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LOTS WITH S_δ-DIAGONALS

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In [L_1] it was shown that a LOTS (=linearly ordered topological space) with a G_δ-diagonal is metrizable. In this note the effect of a generalization of a G_δ-diagonal in a LOTS is studied.

A subset A of a topological space X is an S_δ-set if there is a countable collection $\mathcal{U}$ of open sets that if $x \in A$ and $y \notin A$ then there exists $U \in \mathcal{U}$ such that $x \in U$ and $y \notin U$. If each $u \in \mathcal{U}$ contains A then A is a G_δ-set. Clearly a G_δ-set is an S_δ-set but the set of rational numbers in the real line is an S_δ-set which is not a G_δ-set. S_δ-sets are studied in [BB_1] and [BB_2].

Let ω denote the first infinite ordinal. An open cover $\mathcal{O}$ of X is a countable open point-separating cover if $|\mathcal{O}| \leq \omega$ and if x and y are distinct points of X then there exists $O \in \mathcal{O}$ such that $x \in O$ and $y \notin O$. Obviously each subset of a space with a countable open point-separating cover is an S_δ-set. Notice also that if a space has a weaker second-countable topology, then it has a countable open point separating cover.

A base $\mathcal{B}$ for a topological space X is a σ-disjoint base if $\mathcal{B} = \cup\{\mathcal{B}_n: n \in \omega\}$ such that for each $n \in \omega$ if $B_1, B_2 \in \mathcal{B}_n$, $B_1 \neq B_2$, then $B_1 \cap B_2 = \emptyset$. A σ-disjoint base B has property * if, given $x \in X$, and $y, z \in X$ such that $y \neq z$, then there exists $n \in \omega$ such that $x \in \cup\mathcal{B}_n$ and no two distinct elements of $\{x,y,z\}$ are in the same member of

$\mathcal{B}_n$. Notice that y or z could be x and $\cup\mathcal{B}_n$ need not contain y or z if $x \neq y$ or $x \neq z$.

If X is a LOTS with order $\leq$, then let $\langle a,b \rangle$ denote the ordered pair with its first component a and second component b. A space X has an S_δ-diagonal (G_δ-diagonal) if $\Delta = \{\langle x,x \rangle: x \in X\}$ is an S_δ-subset (G_δ-subset) of $X \times X$. Let $(a,b) = \{x \in X: a < x < b\}$ and $[a,b) = \{x \in X: a \leq x < b\}$ with $(a,b]$ and $[a,b]$ defined in a similar fashion. A subset A of X is an order-convex component if whenever $a,b \in A$, $a < b$, then $[a,b] \subset A$ and A is not a subset of another set with this property.

Theorem 1.1. *A* LOTS X *has an* S_δ*-diagonal if and only if* X *has a* σ*-disjoint base with property* *.

Proof. Let $\{U_1,U_2,\cdots\}$ be a countable collection of open subsets of $X \times X$ that witnesses that $\Delta = \{\langle x,x \rangle: x \in X\}$ is a S_δ set in $X \times X$. No generality is lost if it is assumed that $\{U_1,U_2,\cdots\}$ is closed under finite intersections. For each $n \in \omega\setminus\{0\}$ let $\mathcal{G}_n$ be the set of order-convex components of $\{x \in X: \langle x,x \rangle \in U_n\}$. If $\mathcal{G}_0 = \{\{x\}: x \in X, \{x\}$ open in $X\}$ then $\mathcal{G}_0,\mathcal{G}_1,\mathcal{G}_2,\cdots$ is a σ-disjoint collection for X. To see that $\mathcal{G}_0,\mathcal{G}_1,\cdots$ is a base for X, let $x \in X$ and $a,b \in X$ such that $a \leq x < b$. If $\{x\}$ is open in X then $\{x\} \subset [a,b)$. If $\{x\}$ is not open in X and $a = x$, then, since x is a LOTS, there exists $x^- < x$ such that $(x^-,x) = \emptyset$, then $[a,b) = (x^-,b)$. Choose U_n such that $\langle x,x \rangle \in U_n, \langle x^-,x \rangle \notin U_n$, and $\langle x,b \rangle \notin U_n$. Then neither x^- nor b are in the same order component of $\mathcal{G}_n$ that contains x. All other cases are similar. Hence $\mathcal{G}_0,\mathcal{G}_1,\mathcal{G}_2,\cdots$ is a

σ-disjoint base for X. To see that $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \cdots$ has property *, let $x \in X$ and let y,z be distinct elements of X. If $\{x\}$ is open in X then $\mathcal{G}_0$ witnesses property *. Otherwise choose $n \in \omega$ such that $\langle x,x \rangle \in U_n$, $\langle x,z \rangle \notin U_n$, $\langle x,y \rangle \notin U_n$ and $\langle y,z \rangle \notin U_n$ (with appropriate accommodations being made if either $x = y$ or $x = z$). Then $x \in \cup\mathcal{G}_n$ but no member of $\mathcal{G}_n$ contains two distinct members of $\{x,y,z\}$.

If $\mathcal{B} = \cup\{\mathcal{B}_n: n \in \omega\}$ is a σ-disjoint base for X and $\mathcal{B}$ has property *, let $\mathcal{U}_n = \cup\{B \times B: B \in \mathcal{B}_n\}$. If $\langle x,x \rangle \in \Delta$ and $\langle y,z \rangle \notin \Delta$ choose $n \in \omega$ such that $x \in \cup\mathcal{B}_n$ and no two distinct points of $\{x,y,z\}$ are in the same element of $\mathcal{B}_n$. Then $\langle x,x \rangle \in \mathcal{U}_n$ and $\langle y,z \rangle \notin \mathcal{U}_n$ since if $\langle y,z \rangle \in \mathcal{U}_n$ then there exists $B \in \mathcal{B}_n$ such that $y,z \in B$.

This theorem theorem would be nicer if the answer to the following question was known.

Question 1.1. Does every LOTS with a σ-disjoint base have a σ-disjoint base with property (*)?

In order to give a partial answer, the following folklore observation is needed.

If Z is any set of size $\leq 2^\omega$, then there exists a countable collection of $\mathcal{Z} = \{Z_n: n \in \omega\}$ of subsets of Z such that whenever $z_1, z_2 \in Z$, $z_1 \neq z_2$, then there exists $n \in \omega$ with $z_1 \in Z_n$ and $z_2 \notin Z_n$. If $\mathcal{Z}$ is closed under finite intersections then if $z, z_1, \cdots, z_k$ are distinct elements of Z then there exists n such that $z \in Z_n$ and $\{z_1, \cdots, z_k\} \cap Z_n = \emptyset$.

This observation follows by pretending that Z is a subset of the real line and letting $\mathcal{Z}$ be the set of intersections of Z with members of a countable base for the real line.

Theorem 1.2. *If* X *is a* LOTS *with* $c(X) \leq 2^{\omega}$, *then* X *has a* σ*-disjoint base if and only if* X *has a* σ*-disjoint base with property* (*).

Proof. Let $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega\}$ be a σ-disjoint base for X. Since $C(X) \leq 2^{\omega}$, for each $n \in \omega$ if $\mathcal{B}_n = \{B(n,\alpha):$ $\alpha \in A_n\}$, then $|A_n| \leq 2^{\omega}$. Let $A(n,i)$, $i \in \omega$, be a collection of subsets of A_n satisfying the observation above. If $\mathcal{B}(n,i) = \{B(n,\alpha) \in \mathcal{B}_n : \alpha \in A(n,i)\}$, then $\{\mathcal{B}(n,i): n \in \omega,$ $i \in \omega\}$ is a σ-disjoint base satisfying (*).

Theorem 1.3. *A perfect* LOTS *with an* S_δ*-diagonal is metrizable.*

Proof. A perfect space with a σ-disjoint base in a Moore space and LOTS that are Moore spaces are metrizable.

Theorem 1.4. *If a* LOTS X *has a countable open point-separating cover then* X *has an* S_δ*-diagonal.*

Proof. Let $\mathcal{O} = \{O_1, O_2, \cdots\}$ be a countable open point-separating cover for X that is closed under finite intersections. For each $n \in \omega$ let $U_n = \{\langle x,y \rangle : x,y \in O_n\}$. Let $\langle x,x \rangle \in \Delta$ and $\langle y,z \rangle \notin \Delta$. Choose $n \in \omega$ such that $x \in O_n$, $y \notin O_n$ than $\langle x,x \rangle \in U_n$ and $\langle y,z \rangle \notin U_n$. Thus $\{U_0, U_1, \cdots\}$ witnesses that X has an S_δ-diagonal.

Example 1.1. There is a LOTS Z with an S_δ-diagonal that does not have an open countable point-separating cover.

Let I denote the set of integers and $k = (2^\omega)^+$. Let $Z = k \times I$ ordered lexicographically. Since Z is metrizable it has a G_δ-diagonal. Suppose Z has an open countable point-separating cover $\mathcal{U} = \{U_1, U_2, \cdots\}$. For each $\alpha < k$ let $\mathcal{U}_\alpha = \{U_1 \cap (\{\alpha\} \times I), U_2 \cap (\{\alpha\} \times I), \cdots\}$. Since $k = (2^\omega)^+$ and since I has a countable base there exists ordinals α and β, $\alpha \neq \beta$ such that $\mathcal{U}_\alpha = \mathcal{U}_\beta$. Then for each $m \in I$, $\langle \alpha, m \rangle \in U_k$ if and only if $\langle \beta, m \rangle \in U_k$. This $\mathcal{U}$ is not point-separating.

Of course the situation changes drastically if it only required that X be a GO-space (=subspace of a LOTS [L_2]). The Sorgenfrey Line [S] is a GO-space with a G_δ-diagonal (hence, S_δ-diagonal) that does not have a σ-disjoint base.

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