcal{W}$ is open for any $\mathcal{W} \subseteq \mathcal{V}$.

0.5.5 [L]. A space X is a GO-*space* if and only if X can be embedded in a linearly ordered topological space (LOTS, for short).

0.5.6 [NS]. A space X is a $\sigma\#$-*space* if and only if X has a σ-closure preserving point separating closed cover.

0.5.7 [M]. Let $\kappa \geq 2$ be a cardinal. A space X is κ-*fully normal* if and only if for every open cover $\mathcal{U}$ of X there exists an open cover $\mathcal{V}$ of X, refining $\mathcal{U}$, such that if $\mathcal{W} \subseteq \mathcal{V}$ with $|\mathcal{W}| \leq \kappa$ and $\cap\mathcal{W} \neq \emptyset$, then there is $U \in \mathcal{U}$ with $\cup\mathcal{W} \subseteq U$. Note we call $\mathcal{V}$ a κ-star refinement of $\mathcal{U}$.

0.5.8 [M]. Let $\kappa \geq 2$ be a cardinal. A space X is *almost κ-fully normal* if and only if for every open cover $\mathcal{U}$ of X there exists an open cover $\mathcal{V}$ of X, refining $\mathcal{U}$, such that if $x \in X$ and $A \subseteq st(x, \mathcal{V})$ with $|A| \leq \kappa$, then there is $U \in \mathcal{U}$ with $A \subseteq U$.

0.5.9 [H_1]. A space X is *Δ-normal* (or diagonal normal) if and only if ΔX, the diagonal in $X \times X$, has a closed neighborhood base, i.e. if U is open and $\Delta X \subseteq U$, then there is open V such that $\Delta X \subseteq V \subseteq \overline{V} \subseteq U$.

We follow the usual conventions about cardinals and ordinals, viz, a cardinal is an initial ordinal, ω is the set of natural numbers, etc. All ordinals are assumed to have the order topology.

1. Relationships—Positive Results

The main result of this section is that (*) is equivalent to several other properties for spaces of form $\alpha \times \beta$, α and β being ordinals. Before doing that, we establish several results indicating roughly where (*) is in some of the hierarchies of properties commonly studied.

1.1. *Remark.* Suppose X is regular. If X is metacompact, then X satisfies strong (*), and if X is θ-refinable, then X satisfies (*).

This is easy, and we omit the proof. The need for regularity is evident from the following.

1.2. *Remark.* If X is a T_1-space and satisfies (*), then X is regular.

Proof. Suppose $x \in X$ and $U \subseteq X$ is open with $x \in U$. Then $\mathcal{U} = \{U, X \setminus \{x\}\}$ is an open cover of X, and if $\overline{st(x, \mathcal{V}_{n_x})} \subseteq st(x, \mathcal{U}) = U$, we have regularity.

We now exhibit a separation property which implies (*), and in fact strong (*).

1.3. *Theorem. Every* Δ*-normal space satisfies strong* (*).

Proof. Suppose X is Δ-normal, and let $\mathcal{U}$ be an open cover of X. Let $U^* = \cup\{U \times U: U \in \mathcal{U}\}$ and find $V^* \subseteq X \times X$ open with $\Delta X \subseteq V^* \subseteq \overline{V^*} \subseteq U^*$. Let $\mathcal{V} = \{V: V$ is open, $V \subseteq U$ for some $U \in \mathcal{U}$, $V \times V \subseteq V^*\}$. Suppose $y \in \overline{st(x, \mathcal{V})}$, and let G be a neighborhood of (x,y) in $X \times X$. Now G contains a set of form $G_1 \times G_2$ where G_1, G_2 are open in X, $x \in G_1$, and $y \in G_2$. Hence there is a $V \in \mathcal{V}$ with $x \in V$ and $G_2 \cap V \neq \emptyset$, say $z \in G_2 \cap V$. Then $(x,z) \in (G_1 \times G_2) \cap (V \times V) \subseteq G \cap V^*$. Hence $(x,y) \in \overline{V}^* \subseteq U^*$. So there is $U \in \mathcal{U}$ with $(x,y) \in U \times U$, i.e. $y \in st(x, \mathcal{U})$.

It is shown in [H_1] that almost 2-fully normal spaces are Δ-normal, in [M] that all LOTS are ω-fully normal, and in [L] that all GO-spaces are homeomorphic to closed subspaces of LOTS. In particular, it follows from the above that all GO-spaces, and hence all LOTS, satisfy strong (*).

We now turn to the main result of this section. We will accomplish this theorem via a sequence of small steps, looking at spaces $\alpha \times \beta$ for several special ordinals α and β.

1.4. *Theorem. If* κ *is an uncountable regular cardinal, then* $\kappa \times (\kappa + 1)$ *does not satisfy* (*).

Proof. Let $\mathcal{U} = \{\kappa \times \kappa\} \cup \{[0,\alpha] \times (\alpha,\kappa]: \alpha < \kappa\}$, and suppose $\langle \mathcal{V}_n: n \in \omega \rangle$ is any sequence of open covers refining $\mathcal{U}$. For each $n \in \omega$, for each $\alpha < \kappa$, choose $f_n(\alpha) < \alpha$ such that $(f_n(\alpha),\alpha] \times (f_n(\alpha),\alpha] \subseteq V$ for some $V \in \mathcal{V}_n$. By the Pressing Down Lemma, we choose $\gamma_n < \kappa$ and a stationary set $S_n \subseteq \kappa$ such that $f_n(\alpha) = \gamma_n$ for all $\alpha \in S_n$. Since S_n is cofinal in κ, note that for any $\alpha > \gamma_n$, $(\alpha,\kappa) \in \overline{st((\alpha,\alpha),\mathcal{V}_n)} \setminus st((\alpha,\alpha),\mathcal{U})$. Since $cf(\kappa) > \omega$, $\gamma = \sup\{\gamma_n: n \in \omega\} + 1 < \kappa$ and $(\gamma,\kappa) \in \overline{st((\gamma,\gamma),\mathcal{V}_n)} \setminus st((\gamma,\gamma),\mathcal{U})$ for every $n \in \omega$, so (*) fails.

1.5. *Theorem. If* α, β, γ *are ordinals with* $\alpha = \gamma + 1 < cf(\beta)$, *then* $\alpha \times \beta$ *is* ω*-fully normal.*

Proof. First note that if $cf(\beta) \leq \omega$, then $\alpha \times \beta$ is a finite disjoint union of clopen copies of β, so $\alpha \times \beta$ is ω-fully normal. So we assume $cf(\beta) > \omega$.

Now suppose α is the smallest counterexample, and suppose $\mathcal{U}$ is an open cover of $\alpha \times \beta$. Choose a closed cofinal subset $C \subseteq \beta$ which is order isomorphic to $cf(\beta)$. For each $\delta \in C$ choose $f(\delta) < \gamma$ and $s(\delta) < \delta$ such that $(f(\delta),\gamma] \times (s(\delta),\delta] \subseteq U$ for some $U \in \mathcal{U}$. Applying the Pressing Down Lemma on the uncountable regular cardinal $cf(\beta)$, we can find $\eta < \beta$ and $C' \subseteq C$ with C' cofinal in β and $s(\delta) < \eta$ for all $\delta \in C'$. Since $|C'| = cf(\beta) > \gamma$, there exists $C'' \subseteq C'$, also cofinal in β, and $\mu < \gamma$ such that $f(\delta) = \mu$ for every $\delta \in C''$. Now $\{(\mu,\gamma] \times (\eta,\delta]: \delta \in C''\}$ is an increasing open cover of $(\mu,\gamma] \times (\eta,\beta)$ with uncountable cofinality, hence is an ω-star refinement of itself, and therefore of $\mathcal{U}$. By minimality of α, there is an open ω-star refinement of $\mathcal{U}$ whose union is the clopen subspace $[0,\mu] \times \beta$. By compactness, there is an open ω-star refinement of $\mathcal{U}$ whose union is the clopen subspace $[\mu + 1,\gamma] \times [0,\eta]$. Hence $\alpha \times \beta$ is ω-fully normal. Thus no minimal counterexample can exist, and the result is established.

1.6. *Theorem. If* κ *is regular, then* $\kappa \times \kappa$ *is* ω*-fully normal.*

Proof. If $\kappa \leq \omega$, then $\kappa \times \kappa$ is σ-compact, and the result is true. Suppose $\kappa > \omega$ and $\mathcal{U}$ is an open cover of $\kappa \times \kappa$. For each $\alpha < \kappa$, choose $f(\alpha) < \alpha$ such that $(f(\alpha),\alpha] \times (f(\alpha),\alpha] \subseteq U$ for some $U \in \mathcal{U}$. By the Pressing Down Lemma, there exist $\gamma < \kappa$ and a stationary set $S \subseteq \kappa$ such that $f(\alpha) = \gamma$ for all $\alpha \in S$. Now $\{(\gamma,\alpha] \times (\gamma,\alpha]: \alpha \in S\}$ is an ω-star refinement of $\mathcal{U}$ covering $(\gamma,\kappa) \times (\gamma,\kappa)$, and $\kappa \times [0,\gamma], [0,\gamma] \times [\gamma + 1,\kappa)$ are ω-fully normal by 1.5 and complete a clopen partition of $\kappa \times \kappa$. The result now follows.

1.6.1. *Corollary. For any ordinal* α, $\alpha \times \alpha$ *satisfies* (*) *if and only if* $\mathrm{cf}(\alpha) \leq \omega$ *or* $\mathrm{cf}(\alpha) = \alpha$.

Proof. If $\mathrm{cf}(\alpha) \leq \omega$ or $\mathrm{cf}(\alpha) = \alpha$, then (*) follows by σ-compactness or ω-fully normal, respectively. If $\omega < \mathrm{cf}(\alpha) < \alpha$, then $\alpha \times \alpha$ contains a closed copy of $\mathrm{cf}(\alpha) \times (\mathrm{cf}(\alpha) + 1)$; hence $\alpha \times \alpha$ does not satisfy (*).

1.7. *Lemma. If* $\mathrm{cf}(\alpha) \leq \omega$ *and* $\alpha < \mathrm{cf}(\beta)$, *then* $\alpha \times \beta$ *is* ω*-fully normal.*

Proof. We first note that for any successor $\gamma \leq \alpha$, we have from 1.5 that $\gamma \times \beta$ is ω-fully normal. Suppose $\mathrm{cf}(\alpha) = \omega$. Choose a sequence $\langle \alpha_n : n \in \omega \rangle$ strictly increasing with $\alpha_n \to \alpha$. For each $n \in \omega$, $[\alpha_n + 1, \alpha_{n+1}]$ is homeomorphic to a successor $\gamma_n < \alpha$. Now $\alpha \times \beta$ is homeomorphic to the free union $(\alpha_0 + 1) \times \beta \uplus (\uplus_{n\in\omega}\gamma_n \times \beta)$, and thus is ω-fully normal.

1.8. *Theorem. Suppose* α *and* β *are ordinals with* $\alpha \leq \beta$. *The following are equivalent:*

(1) $\alpha \times \beta$ *is collectionwise normal*

(2) $\alpha \times \beta$ *is normal.*

(3) $\alpha \times \beta$ *is* ω*-fully normal.*

(4) $\alpha \times \beta$ *is almost* ω*-fully normal.*

(5) $\alpha \times \beta$ *is* 2*-fully normal.*

(6) $\alpha \times \beta$ *is almost* 2*-fully normal.*

(7) $\alpha \times \beta$ *is* Δ*-normal.*

(8) $\alpha \times \beta$ *satisfies strong* (*).

(9) $\alpha \times \beta$ *satisifes* (*).

(10) $\alpha \times \beta$ *is orthocompact*

(11) *One of the following is true:*

a) $\mathrm{cf}(\alpha) \leq \omega$ *and* $\mathrm{cf}(\beta) \leq \omega$.

b) $\alpha = \beta = \mathrm{cf}(\alpha)$.

c) $\mathrm{cf}(\alpha) \leq \omega$ *and* $\alpha \leq \mathrm{cf}(\beta)$.

Proof. It is shown in [S] that (2) $\leftrightarrow$ (10) $\leftrightarrow$ (11). It is clear that (3) $\to$ (4) $\to$ (6), (3) $\to$ (5) $\to$ (6), (8) $\to$ (9) and (1) $\to$ (2). It is shown in [Co] that (6) $\to$ (1)

and in [H] that (6) → (7). By 1.3, (7) → (8). Hence, to complete the proof, we need to show that (9) → (11) and (11) → (3).

Suppose (11) is true. If a) holds, then $\alpha \times \beta$ is σ-compact. If $\alpha = \beta = cf(\alpha)$, then $\alpha \times \beta$ is ω-fully normal by 1.6. Suppose c) holds. If $\alpha = cf(\beta)$, then again $\alpha \times \beta$ is σ-compact. If $\alpha < cf(\beta)$, then by 1.7 $\alpha \times \beta$ is ω-fully normal. Hence (11) → (3).

Now suppose (11) fails. First suppose $\alpha = \beta$. By 1.6.1, $\alpha \times \beta$ does not satisfy (*). Suppose $\alpha < \beta$. If $cf(\alpha) > \omega$, then $\alpha \times \beta$ contains a closed copy of $cf(\alpha) \times (cf(\alpha) + 1)$; hence $\alpha \times \beta$ does not satisfy (*). Suppose $cf(\alpha) \leq \omega$. Since c) fails, we have $\alpha > cf(\beta)$. Thus $\alpha \times \beta$ contains a closed copy of $(cf(\beta) + 1) \times cf(\beta)$; hence $\alpha \times \beta$ does not satisfy (*). This exhausts the cases. So (9) → (11), and the proof is complete.

2. Examples

In this section, we present several examples showing where implications between (*) and certain other properties fail. In particular, we were interested in comparisons of (*) with the properties of orthocompactness and being a $\sigma^{\#}$-space. Our reason for being interested in these stems from two results which were conveyed to the authors by D. K. Burke. Thye are (1) a $w\Delta$-space is developable if and only if it is a $\sigma^{\#}$-space [B_3], and (2) any orthocompact space with G_δ-diagonal is a $\sigma^{\#}$-space.

2.1. *Example*. The ordinal space ω_1 is a countably compact space (hence, $w\Delta$-space) satisfying strong (*) (and many other things) which is not a $\sigma\#$-space.

2.2. *Example*. Gruenhage's "orthocompactness-killing" machine applied to ω_1, which we will denote $G(\omega_1)$, satisfies strong (*) (and many other things, see [G]) and is not orthocompact.

Proof. The underlying set of $G(\omega_1)$ is $\omega_1 \cup (\omega_1 \times \omega_1)$. Points of $\omega_1 \times \omega_1$ are isolated. Neighborhoods of $\alpha \in \omega_1$ have form $U \cup (U \times (\omega_1 \setminus F))$ where U is a neighborhood of α in ω_1 and $F \subseteq \omega_1$ is finite. It is shown in [G] that this space is not orthocompact. Junnila has shown in [Ju] that this space is almost 2-fully normal, hence satisfies strong (*).

2.3. *Example* [FL]. There is a Moore space (hence, satisfying (*) and having G_δ-diagonal) which is not orthocompact.

The properties are verified in [FL] where the space is attributed to R. W. Heath. So we shall simply describe it here. Let $R = \{(x,y) \in \mathbb{Q}^2 : y > 0\}$. Now $X = R \cup [(\mathbb{R}\setminus\mathbb{Q}) \times \{0\}]$. Points of R have Euclidean neighborhoods. For $x \in \mathbb{R}\setminus\mathbb{Q}$, $\{\{(x,0)\} \cup \{(x+k,h) \in R : |k| < |h| < \varepsilon\} : \varepsilon > 0\}$ is a neighborhqod base at $(x,0)$.

2.4. *Example* [My]. Mysior's celebrated "simple" example of a regular space which is not completely regular is shown in [H_1] to be Δ-normal. So we have that while

(*) implies regular, even the stronger Δ-normal does not imply completely regular. As the properties we want are proved in [H_1] and [My], we again simply describe the space.

Let X be the closed upper half-plane plus a point ∞. Points above the x-axis are isolated. A neighborhood base at (x,0) is $\{\{(x,y): 0 \leq y < \frac{1}{n}\} \cup \{(x + 1 + y, y): 0 < y < \frac{1}{n}\}: 0 < n \in \omega\}$. A neighborhood base at ∞ is $\{\{\infty\} \cup \{(x,y): x > n\}: n \in \omega\}$.

2.5. *Example* [S]. There is a T_2 orthocompact space which fails to be regular (hence cannot satisfy (*)).

Again, the relevant properties are noted in [S], so we simply describe the space. Let $X = \mathbb{R}$. As a base for the topology on X, take $\{\{x\}: x \in \mathbb{R}\setminus\mathbb{Q}\} \cup \{V\setminus A: V$ Euclidean open, $A \subseteq \mathbb{Q}\}$.

This is, of course, the "wrong" reason for orthocompactness to not imply (*). We really want a regular example, and we will give one shortly. First, let's look at another example, which has been attributed to R. H. Bing, which shows that even very strong separation properties may not imply (*).

2.6. *Example*. There is a monotonically normal, hereditarily countably paracompact space which does not satisfy (*).

Proof. Let X be the space obtained from $\omega_1 \times (\omega_1 + 1)$ by isolating all points of $\omega_1 \times \omega_1$. It is shown in [Co] that X is collectionwise normal, and in [vD] that it has the properties listed above. We show that it does not satisfy (*). Let $\mathcal{U} = \{[0,\alpha] \times (\alpha,\omega_1]: \alpha < \omega_1\} \cup \{\{x\}: x \in \omega_1 \times \omega_1\}$. Suppose $\langle \mathcal{V}_n: n \in \omega \rangle$ is a sequence of open covers, each refining $\mathcal{U}$. Suppose $n \in \omega$. For each $\alpha < \omega_1$, choose $f_n(\alpha) < \alpha$, $s_n(\alpha) \in \omega_1$, and $V_{n\alpha} \in \mathcal{V}$ such that $(f_n(\alpha),\alpha] \times (s_n(\alpha),\omega_1] \subseteq V_{n\alpha}$. By the Pressing Down Lemma, there is $\beta_n \in \omega_1$ and a stationary $S_n \subseteq \omega_1$ with $f_n(\alpha) < \beta_n$ for all $\alpha \in S_n$. Let $\beta = \sup\{\beta_n: n \in \omega\} + 1$. Let $C = \{\delta \in \omega_1$: for each $n \in \omega$ ($S_n \cap \delta$ is cofinal in δ and $s_n(\alpha) < \delta$ for each $\alpha < \delta$)$\}$. Now C is c.u.b. in ω_1, so we choose a limit ordinal $\delta \in C$ with $\delta > \beta$. For each $n \in \omega$, we see that for each $\alpha \in (\beta,\delta) \cap S_n$, $f_n(\alpha) < \beta$ and $s_n(\alpha) < \delta$. Thus $(\beta,\delta) \in (f_n(\alpha),\alpha] \times (s_n(\alpha),\omega_1] \subseteq V_{n\alpha}$. Hence we have $\{(\alpha,\omega_1): \alpha \in (\beta,\delta) \cap S_n\} \subseteq st((\beta,\delta),\mathcal{V}_n)$, and so $(\delta,\omega_1) \in \overline{st((\beta,\delta),\mathcal{V}_n)}$ for every $n \in \omega$, but $(\delta,\omega_1) \notin st((\beta,\delta),\mathcal{U})$. Hence (*) cannot be satisfied.

2.7. *Example*. $\omega_1 \times (\omega_1 + 1)$ is countably compact (hence, a wΔ-space) but does not satisfy (*).

The properties claimed are clear, see 1.4. We would also remark that this shows that (*) is not preserved by perfect preimages, even though the range space, in this case ω_1, may have many strong properties.

Our last two examples are what have come to be called "ψ-like spaces," named for the famous Isbell-Mrowka space ψ found in [GJ] exercise 5.I.

2.8. *Definition*. A ψ-like space is a space $\psi(\mathcal{A},X) = \mathcal{A} \cup X$ where $\mathcal{A}$ is an almost disjoint collection of countable subsets of X. Points of X are isolated. If

$A \in \mathcal{A}$, then $\{\{A\} \cup (A \setminus F): F$ is finite$\}$ is a neighborhood base at A.

2.9. *Lemma. Every ψ-like space is orthocompact.*

Proof. If $\mathcal{U}$ is an open cover of $\psi(\mathcal{A},X)$, then $\mathcal{U}$ has an open refinement $\mathcal{V} = \{\{x\}: x \in X\} \cup \{\{A\} \cup (A \setminus F_A): A \in \mathcal{A}\}$ where each F_A is an appropriately chosen finite set. Clearly, $\mathcal{V}$ is interior preserving.

2.10. *Example.* There is an almost disjoint family $\mathcal{A}$ on ω such that $\psi(\mathcal{A},\omega)$ is an orthocompact Moore space which does not satisfy strong (*).

Proof. It follows from 2.9 that any $\psi(\mathcal{A},\omega)$ is an orthocompact Moore space (developability is trivial). We construct $\mathcal{A} = \{A_\alpha: \alpha < \omega_1\}$ by induction. Let $\{A_n: n \in \omega\}$ be a partition of ω into infinite sets. For $\alpha \geq \omega$, suppose we have $\{A_\gamma: \gamma < \alpha\}$ defined. We now define A_α. Write $\alpha = \lambda_\alpha + m_\alpha$ where λ_α is a limit ordinal and $m_\alpha \in \omega$. Since $\alpha < \omega_1$, we let $\{\gamma_i: i \in \omega\}$ be a one-to-one counting of α. Choose $A_\alpha \subseteq \omega$ such that the following are satisfied: (1) $m_\alpha \notin A_\alpha$ and (2) $|A_\alpha \cap A_{\gamma_i}| = i$ for each $i \in \omega$. By the almost disjointness of $\{A_\gamma: \gamma < \alpha\}$, it is clear that we can construct such an A_α. Now let $\mathcal{U} = \{\omega\} \cup \{\{A\} \cup A: A \in \mathcal{A}\}$. If $\mathcal{V}$ is any open refinement of $\mathcal{U}$, then $\mathcal{V}$ has an open refinement of form $\{\{A_\alpha\} \cup (A_\alpha \setminus n_\alpha): \alpha \in \omega_1\} \cup \{\{i\}: i \in \omega\}$, and stars decrease with refinements, so we will assume $\mathcal{V}$ has this form. Fix m and n such that the set $I = \{\alpha: n_\alpha = n, m \in A_\alpha \setminus n\}$ is uncountable. Now choose $\alpha \geq \omega$ such that $|I \cap \alpha| \geq \omega$ and $m = m_\alpha$. Now by (1), $m \notin A_\alpha$, so $A_\alpha \notin st(m,\mathcal{U})$.

However by (2) for each $k \in \omega$ there is $\gamma \in I \cap \alpha$ with $(A_\alpha \setminus k) \cap (A_\gamma \setminus n) \neq \emptyset$. Hence $A_\alpha \in \overline{\bigcup_{\gamma \in I \cap \alpha} (A_\gamma \setminus n)} \subseteq \overline{st(m, \mathcal{V})}$, so strong (*) fails.

2.11. *Example* [B_1], [B_4]. If $Z = {}^{\omega}2$ is the Cantor set, and $\mathcal{A}$ is a maximal almost disjoint collection of convergent sequences in Z, then $\psi(\mathcal{A}, Z)$ is locally compact, orthocompact, a $\sigma^{\#}$-space, and has G_δ-diagonal, but does not satisfy (*).

Proof. Orthocompactness follows from 2.9. The $\sigma^{\#}$-space property follows since the space is orthocompact and has G_δ-diagonal. It is noted in [B_1], and is not terribly difficult, that the space does not have G_δ^*-diagonal, hence it could not satisfy (*) since it has G_δ-diagonal.

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