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COMPOSANTS OF INDECOMPOSABLE STONE-CECH REMAINDERS

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This article concerns the properties of certain spaces which can occur as Stone-Čech remainders of locally compact Hausdorff spaces. I want to thank B. Diamond for two very useful conversations on this topic, and for calling to my attention Lemma 1 which made this work possible.

A *continuum* is a compact, connected Hausdorff space. Let Y be a continuum. Y is *irreducible between* the points $a,b \in Y$ if no proper subcontinuum of Y contains both of them. This will be denoted by $Y = [a,b]$, with the understanding that if $a,b \in \mathbb{R}$, the usual meaning applies. Y is connected *im Kleinen* at $p \in Y$ provided every neighborhood of p contains a neighborhood of p which is connected and closed in Y. Y is *indecomposable* if it is not the union of two of its proper subcontinua, or equivalently if every proper subcontinuum of Y is nowhere dense. If $p \in Y$, the *composant of* p in Y, denoted $C(Y;p)$, is defined by

$$C(Y;p) = \{y \in Y \mid Y \neq [p,y]\}.$$

$C(Y;p)$ is then the union of all the proper subcontinua of Y containing p. If Y is nondegenerate and indecomposable, the sets $C(Y;p)$ partition Y; that is, $y \in C(Y;p)$ is an equivalence relation. $\mathcal{C}(Y)$ will denote the set of composants of Y. Nondegenerate metrizable indecomposable continua have been known since the 1920's to have exactly c composants

[9]. For the nonmetric case, the situation is more complicated. It is known that there exist indecomposable continua X such that $\mathcal{C}(X)$ has cardinality 1, 2 or 2^m for any infinite cardinal number m [3], [12]. Whether other numbers are possible is open.

For any completely regular space X, βX will denote its Stone-Čech compactification and X^* will denote the remainder $\beta X - X$. A will always denote $(0,1]$ and I will denote $[0,1]$. A^* is an indecomposable continuum [1], [2] or [13], but the cardinaltiy of $\mathcal{C}(A^*)$ depends on your set theory; it is known that it can be either one or 2^c [5], [10], [11]. The purpose of this paper is to show that for many other non-pseudocompact X with X^* an indecomposable continuum $\mathcal{C}(X^*)$ and $\mathcal{C}(A^*)$ are equipollent.

Dickman [7] showed that a half open interval is essentially the only locally connected and locally compact metric space with an indecomposable continuum as its Stone-Čech remainder; however, L. R. Rubin and the author demonstrated the existence of a broader class of objects, called waves, with this property [4].

Definition. A *wave* from a to b is a topological pair (Y,X) such that Y is a continuum irreducible between a and b, Y is both connected im Kleinen and first countable at b, and $X = Y - \{b\}$.

Theorem 1 [4]. *If* (Y,X) *is a wave from* a *to* b, *then* X^* *is an indecomposable continuum.*

An indecomposable continuum of this type will be called a *wave remainder*.

Theorem 2. *There exist wave remainders of arbitrarily large cardinality.*

Proof. Given a limit ordinal number m, perform a long line construction on the ordinal $\alpha = m \times \omega$. That is, define X to be the set $\alpha \times [0,1)$ with the lexicographic order topology, and let Y be the one point compactification of X. Let S_0 denote the closure of the subset of X, $\{(\beta,t) | \beta < m\}$, and let S denote S_0 with its top and bottom points identified. It is easy to see, using a spiral-like construction in $Y \times S$, that X has a compactification with remainder S, and since $\beta X - X$ admits a continuous map onto S, it has cardinality at least as large as S. S, however, has cardinality at least that of m, so the proof is done.

The principal result here is:

Theorem 3. *If* X^* *is any wave remainder, then* $\mathcal{C}(A^*)$ *and* $\mathcal{C}(X^*)$ *are equipollent.*

To prove this, a number of Lemmas are needed.

Lemma 1. *Let* X *and* Y *be completely regular spaces and let* $f: X \to Y$ *be a monotone quotient map. Then* $\beta f: \beta X \to \beta Y$ *is a monotone map also.*

Proof. This is a special case of B. Diamond's theorem 4.7 of [6, p. 76].

Lemma 2. *If* X *is a locally compact space and* $\mathcal{D}$ *is a decomposition of* X *into compact sets such that the nondegenerate members of* $\mathcal{D}$ *form a neighborhood finite collection, then the quotient map* $q\colon X \to \frac{X}{\mathcal{D}}$ *is perfect. Consequently,* $\beta q(X^*) = (\frac{X}{\mathcal{D}})^*$ *and* $(\beta q)^{\leftarrow}((\frac{X}{\mathcal{D}})^*) = X^*$.

Proof. Each point inverse is clearly compact, and $\mathcal{D}$ is upper semicontinuous, since if $A \in \mathcal{D}$ and U is open with $A \subseteq U$, $U - \cup\{B \in \mathcal{D} | B \neq A$ and B is nondegenerate$\}$ is a saturated open set containing A and contained in U. The last sentence follows from Lemma 1.5 of [8, p. 87] and the definition of compactification. (Henriksen and Isbell use the term *fitting map* for what is nowadays commonly called a perfect map.)

Lemma 3. *Suppose* S *and* Z *are indecomposable continua,* Z *is nondegenerate, and* $f\colon S \to Z$ *is a monotone onto map. Then* f *induces a bijection between* $\mathcal{C}(S)$ *and* $\mathcal{C}(Z)$.

Proof. Let C be any composant of Z. Then

$$C = \cup\{W | p \in W,\ W \text{ a proper subcontinuum of } Z\}$$

for some $p \in Z$. Thus,

$$f^{\leftarrow}(C) = \cup\{f^{\leftarrow}(W) | p \in W;\ W \text{ a proper subcontinuum of } Z\}.$$

Since for $W \neq Z$, $f^{\leftarrow}(W) \neq S$, it follows that $f^{\leftarrow}(C)$ is a subset of a single composant of S. Define $H\colon \mathcal{C}(Z) \to \mathcal{C}(S)$ by H(C) = the composant of S containing $f^{\leftarrow}(C)$. Then, if $x \in S$, $f(x) \in Z$ and thus $f(x) \in D$ for some composant D of Z. Thus, $f^{\leftarrow}(D) \subseteq C(S;x)$, so that H is surjective.

Suppose W is a proper subcontinuum of S and that $f(W) = Z$. Since Z is nondegenerate, there is a nonempty, nondense open $U \subseteq Z$. W is nowhere dense in S, so neither $f^{\leftarrow}(U)$ nor $f^{\leftarrow}(Z - \overline{U})$ is a subset of W. By monotonicity, $W \cup f^{\leftarrow}(Z - U)$ and $W \cup f^{\leftarrow}(\overline{U})$ are proper subcontinua of S whose union is S, a contradiction to the indecomposability of S. Consequently, for each proper subcontinuum W of S, $f(W) \neq Z$.

Therefore, for any $D \in \mathcal{C}(S)$, $f(D)$ is contained in a single composant of Z. (Since f commutes with unions, the same argument works as for $f^{\leftarrow}$ above.) If for two composants C_1 and C_2 of Z, $H(C_1) = H(C_2)$, then $f^{\leftarrow}(C_1 \cup C_2) \subseteq H(C_1)$ and so $C_1 \cup C_2 \subseteq f(H(C_1)) \subseteq C_3$ for some single composant C_3 of Z. This is possible only if $C_1 = C_2 = C_3$; therefore, H is injective and hence bijective.

Definition. A wave (Y,X) from a to b has a *cofinal sequence of cutpoints* provided that there is a sequence $\langle b_n \rangle_{n=0}^{\infty}$ converging to b such that $b_0 = a$ and for each $n \geq 1$, b_n separates b_{n-1} from b.

Remark. This is a fairly strong property. It is easy to string together a sequence of indecomposable continua with more than three composants to form a wave in which no connected, nowhere dense set separates.

Lemma 4. *Let* (Y,X) *be a wave from* a *to* b. *Then there is a descending sequence of continua* $\langle W_i \rangle_{i=0}^{\infty}$ *such that* $W_0 = Y \neq W_1$; *for each* $i \geq 1$, $W_i \subseteq \text{Int}(W_{i-1})$; *and* $\bigcap_{i=0}^{\infty} W_i = \{b\}$. *The* W_i*'s,* $i \geq 1$, *can be chosen to have one-point boundaries if and only if* (Y,X) *has a cofinal sequence of cutpoints.*

Proof. First countability and connectedness im Kleinen at b enable one to do a simple recursive construction of the W_i's. Irreducibility is used for the last sentence.

Convention. If (Y,X) is a wave from a to b and $\mathcal{D}$ is an upper-semicontinuous decomposition of Y with {b} a degenerate element of $\mathcal{D}$, then $\frac{X}{\mathcal{D}}$ will be used to denote the image of X under the quotient map $Y \to \frac{Y}{\mathcal{D}}$. $\mathcal{D} - \{b\}$ is an upper-semicontinuous decomposition of X in this case.

Lemma 5. *Let* (Y,X) *be a wave from* a *to* b. *Then there is a monotone decomposition* $\mathcal{D}$ *of* Y *such that every nondegenerate member of* $\mathcal{D}$ *is a subset of* X, *the nondegenerate members of* $\mathcal{D}$ *form a neighborhood finite family in* X, *and* $(\frac{Y}{\mathcal{D}}, \frac{X}{\mathcal{D}})$ *is a wave from* [a] *to* [b] *with a cofinal sequence of cutpoints.*

Proof. Define $D_n = \overline{W_n - W_{n+1}}$, and let $\mathcal{D}$ be the decomposition with nondegenerate elements $\{D_n | n \text{ odd}\}$. By irreducibility, each D_n is connected and becomes a cutpoint of the quotient $\frac{Y}{\mathcal{D}}$, as required. Both {a} and {b} are degenerate elements of $\mathcal{D}$, making the necessary verifications easy.

Lemma 6. *Let* (Y,X) *be a wave from* a *to* b *and let* $\mathcal{D}$ *be an upper semicontinuous monotone decomposition of* (Y,X) *with the nondegenerate elements forming a neighborhood finite collection in* X. *Then* X* *has the same number of composants as* $(\frac{X}{\mathcal{D}})^*$.

Proof. If $q\colon X \to \frac{X}{D}$ is the quotient map, then $\beta q(X^*) = (\frac{X}{D})^*$ and $(\beta q)^{\leftarrow}((\frac{X}{D})^*) = X^*$, and thus $(\beta q)|X^*$ is monotone by Lemma 1. By Lemma 3, $(\beta q)|X^*$ induces a bijection between the set of composants of X^* and that of $(\frac{X}{D})^*$, completing the argument.

Definition. A wave (Y,X) from a to b has a *cofinal sequence of closed intervals* if there is a descending sequence of continua, $\langle W_i \rangle_{i=0}^{\infty}$ with $W_0 = Y \neq W_1$; for each $i \geq 1$, $W_i \subseteq \mathrm{Int}(W_{i-1})$; $\cap_{i=0}^{\infty} W_i = \{b\}$; and for each odd i, $\overline{W_i - W_{i+1}}$ is homeomorphic to I.

Lemma 7. *Let* (Y,X) *be a wave from* a *to* b, *with a cofinal sequence of cutpoints. Then there is a bijection between* $\mathcal{C}(X^*)$ *and* $\mathcal{C}(A^*)$.

Proof. Suppose $\{W_i\}_{i=0}^{\infty}$ is a descending sequence of continua in Y such that $W_0 = Y$, and for $i \geq 1$, W_i has boundary $\{b_i\}$, and $\cap_{i=0}^{\infty} W_i = \{b\}$. Define $L_i \subseteq Y$ by $L_i = \overline{W_{i-1} - W_i}$. Now, define $\hat{X} \subseteq Y \times I$ by

$$\hat{X} = (\cup_{i=1}^{\infty}(L_i \times \{\tfrac{1}{i}\})) \cup (\cup_{i=1}^{\infty}(\{b_i\} \times [\tfrac{1}{i+1},\tfrac{1}{i}])).$$

The only limit point of $\hat{X}$ which does not belong to $\hat{X}$ is $(b,0)$. Thus, if $\hat{Y} = \hat{X} \cup \{(b,0)\}$, it is easy to see that $(\hat{Y},\hat{X})$ is a wave from $(a,1)$ to $(b,0)$ with a cofinal sequence of closed intervals, $\langle \{b_i\} \times [\frac{1}{i+1},\frac{1}{i}] \rangle_{i=1}^{\infty}$. Shrinking each of them to a point is accomplished by restricting the projection $Y \times I \to Y$ to $\hat{Y}$, so that the quotient of $(\hat{Y},\hat{X})$ so obtained is (Y,X).

Thus $\mathcal{C}(\hat{X}^*)$ and $\mathcal{C}(X^*)$ are equipollent.

Continuing, the projection $Y \times I \to I$ restricted to $\hat{Y}$ is also monotone and has the effect of shrinking each $L_i \times \{\frac{1}{i}\}$ to a point. Thus, $(\hat{Y},\hat{X})$ also admits a monotone quotient map onto (I,A), so that $\mathcal{C}(A^*)$ and $\mathcal{C}(\hat{X}^*)$ are also equipollent. Thus, the set of composants of X^* and that of A^* are also equipollent, by transitivity.

Proof of Theorem 3. This is now immediate from Lemmas 5, 6, and 7.

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