TOPOLOGY PROCEEDINGS
Volume 12, 1987
Pages 193–196

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

Research Announcement

NON-HOMOGENEITY OF INTERMEDIATE UNIVERSAL CONTINUA

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For each pair of non-negative integers n and k, $k > n$, Menger [6] has described an n-dimensional continuum M_n^k in E^k which is universal with respect to containing homeomorphic copies of every n-dimensional continuum which can be embedded in E^k. For any $k > 0$, M_0^k is the standard Cantor set. M_1^2 is the Sierpiński universal plane curve [8], [9], and $M_1^k (k \geq 3)$ is the Menger universal curve.

The Cantor set is clearly homogeneous. It is a classical result that the Sierpiński curve is not homogeneous. Anderson [1] proved in 1958 that the Menger universal curve is homogeneous and gave a characterization of the continuum.

In early 1983 the author announced informally [5] that M_n^k is never homogeneous for $n > 0$, $k < 2n + 1$, and indicated an argument for the result. The result was not written up for publication at the time since the question of homogeneity for the case $k \geq 2n + 1$, when no embedding restrictions apply to the universality of M_n^k, remained open. Since then, Bestvina [2] has given a very nice proof of the homogeneity of M_n^k for $k \geq 2n + 1$ and a complete characterization of such spaces. Several persons have expressed interest in the result announced earlier for $k < 2n + 1$, or uncertainty about the status of this problem. It is the purpose of this note to clarify this.

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*This work was supported in part by N.S.F. grant DMS 83-00826.

Alternative constructions of universal continua have been given by Lefschetz [4] and Pasynkov [7]. The space M_n^k in Lefschetz' construction is the intersection of a nested sequence of k-manifolds with boundary. Let $M_{(0,n)}^k$ be a k-simplex. Inductively, let $M_{(i+1,n)}^k$ be the star of the n-skeleton of $M_{(i,n)}^k$ in the second barycentric subdivision of $M_{(i,n)}^k$. Set $M_n^k = \cap_{i=0}^{\infty} M_{(i,n)}^k$. It remains an open problem to characterize M_n^k for $k < 2n + 1$ as well as to determine whether the constructions of Lefschetz, Menger, and Pasynkov yield homeomorphic continua. (Some results are known for the case $k = n + 1$ [3],[9].) Our argument is described in terms of Lefschetz' construction, but can be appropriately modified for the other two.

Theorem. M_n^k *is not homogeneous for* $n > 0$, $k < 2n + 1$.

Sketch of proof. The case $k = n + 1$ is already known from arguments similar to that for the Sierpiński curve, i.e. local separation vs. its lack. However, this is also covered by the general case, which uses a generalization of these arguments.

Let $B = \{x \in M_n^k | x$ is in the boundary of some $M_{(i,n)}^k$ in the defining sequence for $M_n^k\}$, and let $I = \{x \in M_n^k | x$ is in the interior of every $M_{(i,n)}^k$ in the defining sequence for $M_n^k\}$.

There are two alternative but closely related methods to topologically distinguish between points in B and points in I.

The first involves local linking properties. If $x \in B$, there is an n-sphere Σ in M_n^k containing x (and a subset of the boundary of some $M^k_{(i,n)}$) such that every $(k - n - 1)$-sphere in $M^k_n - \Sigma$ is null-homotopic in $M^k_n - \Sigma$. No point in I has this property. If $y \in I$ and S is any n-sphere in M^k_n containing y, there is a $(k - n - 1)$-sphere T in $M^k_n - S$ such that every null homotopy of T in M^k_n intersects S. (One can if desired restrict one's attention to S being as nice as desired in terms of ulc properties, etc., since it is to be compared to the Σ containing x, which can be chosen to have any such desired properties.)

Alternatively, one can use a variation on the disjoint disk property. If $x \in B$, there is an embedding $f: C^n \to M^k_n$ with $f(0) = x$, where C^n is an n-simplex and 0 its barycenter, such that for any map $g: C^{k-n} \to M^k_n$ and any $\varepsilon > 0$ there are maps $\tilde{f}: C^n \to M^k_n$ and $\tilde{g}: C^{k-n} \to M^k_n$, with $\text{dist}(f,\tilde{f}) < \varepsilon$ and $\text{dist}(g,\tilde{g}) < \varepsilon$ such that $\tilde{f}(C^n) \cap \tilde{g}(C^{k-n}) = \phi$.

No point of I has this property. If $y \in I$ and $h: C^n \to M^k_n$ is any embedding with $h(0) = y$, when there exist $j: C^{k-n} \to M^k_n$ and $\varepsilon > 0$, such that if $\tilde{h}: C^n \to M^k_n$ and $\tilde{j}: C^{k-n} \to M^k_n$ are two maps with $\text{dist}(h,\tilde{h}) < \varepsilon$ and $\text{dist}(j,\tilde{j}) < \varepsilon$ then $\tilde{h}(C^n) \cap \tilde{j}(C^{k-n}) \neq \phi$.

For the first argument, the non-existence of null-homotopies of n spheres in any M^k_n prevents it from applying to M^{2n+1}_n. For the second argument, the fact that M^{2n+1}_n has the disjoint n-cell property, and $k - n > n$ in this case, prevents the argument from applying for $k > 2n$. It is the fact that M^{2n+1}_n satisfies the disjoint n-cell property which is central in the characterization of M^{2n+1}_n and the proof of its homogeneity.

Since the above arguments distinguish points based on local properties, they also show that no Sierpiński manifold (i.e. continuum locally homeomorphic to M_n^k for some $n > 0$, $n < k < 2n + 1$) is homogeneous. Left open are the questions of how many orbits M_n^k has under the action of its homeomorphism group and whether either of the sets B or I constitutes an orbit, as well as homogeneity properties of M_n^k under classes of maps more general than homeomorphisms.

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