TOPOLOGY PROCEEDINGS

Volume 12, 1987

Pages 159–171

http://topology.auburn.edu/tp/

ON THE COUNTABLE BOX PRODUCT OF COMPACT ORDINALS

by

Soulian Yang and Scott W. Williams

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

ON THE COUNTABLE BOX PRODUCT OF COMPACT ORDINALS

Soulian Yang[1] and Scott W. Williams[2]

If X is a topological space, then $\square^\kappa X$ (the box product of κ many copies of X) denotes the product $\Pi^\kappa X$ with the topology induced by the family of all sets of the form $\Pi_{\alpha\in\kappa} U_\alpha$, where each U_α is an open set in X. For a recent survey on box products, see [Wi2].

Consider the following theorem due to M. E. Rudin:

0.1 *Theorem. Assume the Continuum Hypothesis holds. Then, for each ordinal* λ, $\square^\omega \lambda + 1$ *is paracompact.*

The conclusion to this theorem has been expanded to the larger class of compact spaces ([Kul]) and ω_1 many factors ([Wi3]). Under the set-theoretic statement--there is κ-scale in ${}^\omega\omega$--the best result was "$\square^\omega \omega_1 + 1$ is paracompact" ([Wil]). We offer our main result:

0.2 *Theorem. Suppose that for some cardinal* κ *there is a* κ*-scale in* ${}^\omega\omega$. *Then, for each ordinal* λ, $\square^\omega \lambda + 1$ *is paracompact.*

1. Preliminaries

Given a set X, ${}^\omega X$ is the set of functions from ω to the set X. For f and g in ${}^\omega X$, define $f =^* g$ if they differ on

---

[1]Author's work completed while visiting S.U.N.Y. Buffalo.

[2]Author's work partially supported by N.S.F. grant.

only finitely many coordinates. We denote the resulting quotient set by $\nabla^{\omega}X$ and write $[f] = \{g: g =^* f\}$.

Suppose X is an ordinal set. There are two very different but similarly defined orders on $\nabla^{\omega}X$. First of all, define $f \leq^* g$ $(f,g \in \square^{\omega}X)$ provided that $f(n) > g(n)$ for only finitely many $n \in \omega$; define $f <^* g$ provided that $f(n) \geq g(n)$ for only finitely many $n \in \omega$. Then we define

$$[f] \leq^* [g] \text{ if } f \leq^* g;$$

$$[f] <^* [g] \text{ if } f <^* g.$$

It is trivial that $[f] = [g]$ iff $f =^* g$ and both orders, $\leq^*$ and $<^*$, are partial orders on $\nabla^{\omega}X$.

In this paper, for each $x \in \nabla^{\omega}X$*, we fix some* $f_x \in x$ *and identify* $x = [f_x]$ *with* f_x.

Suppose, $f,g \in \nabla^{\omega}X$. We define

$$[f,g] = \{h \in \nabla^{\omega}X: f \leq^* h \leq^* g\} = \nabla_{n\in\omega}[f(n),g(n)],$$

and call $[f,g]$ *basic set* iff both sets $\{n: g(n)$ is limit ordinal, $f(n) = g(n)\}$ and $\{n: f(n)$ is a limit ordinal$\}$ are finite.

Suppose κ is a cardinal. The statement there is a κ-scale in ${}^{\omega}\omega$ means there is an order preserving injection from κ into ${}^{\omega}\omega$ whose range is confinal in $({}^{\omega}\omega,<^*)$.

Suppose Z is a topological space. Then $\nabla^{\omega}Z$ denoted the quotient space induced by $=^*$ on $\square^{\omega}Z$. This is known as the nabla product. We make strong use of an important lemma due to K. Kunen (see [Wi2]):

1.1 *Lemma. If* Z *is locally compact and paracompact then*
(1) $\nabla^{\omega}Z$ *is paracompact iff* $\square^{\omega}Z$ *is paracompact;*
(2) $\nabla^{\omega}Z$ *is a* P*-space (every* G_{δ}*-set is open).*

So we need only to prove $\nabla^{\omega}\lambda + 1$ is paracompact in order to prove 0.2.

1.2 *Definition.* A space X is called specially paracompact provided that each open cover of X has a refinement consisting of pairwise disjoint basic sets.

The symbol $\#(\alpha)$ denotes the statement: $\nabla^{\omega}\alpha + 1$ is specially paracompact. According to 1.1 $\#(\alpha)$ implies $\nabla^{\omega}\alpha + 1$ is paracompact.

1.3 *Theorem. If there is a κ-scale in ${}^{\omega}\omega$, then $\#(\omega_1)$ is true.*

This theorem has been proved by Williams in [Wil]. But he stated in [Wil] a weaker proposition: $\nabla^{\omega}\omega_1 + 1$ is paracompact if $\exists$ is a κ-scale in ${}^{\omega}\omega$. In fact, his proof really is a stronger one.

1.4 *Lemma. Suppose λ is an ordinal. Then every clopen set in $\nabla^{\omega}\lambda + 1$ is a union of pairwise disjoint basic sets.*

This result is implicitly proved in [Ru]. But it is not so easy to extract from Rudin's paper. Fortunately, in this paper, we need only some particular cases of the lemma: first case, the clopen set is a difference set between two basic sets; second case, the clopen set is an intersection of countably many basic sets. Both are not so hard to prove. We leave it to the readers.

2. Tops Refinement

2.1 *Definition.* A set $M \subset \omega \times (\lambda + 1)$ is called a matrix provided that there is $b \in \lambda + 1$ for each $n < \omega$ such that $\langle n,b \rangle \in M$.

2.2 *Definition.* Suppose $f \in \nabla^{\omega}\lambda + 1$. If there is a member $\langle n,b \rangle$ of M for all but finitely many n, such that $f(n) = b$, then we say that f is on the matrix M. The set $\{f \in \nabla^{\omega}\lambda + 1\colon f \text{ is on the matrix } M\}$ is denoted by D(M).

2.3 *Lemma. Assume* $\#(\omega_1)$ *is true. If a matrix* M *is countable and* D(M) *is closed, then* D(M) *is specially paracompact.*

Proof. Since M is countable and D(M) is closed, it is easy to find an embedding map E from a basic set $[\overline{0},g] \subset \nabla^{\omega}\omega_1 + 1$ into $\nabla^{\omega}\lambda + 1$, such that $E([\overline{0},g]) = D(M)$, where $\overline{0} = \langle 0,0,\cdots \rangle$ and $g <^* \overline{\omega}_1 = \langle \omega_1,\omega_1,\cdots \rangle$. In fact, $M_n = \{b \in \lambda + 1\colon \langle n,b \rangle \in M \cap (\{n\} \times (\lambda + 1))\}$ is countable and is a closed set for all but finitely many n. Let η_n be the order type of M_n. If M_n is closed, then η_n is a successor ordinal, $\eta_n = \mu_n + 1$. Let $g = \langle \mu_0,\mu_1,\cdots,\mu_n,\cdots \rangle$. Obviously, we can define an embedding map $E\colon [\overline{0},g] \to \nabla^{\omega}\lambda + 1$ satisfying $E([\overline{0},g]) = D(M)$ in natural way. Moreover, $g <^* \overline{\omega}_1$ since $\mu_n < \omega_1$. $[\overline{0},g]$ is specially paracompact since $\#(\omega_1)$ and $[0,g]$ is closed in $\nabla^{\omega}\omega_1 + 1$. It implies D(M) is specially paracompact.

Suppose $B \subset \nabla^{\omega}\lambda + 1$. Remember that we have fixed $f_x \in x$ for each $x \in \nabla^{\omega}\lambda + 1$. Let

$$B_n = \{f_x(n)\colon x \in B\}.$$

If B is countable, then the matrix

$$M(B) = \cup_{n<\omega}\{\langle n,p\rangle : p \in \overline{B}_n\}$$

is also countable, where $\overline{B}_n$ denotes the closure of B_n in $\lambda + 1$. Moreover, $D(M(B))$ is a closed set since $D(M(B)) = \nabla_{n\in\omega}\overline{B}_n$.

2.4 *Definition.* Suppose that $[a,b]$ is a basic set in $\nabla^{\omega}\lambda + 1$, $B \subset \nabla^{\omega}\lambda + 1$ is countable and $\mathcal{R}$ is an open cover of $[a,b]$. The notion of tops refinement of $\mathcal{R}$ relative to B and $[a,b]$ is defined by the following cases:

Case 1. $D(M(B)) \cap [a,b] = \emptyset$. There is an open set $U \in \mathcal{R}$ such that $b \in U$. We choose a basic set V satisfying $b \in V$ and $V \subset U \cap [a,b]$. In this case, the tops refinement is a singleton basic set $\{V\}$.

Case 2. $D(M(B)) \cap [a,b] \neq \emptyset$. Assume $\#(\omega_1)$. By 2.3, $D(M(B))$ is specially paracompact since $D(M(B))$ is closed. Then $D(M(B)) \cap [a,b]$ also is specially paracompact. Hence there is a refinement $\mathcal{U}$ of $\mathcal{R}$ consisting of pairwise disjoint basic sets which cover $D(M(B)) \cap [a,b]$. We call $\mathcal{U}$ tops refinement of $\mathcal{R}$ relative to B and $[a,b]$.

Since a tops refinement might not cover $[a,b]$, we notice that a tops refinement need not be a refinement in the usual sense. However, we wish to make use of those sets not belonging to the tops refinement.

2.5 *Lemma. Suppose* $B \subset \nabla^{\omega}\lambda + 1$ *is countable,* $[a,b]$ *is a basic set in* $\nabla^{\omega}\lambda + 1$ *and* $\mathcal{R}$ *is an open cover of* $[a,b]$. *Then there is a tops refinement* $\mathcal{U}$ *of* $\mathcal{R}$ *relative to* B *and* $[a,b]$, *and there is a partition* $\mathcal{P}$ *of* $[a,b]$ *into basic sets such that* $\mathcal{U} \subset \mathcal{P}$.

Proof. By 2.3 D(M(B)) is specially paracompact. Then $K = D(M(B)) \cap [a,b]$ is specially paracompact. For K, as a subspace of $\nabla^{\omega}\lambda + 1$, there is a refinement $\mathcal{V}$ of $\{U \cap K: U \in \mathcal{R}\}$ consisting of pairwise disjoint basic sets in K. The tops refinement $\mathcal{U}$ of $\mathcal{R}$ relative to B and [a,b] can be induced by $\mathcal{V}$ in the following way.

First of all, we define a map from $\mathcal{V}$ into the family of all basic sets in $\nabla^{\omega}\lambda + 1$. If $V = [r,s] \cap K \in \mathcal{V}$, $r,s \in K$, we define $\phi(V) = [\bar{r},s] \subset [a,b]$ by the following clauses:

(1) $r(n) < s(n)$. Define $\bar{r}(n)$ from $r(n)$. If $r(n)$ is a successor ordinal; define

$$\bar{r}(n) = \sup\{p \in \lambda + 1: p < r(n) \ \& \ p \in K_n\} + 1$$

if $r(n)$ is a limit ordinal and $S = \{p \in \lambda + 1: p < r(n) \ \& \ p \in K_n\} \neq \emptyset$; define $\bar{r}(n) = a(n)$ if $r(n)$ is a limit ordinal and $S = \emptyset$.

(2) $r(n) = s(n)$. In this case $s(n)$ is an isolated point of K_n. Define $\bar{r}(n) = s(n)$ if $s(n)$ is still an isolated point in $\lambda + 1$. If $s(n)$ is a limit ordinal, then we define $\bar{r}(n)$ in the same way as we did in (1).

We claim that:

(a) $V = \phi(V) \cap K$ for every $V \in \mathcal{V}$;

(b) $\phi(\mathcal{V}) = \{\phi(V): V \in \mathcal{V}\}$ is pairwise disjoint;

(c) $\cup\phi(\mathcal{V})$ is a basic set.

The clauses (a) and (b) are trivial. We prove (c). In fact, let $h_n = \sup K_n$, then

$$[a,h] = \cup\phi(\mathcal{V}),$$

where $h = X_{n<\omega}h_n$. Let us prove this equality. Suppose $x \in [a,h]$. Then $a(n) \leq x(n) \leq h(n)$ for almost all n. Let

$m(n) = \min P$, if $P = \{\alpha \in K_n : \alpha > x(n)\} \neq \emptyset$;

$m(n) = h(n)$, if $P = \emptyset$; $m = \langle m(0), m(1), \cdots, m(n), \cdots \rangle$.

Since $m \in K$, there is a $V = [r,s] \cap K$ such that $m \in V$. Then

$$\bar{r}(n) \leq r(n) \leq x(n) \leq m(n) \leq s(n)$$

according to (1), (2) and $r,m,s \in K$. It means $x \in \phi(V)$. Then $x \in \cup\phi(\mathcal{V})$. So far we have proved the inclusion "$\subset$". The other inclusion "$\supset$" is trivial.

Notice that every $\phi(V)$ contains only one member V of $\mathcal{V}$, and $V \subset U$ for some $U \in \mathcal{R}$. Without loss of generality, we suppose U is a basic set. Then we can find naturally a basic set $\psi(V)$ in $V^{\omega}\lambda + 1$ such that

$$V \subset \psi(V) \subset \phi(V) \text{ and } \psi(V) \subset U.$$

Let

$$\mathcal{U} = \{\psi(V): V \in \mathcal{V}\},\ \mathcal{H} = \{\phi(V) \setminus \psi(V): V \in \mathcal{V}\} \cup \{[a,b] \setminus [a,h]\}$$

Then

$$[a,b] = [a,h] \cup ([a,b] \setminus [a,h]) = (\cup\mathcal{U}) \cup (\cup\mathcal{H}).$$

By 1.4, every member of $\mathcal{H}$ is a union of pairwise disjoint basic sets. Hence there is a collection $\mathcal{G}$ of pairwise disjoint basic sets such that $\cup\mathcal{G} = \cup\mathcal{H}$. Then $\mathcal{P} = \mathcal{U} \cup \mathcal{G}$ is a partition of $[a,b]$ and $\mathcal{U} \subset \mathcal{P}$.

2.6 *Definition.* Every element of $\mathcal{P} \setminus \mathcal{U}$ is called an uncovered tape.

3. The Proof of the Main Theorem

We assume that there is a κ-scale in ${}^{\omega}\omega$. We are going to prove that $\#(\lambda)$ is true for every ordinal λ.

For simplicity, let $X = \nabla^{\omega}\lambda + 1$. Suppose $\mathcal{R}$ is an open cover of X. We intend to build a tree T consisting of basic sets (exactly, of uncovered tapes). The tree T is ordered by > and the height of T is ω_1, where the order > is defined by

$$[a,b] > [r,s] \text{ iff } a \leq^* r \leq^* s \lneq^* b$$

($s \lneq^* b$ means $s \leq^* b$ and $s \neq^* b$). Simultaneously, we will construct a collection G_α of basic sets for each ordinal $\alpha < \omega_1$ so that $\mathcal{U} = \cup\{G_\alpha : \alpha < \omega_1\}$ is a refinement of $\mathcal{R}$ covering X. All of them subject to the following restrictions:

(3.1) The level 0, which is denoted by T_0, of T is $\{X\}$ and $G_0 = \emptyset$.

(3.2) T_α denotes the α'th level of T. Then $(\cup T_\alpha) \cup (\cup G_\alpha) = X$ and the elements of $T_\alpha \cup G_\alpha$ are pairwise disjoint for every $\alpha < \omega_1$.

(3.3) For each $\alpha < \omega_1$ and $V \in G_\alpha$ there is an $U \in \mathcal{R}$ such that $V \subset U$.

(3.4) The elements of $\cup_{\alpha<\eta} G_\alpha$ are pairwise disjoint for all $\eta < \omega_1$.

(3.5) $\alpha < \beta < \omega_1$ implies $G_\alpha \subset G_\beta$ (Then we have $\cup(G_\beta \setminus G_\alpha) \subset \cup T_\alpha$, $\alpha < \beta$, which follows from (3.2) and the inclusion $G_\alpha \subset G_\beta$).

(3.6) Suppose $\mathcal{A}$ is a branch of η'th subtree $\cup\{T_\alpha : \alpha < \eta\}$, $\eta < \omega_1$, then the length of $\mathcal{A}$ is η, and the intersection $\cap\mathcal{A}$ is non-empty.

Suppose $\mu < \omega_1$. We assume inductively that, for each $\xi < \mu$, T_α, G_α, $\alpha < \xi$ have been built and satisfy (3.1)-(3.6) by taking ξ instead of ω_1 in the statements. It is easy to check that T_α, G_α, $\alpha < \mu$, also satisfy (3.1)-(3.6) by taking μ instead of ω_1 in the statements.

Now we built T_μ and G_μ by the following way.

First of all, we claim that

$$\cap_{\alpha<\mu}(\cup T_\alpha) = \cup\{\cap\mathcal{A}: \mathcal{A} \in \mathcal{B}\}, \tag{1}$$

where $\mathcal{B}$ is the collection of all branches of the μ'th subtree $\cup\{T_\alpha: \alpha < \mu\}$. It follows trivially from (3.6) and (3.2).

Suppose $\mathcal{A} = \{V_\alpha = [a_\alpha, b_\alpha] \in T_\alpha: \alpha < \mu\}$ is a branch. We conclude that $A = \cap\mathcal{A} \neq \emptyset$, because there is no (ω,ω^*) gap in ${}^\omega\omega$ and $\mu < \omega_1$. Moreover, the set A is clopen since $\mu < \omega_1$ and X is a P-space (by 1.1). Then $A = \cup S_A$ where S_A is a collection of pairwise disjoint basic sets (by 1.4).

Let $B = \{b_\alpha: [a_\alpha, b_\alpha] \in \mathcal{A}\}$. By 2.5 there is a tops refinement $\mathcal{U}_C$ of $\mathcal{R}$ relative to B and C, $C \in S_A$, and there is a partition $\mathcal{P}_C$ of C such that $\mathcal{U}_C \subset \mathcal{P}_C$. Let

$$W = \{A = \cap\mathcal{A}: \mathcal{A} \in \mathcal{B}\}.$$

We define

$$\begin{aligned} G_\mu &= (\cup\{\mathcal{U}_C: C \in S_A \;\&\; A \in W\}) \cup (\cup\{G_\alpha: \alpha < \mu\}), \\ T_\mu &= \cup\{\mathcal{P}_C \setminus \mathcal{U}_C: C \in S_A \;\&\; A \in W\}. \end{aligned} \tag{2}$$

If $[r,s] \in T_\mu$, then $[r,s] < V_\alpha$ for all $V_\alpha \in \mathcal{A}$. In fact, $[r,s] \in \mathcal{P}_C$ for some $C \in S_A$ and some $A \in W$, then $[r,s] \subset A \subset V_\alpha$ for every $V_\alpha \in \mathcal{A}$. On the other hand, $[r,s] \not\subset \mathcal{U}_C$ implies $s \not\subset \cup\mathcal{U}_C$. But the top of $V_\alpha = [a_\alpha, b_\alpha]$, b_α, is an element of $\cup\mathcal{U}_C$. Hence $s \neq^* b$.

The rest of the job is to check if T_α, G_α, $\alpha < \mu + 1$ still satisfy (3.1)-(3.6) by taking $\mu + 1$ instead ω_1. We check the clause (3.2) and leave the rest to the readers.

It is easy to check that

$$(\cap_{\alpha<\mu}(\cup T_\alpha)) \cup (\cup_{\alpha<\mu}(\cup G_\alpha)) = X. \quad (3)$$

Now we prove

$$(\cup T_\mu) \cup (\cup G_\mu) = X,$$

i.e. (3.2) holds. In fact, if $x \in X$, then either $x \in \cap_{\alpha<\mu}(\cup T_\alpha)$ or $x \notin \cap_{\alpha<\mu}(\cup T_\alpha)$. If $x \in \cap_{\alpha<\mu}(\cup T_\alpha)$, then $x \in \cup(\cap\mathcal{A} : \mathcal{A} \in \mathcal{B})$ by the equality (1). Thus

$$x \in \cap\mathcal{A} = A,$$

$$x \in C,$$

where $\mathcal{A} \in \mathcal{B}$, $C \in S_A$ and $A = \cup S_A$. Because $C = \cup\mathcal{P}_C$ and $\mathcal{U}_C \subset \mathcal{P}_C$, we have

$$x \in (\cup(\mathcal{P}_C \setminus \mathcal{U}_C)) \cup (\cup\mathcal{U}_C).$$

So

$$x \in (\cup T_\mu) \cup (\cup G_\mu). \quad (4)$$

If $x \notin \cap_{\alpha<\mu}(\cup T_\alpha)$, then the fact (4) follows from (3) and (2).

The clauses (3.2), (3.3) imply that $\mathcal{U} = \cup\{G_\alpha : \alpha < \omega_1\}$ is a refinement of $\mathcal{R}$ and the elements of $\mathcal{U}$ are pairwise disjoint. Is $\mathcal{U}$ a cover of X? We have to prove the following theorem in order to answer the question.

3.7 *Theorem. Suppose* $\mathcal{A} = \{V_\alpha = [a_\alpha, b_\alpha] \in T_\alpha : \alpha < \omega_1\}$ *is a branch of the tree* T. *Then* $\cap\mathcal{A} = \emptyset$.

Proof. If the conclusion is false, then there is a point $x \in \cap\mathcal{A}$. Let B_μ denote the set $\{b_\alpha : \alpha < \mu\}$. $M(B_\mu)\restriction x$ denote such a submatrix of $M(B_\mu)$ that

$$M(B_\mu)\restriction x = \{\langle n,j\rangle \in M(B_\mu) : n < \omega,\ j > x(n)\}.$$

We say that *a point* $b \in X$ *extends a matrix* M if there is an infinite set $E \subset \omega$ such that $b(n) < j$ for all $n \in E$ and $\langle n,j\rangle \in M$.

We conclude that, for each $\mu < \omega_1$, b_μ extends $M(B_\mu)\restriction x$. In fact, since $[a_\mu, b_\mu] \in T_\mu$, there is a basic set $C \subset \cap\{V_\alpha \in T_\alpha : \alpha < \mu\}$ such that

$$[a_\mu, b_\mu] \in \mathcal{P}_C \setminus \mathcal{U}_C.$$

Because $\mathcal{U}_C$ covers $D(M(B_\mu)) \cap C$, and $\mathcal{P}_C$ is a partition of C, we have

$$[a_\mu, b_\mu] \cap D(M(B_\mu)) = \emptyset.$$

Hence

$$[x,b] \cap D(M(B_\mu)) = \emptyset. \tag{5}$$

If the assertion fails then for every infinite $E \subset \omega$ there is some $n \in E$ and $\xi \in \overline{(B_\mu)_n}$ with $x(n) < \xi \leq b(n)$. So we can find an $m \in \omega$, for each $n > m$, there is $\xi_n \in \overline{(B_\mu)_n}$ with $x(n) < \xi_n \leq b(n)$. Let $f(n) = \xi_n$ for every $n > m$. Then

$$f \in [x,b] \cap D(M(B_\mu)).$$

It is contradictory to (5).

There are only ω_1 many b_μ's. So the extending will go ω_1 many times. It is impossible. Why? Suppose $\mu_{nj} < \omega_1$ is an ordinal. We define inductively

$$\mu_{nj+1} = \begin{cases} \min\{\eta : x(n) < b_\eta(n) < b_{\mu_{nj}}(n)\}, \\ \qquad \text{if } \exists \eta > \mu_{nj}(x(n) < b_\eta(n) < b_{\mu_{nj}}(n)) \\ \mu_{nj}, \text{ if } \nexists \eta > \mu_{nj}(x(n) < b_\eta(n) < b_{\mu_{nj}}(n)) \end{cases}$$

and $b_{\mu_{n0}} = b_0$. Then

$$x(n) < \cdots \leq b_{nj}(n) \leq \cdots \leq b_{\mu_{n1}}(n) \leq b_{\mu_{n0}}(n), \tag{*}$$

for every $n < \omega$. There just are finitely many "<" appearing in the line (*) since every $b_{\mu_{nj}}$ is an ordinal. So there is a minimum among $b_{\mu_{nj}}$'s $(j = 0,1,\cdots)$. Let

$$b(n) = \min\{b_{\mu_{nj}}(n): j < \omega\},$$

$$\mu_n = \min\{\mu_{nj}: b_{\mu_{nj}}(n) = b(n)\}.$$

It is clear that for each $n < \omega$ there is not any ordinal $\eta > \mu_n$ such that

$$x(n) < b_\eta(n) < b_{\mu_n}(n).$$

Let

$$\gamma = \sup\{\mu_n: n < \omega\}.$$

$\gamma < \omega_1$ since every $\mu_n < \omega_1$. If the extending goes ω_1 many times, then b_γ extends $M(B_\gamma)$. It implies that there is an infinite set $E \subset \omega$ such that

$$x(n) < b_\gamma(n) < b_{\mu_n}(n), \; n \in E,$$

since every b_{μ_n} is on the matrix $M(B_\gamma)$. It is a contradiction.

It is similar to the equality (3) that the following equality holds

$$(\cap_{\alpha<\omega_1}(\cup T_\alpha)) \cup (\cup_{\alpha<\omega_1}(\cup G_\alpha)) = X.$$

The theorem 3.7 implies

$$\cap_{\alpha<\omega_1}(\cup T_\alpha) = \emptyset.$$

So

$$\cup_{\alpha<\omega_1}(\cup G_\alpha) = X.$$

i.e. $\mathcal{U}$ covers X.

References

[Ku1] K. Kunen, *On paracompactness of box products of compact spaces*, Trans. A.M.S. 240 (1978), 307-316.

[Ku2] ________, *Set Theory, An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics 102, North-Holland (1980).

[Ro] J. Roitman, *Paracompact box products in forcing extensions*, Fund. Math. 102 (1978), 219-228.

[Ru] M. E. Rudin, *Countable box products of ordinals*, Trans. A.M.S. 192 (1974), 121-128.

[vD] E. K. van Douwen, *Covering and separation properties of box products*, Surveys in general topology, Academic Press (1980), 55-129.

[We] W. A. R. Weiss, *Versions of Martin's axiom*, Handbook of set-theoretic topology, North-Holland (1984), 827-886.

[Wi1] W. Williams, *Boxes of compact ordinals*, Top. Proc. 1 (1977), 631-642.

[Wi2] ________, *Box products*, Handbook of set-theoretic topology, North-Holland (1984), 169-200.

[Wi3] ________, *Paracompactness in box products*, Proc. of the 12th Winter School on Abstract Analysis, Rend. del Circ. Matem. di Palermo (1984), 313-328.

Beijing Teacher's College

Beijing P.R. China

and

S.U.N.Y. Buffalo

Buffalo, New York 14214