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PRIME WALLMAN COMPACTIFICATION

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0. Introduction

It has been shown previously, Carlson [7], that there exists a one-to-one correspondence between the nonempty closed subsets of the Wallman compactification of a space and the balanced closed filters on the space. In this paper it is shown that there exists a one-to-one correspondence between the nonempty closed subsets of the prime Wallman compactification of the space and the closed filters on the space. Moreover, it is shown that the absolute of a space, the hyperabsolute of a space and, if the space is Hausdorff, the Fomin H-closed extension of a space are all homeomorphic to subspaces of the prime Wallman compactification of that space. A natural representation of the prime Wallman compactification of a space in terms of its prime open filters is provided.

1. Preliminaries

By a space we mean a T_1 topological space. For a space X , let $v(X)$ denote the collection of all prime open filters on X ; $w(X)$ the collection of all prime closed filters on X . Let $E(X)$ denote the collection of all fixed open ultrafilters on X ; $\theta(X)$ the collection of all open ultrafilters on X ; and $W(X)$ the collection of all closed ultrafilters on X .

For F a closed set in X , let F^* denote the collection of all closed ultrafilters on X containing F . The collection

of all such F^* forms a base for the closed sets on $W(X)$. $W(X)$ with this topology is the Wallman compactification of X . A similar construction yields a topology on $w(X)$, known as the prime Wallman compactification of X .

A topology on $v(X)$ is provided by setting O^* equal to the collection of all open prime filters containing the open set O . The O^* then form a base for the open sets for a topology on $v(X)$. In this paper, $v(X)$ will always be assumed to have this topology. Moreover, $E(X)$ and $\Theta(X)$ with the subspace topology are known as the absolute of X and the hyperabsolute of X , respectively. It will be shown that $v(X)$ and $w(X)$ are homeomorphic.

1.1 *Definition.* Let X be a topological space. An open grill is a nonempty collection \mathcal{G} of open sets satisfying:

$$(A) \phi \notin \mathcal{G}$$

$$(B) O \in \mathcal{G}, Q \text{ open and } Q \supset O \text{ implies } Q \in \mathcal{G}.$$

$$(C) \text{ For open sets } O \text{ and } Q: O \cup Q \in \mathcal{G} \text{ iff } O \in \mathcal{G} \text{ or } Q \in \mathcal{G}.$$

Closed grills are defined similarly. Let \mathcal{G} be a closed grill and \mathcal{H} be an open grill. Set:

$$(A) O(\mathcal{G}) = \{O: O \text{ is open and } X-O \notin \mathcal{G}\}$$

$$(B) \mathcal{F}(\mathcal{H}) = \{F: F \text{ is closed and } X-F \notin \mathcal{H}\}$$

Easily prime open (closed) filters are open (closed) grills and the operators \mathcal{F} and O will be used on prime open and closed filters, respectively, most frequently.

1.2 *Definition.* Let \mathcal{O} and \mathcal{F} be open and closed filters, respectively. Set:

(A) $\mathcal{G}(\mathcal{O}) = \{F: F \text{ is closed and there exists } O \in \mathcal{O} \text{ with } F \supset O\}$

(B) $\mathcal{S}(\mathcal{F}) = \{O: O \text{ is open and there exists } F \in \mathcal{F} \text{ with } O \supset F\}$

(C) An open (closed) filter is said to be balanced provided it is equal to the intersection of the family of open (closed) ultrafilters that contain it.

(D) An open filter \mathcal{O} is said to be a closed generated open filter provided there exists a closed filter \mathcal{F} with $\mathcal{O} = \mathcal{S}(\mathcal{F})$.

(E) A closed filter \mathcal{F} is said to be an open generated closed filter provided there exists an open filter \mathcal{O} with $\mathcal{F} = \mathcal{G}(\mathcal{O})$.

1.3 *Theorem.* Let X be a topological space and K be a prime closed filter and \mathcal{P} a prime open filter. Then:

(1) $\mathcal{O}(K)$ is a prime open filter.

(2) $\mathcal{F}(\mathcal{P})$ is a prime closed filter.

Let K be a closed filter and \mathcal{P} be an open filter. Then:

(3) $\mathcal{S}(K)$ is an open filter.

(4) $\mathcal{G}(\mathcal{P})$ is a closed filter.

(5) $\mathcal{O}(K)$ is an open grill.

(6) $\mathcal{F}(\mathcal{P})$ is a closed grill.

Let \mathcal{G} be a closed grill and \mathcal{H} be an open grill. Then:

(7) $\mathcal{O}(\mathcal{G})$ is an open filter.

(8) $\mathcal{F}(\mathcal{H})$ is a closed filter.

Let \mathcal{F}_1 and \mathcal{F}_2 (O_1 and O_2) be closed (open) filters. Then:

- (9) $\mathcal{F}_1 \subset \mathcal{F}_2$ iff $O(\mathcal{F}_2) \subset O(\mathcal{F}_1)$.
 (10) $\mathcal{F}_1 \subset \mathcal{F}_2$ iff $S(\mathcal{F}_1) \subset S(\mathcal{F}_2)$.
 (11) $O_1 \subset O_2$ iff $\mathcal{F}(O_2) \subset \mathcal{F}(O_1)$.
 (12) $O_1 \subset O_2$ iff $\mathcal{G}(O_1) \subset \mathcal{G}(O_2)$.
 (13) $\mathcal{F}_1 = \mathcal{F}_2$ iff $O(\mathcal{F}_1) = O(\mathcal{F}_2)$.
 (14) $O_1 = O_2$ iff $\mathcal{F}(O_1) = \mathcal{F}(O_2)$.
 (15) $O_1 = O(\mathcal{F}(O_1))$.
 (16) $\mathcal{F}_1 = \mathcal{F}(O(\mathcal{F}_1))$.

1.4 Corollary. The mappings $\mathcal{F}: v(X) \rightarrow w(X)$ and $O: w(X) \rightarrow v(X)$ are inverse mappings.

2. Prime Wallman Compactification

2.1 Notation. Let (X, τ) be a T_1 topological space.

Set:

$$w(X) = \{K: K \text{ a prime closed filter on } X\}$$

$$W(X) = \{N: N \text{ a closed ultrafilter on } X\}$$

For F closed in X , set $F^* = \{K \in w(X): F \in K\}$. Then $\{F^*: F \text{ closed in } X\}$ forms a base for the closed sets for a topology on $w(X)$. When no confusion can result, $w(X)$ will be denoted by wX . wX with this topology is called the prime Wallman compactification of X .

2.2 Definition. Let O be open in X and $F = X - O$. Set:

$$O^* = \{K \in wX: O \in O(K)\}.$$

2.3 Theorem. $\{O^*: O \text{ open in } X\}$ is a base for the open sets in wX .

2.4 Corollary. Let O be open in X and $F = X - O$. Then $O^* = wX - F^*$.

Proof. Since $\{F^*: F \text{ closed in } X\}$ is a base for the closed sets in wX it follows that $\{wX - F^*: F \text{ closed in } X\}$ is a base for the open sets in wX . Let F be closed in X , it suffices to show that $O^* = wX - F^*$ where $O = X - F$. Let $K \in O^*$. Then $O \in O(K)$ which implies that $X - O \notin K$ and thus $F \notin K$ and hence $K \in wX - F^*$.

Suppose $K \in wX - F^*$. Then $K \notin F^*$ and $F \notin K$ and thus $O = X - F \in O(K)$. Therefore, $K \in O^*$. Hence $O^* = wX - F^*$.

For a T_1 space X , the mapping $x \rightarrow \mathcal{M}_x$, where \mathcal{M}_x is the unique closed ultrafilter containing $\{x\}$ embeds X into wX . Using this mapping we will identify a subset in X with the appropriate subset in wX .

2.5 Theorem. Let F be closed in X . Then $cl_{wX}(F) = F^*$.

Proof. Let F be closed in X . By the above identification, $F = \{\mathcal{M}_x: x \in F\}$ and easily $F \subset F^*$ which is closed in wX . Thus, $cl_{wX}(F) \subset F^*$.

Suppose $K \notin cl_{wX}(F)$. Then there exists a basic open set O^* such that $K \in O^*$ and $O^* \cap \{\mathcal{M}_x: x \in F\} = \emptyset$. Let $G = X - O$. Then, by Corollary 2.4, $O^* = wX - G^*$. Now $K \in O^*$ implies $G = X - O \notin K$. Suppose $K \in F^*$. Then $F \in K$ since $O^* \cap \{\mathcal{M}_x: x \in F\} = \emptyset$ it follows that $\mathcal{M}_x \notin O^*$ for each $x \in F$. Hence $X - O \in \mathcal{M}_x$ for each $x \in F$ and we have $F \subset X - O = G$. If $F \in K$ we must have $G \in K$ but $O \in O(K)$ which implies that $G = X - O \notin K$. Therefore, $F \notin K$ and $K \notin F^*$. Hence $F^* \subset cl_{wX}(F)$.

$W(X)$, with the subspace topology, is the usual Wallman compactification. It is shown in Carlson [7], that there exists a one-to-one correspondence between the nonempty closed subsets of $W(X)$ and the balanced closed filters. It is the purpose of this section to generalize this result to the prime Wallman compactification. Additional information on the prime Wallman compactification can be found in [12].

The following lemma can be proved using Zorn's lemma.

2.6 Lemma. Let \mathcal{F} be a closed filter on a space X and K a nonempty closed set not in \mathcal{F} . Then there exists a prime closed filter \mathcal{K} such that $\mathcal{F} \subset \mathcal{K}$ and $K \notin \mathcal{K}$.

Recall that a balanced closed filter is one that is equal to the intersection of the family of closed ultrafilters that contain it. It is natural to consider a corresponding concept for prime closed filters. However, as the following theorem states, which can be proved using the above lemma, every closed filter is the intersection of the family of prime closed filters that contain it. The author assumes the result is well known but is unaware of a reference for it.

2.7 Theorem. Every closed filter is the intersection of the family of prime closed filters that contain it.

We will now show that there exists a one-to-one correspondence between the nonempty closed subsets of $W(X)$ and the closed filters on X .

2.8 Theorem. Let $\hat{F} = \{K_\alpha : \alpha \in \Omega\}$ be a nonempty closed subset of wX . Set $\mathcal{J} = \bigcap \{K_\alpha : \alpha \in \Omega\}$. Then \mathcal{J} is a closed filter and $\hat{F} = \{K \in wX : \mathcal{J} \subset K\}$.

Proof. Easily \mathcal{J} is a closed filter on X . Set $G = \{K \in wX : \mathcal{J} \subset K\}$.

Let $K \in G$. Suppose $K \notin \hat{F}$. Then there exists a closed set $F \subset X$ such that $\hat{F} \subset F^*$ and $K \not\subset F^*$. Then $F \not\subset K$ but $F \in K_\alpha$ for each $\alpha \in \Omega$. Hence $F \in \mathcal{J} \subset K$ and we have a contradiction. Therefore, $G \subset \hat{F}$.

Let $K \in \hat{F}$. Easily $\mathcal{J} = \bigcap F \subset K$ which implies $K \in G$. Hence $\hat{F} \subset G$.

2.9 Theorem. Let $\hat{F} = \{K_\alpha : \alpha \in \Omega\}$ be a nonempty closed subset in wX and $\mathcal{J} = \bigcap \{K_\alpha : \alpha \in \Omega\}$. Then:

$$(1) \hat{F} = \{F^* : F \in \mathcal{J}\}$$

$$(2) \hat{F} = \{\text{cl}_{wX}(F) : F \in \mathcal{J}\}$$

$$(3) \mathcal{J} = \{F : F \text{ is closed in } X \text{ and } \hat{F} \subset F^*\}$$

$$(4) \mathcal{J} = \{F : F \text{ is closed in } X \text{ and } \hat{F} \subset \text{cl}_{wX}(F)\}$$

Proof. By Theorem 2.8, $\hat{F} = \{K \in wX : \mathcal{J} \subset K\}$. Set $\hat{G} = \{F^* : F \in \mathcal{J}\}$. Let $K \in \hat{G}$. Then $K \in F^*$ for each $F \in \mathcal{J}$ and thus $F \in K$ for each $F \in \mathcal{J}$. Hence $\mathcal{J} \subset K$ and $K \in \hat{F}$. Conversely, if $K \in \hat{F}$, then $\mathcal{J} \subset K$ and $F \in K$, and thus $K \in F^*$ for each $F \in \mathcal{J}$. Therefore, $K \in \hat{G}$. Thus (1) holds and (2) follows by Theorem 2.5.

Let $\mathcal{G} = \{F : F \text{ is closed in } X \text{ and } \hat{F} \subset F^*\}$. Let $F \in \mathcal{G}$. Then $\hat{F} \subset F^*$, $K_\alpha \in F^*$, or equivalently, $F \in K_\alpha$ for each $\alpha \in \Omega$. Thus $F \in \mathcal{J}$. Now let $F \in \mathcal{J}$. Then $F \in K_\alpha$ and hence $K_\alpha \in F^*$ for each $\alpha \in \Omega$. Thus $\hat{F} \subset F^*$ and (3) holds and (4) follows by Theorem 2.5.

2.10 *Theorem.* Let \mathcal{F} be a closed filter on X and $\hat{F} = \{K \in wX: \mathcal{F} \subset K\}$. Then \hat{F} is a nonempty closed subset of wX .

Proof. Let $\hat{G} = \cap\{F^*: F \in \mathcal{F}\}$. Easily \hat{G} is closed in wX and thus it suffices to show that $\hat{G} = \hat{F}$. Let K be a prime closed filter and $K \in \hat{G}$. Then $K \in F^*$ for each $F \in \mathcal{F}$. Thus, $F \in K$ for each $F \in \mathcal{F}$ and hence $\mathcal{F} \subset K$. Therefore, $K \in \hat{F}$ and $\hat{G} \subset \hat{F}$. Let $K \in \hat{F}$. Then $\mathcal{F} \subset K$. Hence, for each $F \in \mathcal{F}$, it follows that $K \in F^*$ since $F \in K$. Therefore, $K \in \hat{G}$ and $\hat{F} \subset \hat{G}$.

2.11 *Theorem.* Let \mathcal{S} denote the family of closed filters on X and \mathcal{C} denote the family of nonempty closed subsets of wX . Define $h: \mathcal{S} \rightarrow \mathcal{C}$ by $h(\mathcal{F}) = \{K \in wX: \mathcal{F} \subset K\}$. Then h is a one-to-one surjective mapping.

Proof. Let \mathcal{F} be a closed filter on X . By Theorem 2.10, $\{K \in wX: \mathcal{F} \subset K\}$ is a nonempty closed subset of wX . Hence h defines a map from \mathcal{S} into \mathcal{C} . h is surjective by Theorem 2.8.

h is one-to-one. Suppose $h(\mathcal{F}_1) = h(\mathcal{F}_2)$. Then $\{K \in wX: \mathcal{F}_1 \subset K\} = \{K \in wX: \mathcal{F}_2 \subset K\}$. Since each closed filter is the intersection of the family of prime closed filters that contain it, we have

$$\mathcal{F}_1 = \cap\{K \in wX: \mathcal{F}_1 \subset K\} = \cap\{K \in w(X): \mathcal{F}_2 \subset K\} = \mathcal{F}_2.$$

Therefore, h is a one-to-one mapping.

3. Homeomorph of the Prime Wallman Compactification

For O open in a T_1 topological space X , set $O^* = \{\mathcal{P} \in v(X): O \in \mathcal{P}\}$. Let $v(X)$ have the topology generated by the base $\{O^*: O \text{ open in } X\}$.

Define $f: wX \rightarrow v(X)$ by $f(K) = O(K)$.

3.1 Theorem. f is a homeomorphism from the Prime Wallman compactification wX onto the space $v(X)$.

Proof. By Theorem 1.3 and Corollary 1.4, the mapping f is a one-to-one mapping from wX onto $v(X)$ and $f^{-1}(\mathcal{P}) = \mathcal{J}(\mathcal{P})$ for each $\mathcal{P} \in v(X)$.

f is continuous. Let O^* be a basic open set in $v(X)$. Then: $f^{-1}(O^*) = f^{-1}(\{\mathcal{P} \in v(X) : O \in \mathcal{P}\}) = \{\mathcal{J}(\mathcal{P}) : O \in \mathcal{P}\}$. Let $F = X - O$ and $F^* = \{K \in wX : F \in K\}$. Now F^* is closed in wX .

Claim: $wX - F^* = f^{-1}(O^*)$. Let $K \in wX - F^*$. Then $F \notin K$. Now $f(K) = O(K) = \{Q : Q \text{ open in } X \text{ and } X - Q \notin K\}$. Hence $O \in f(K)$ and thus $f(K) \in O^*$. Therefore, $K \in f^{-1}(O^*)$ and $wX - F^* \subset f^{-1}(O^*)$.

Let $K \in f^{-1}(O^*) = \{\mathcal{J}(\mathcal{P}) : O \in \mathcal{P}\}$. Thus, there exists a prime open filter \mathcal{P} such that $K = \mathcal{J}(\mathcal{P})$ and $O \in \mathcal{P}$. Now $\mathcal{P} = O(K) = \{Q : Q \text{ open in } X \text{ and } X - Q \notin K\}$. Since $O \in \mathcal{P}$ it follows that $F = X - O \notin K$. Hence $K \in wX - F^*$ and $f^{-1}(O^*) \subset wX - F^*$.

f^{-1} is continuous. It suffices to show that for any closed set F in X that $f(wX - F^*)$ is open in $v(X)$. Let $O = X - F$.

Claim: $f(wX - F^*) = O^*$. Let $\mathcal{P} \in O^*$. Set $K = \mathcal{J}(\mathcal{P})$. Now $K = \{G : G \text{ closed in } X \text{ and } X - G \notin \mathcal{P}\}$. Since $O \in \mathcal{P}$, it follows that $F = X - O \notin K = \mathcal{J}(\mathcal{P})$ and hence $K \in wX - F^*$. Thus $\mathcal{P} \in f(wX - F^*)$.

Let $\mathcal{P} \in wX - F^*$. Then there exists $K \in wX - F^*$ such that $\mathcal{P} = O(K)$. Now $F \notin K$ which implies that $O = X - F \in O(K) = \mathcal{P}$. Hence $\mathcal{P} \in O^*$.

3.3 *Comment.* We have shown that $v(X)$ provides another method for constructing the prime Wallman compactification of a T_1 topological space X . We will show, in section 4, that $v(X)$ contains, in a natural way, both the absolute of X and the hyperabsolute of X , and if X is Hausdorff it contains the Fomin H -closed extension of X .

It follows from Theorem 1.3 that there exists a one-to-one correspondence between the closed filters and open grills on X . By Theorems 1.3, 2.3, 2.4, 2.5, and 3.2 we have the following results.

3.4 *Theorem.* Every open grill is the union of the family of prime open filters it contains.

3.5 *Theorem.* Let X be a T_1 topological space. Then:

(1) There exists a one-to-one correspondence between the nonempty closed sets in $v(X)$ and the open grills on X .

(2) Let \mathcal{G} be an open grill on X . Then $\{\mathcal{P} \in v(X) : \mathcal{P} \subset \mathcal{G}\}$ is a nonempty closed subset of $v(X)$.

(3) Each nonempty closed set in $v(X)$ is of the form $\{\mathcal{P} : \mathcal{P} \subset \mathcal{G}\}$ for some open grill \mathcal{G} on X .

4. Special Subspaces of the Prime Wallman Compactification

4.1 *Notation.* Let X be a T_1 topological space. Set:

$v(X) = \{\mathcal{P} : \mathcal{P} \text{ a prime open filter on } X\}$

$E(X) = \{\mathcal{M} : \mathcal{M} \text{ a fixed open ultrafilter on } X\}$

$\Theta(X) = \{\mathcal{M} : \mathcal{M} \text{ an open ultrafilter on } X\}$

$M(X) = \{\mathcal{P} : \mathcal{P} \text{ a minimal prime open filter on } X\}$

$N(X) = \{\mathcal{P} : \mathcal{P} \text{ a fixed minimal prime open filter on } X\}$

Let each of these sets be endowed with the subspace topology from $v(X)$. Then $E(X)$ and $\Theta(X)$ are the absolute and hyperabsolute of X , respectively. Thus, we have the following theorem.

4.2 Theorem. The absolute of X and the hyperabsolute of X are embedded in the Prime Wallman Compactification of X .

If X is T_1 , then X is homeomorphic to $\mathcal{S} = \{M_x : M_x \text{ the closed ultrafilter converging to } x\}$ considered as a subspace of wX . Easily, \mathcal{S} is homeomorphic to $N(X)$ under the mapping f of Theorem 3.2.

Set $F(X) = N(X) \cup (\Theta(X) - E(X))$ with the subspace topology from $v(X)$. We then have the following theorems.

4.3 Theorem. Let X be a Hausdorff topological space. Then $F(X)$ is homeomorphic to the Fomin H-closed extension of X .

Using the homeomorphism between $w(X)$ and $v(X)$ it is an easy matter to identify these special subspaces in $w(X)$. Specifically:

$$w\Theta(X) = \mathcal{F}(\Theta(X)) = \{K : K \text{ is a minimal prime closed filter}\}$$

$$wE(X) = \mathcal{F}(E(X)) = \{K : K \text{ is a fixed minimal prime closed filter}\}$$

$$wF(X) = \mathcal{F}(F(X)) = \{K : K \text{ is a fixed closed ultrafilter or } K \text{ is a free minimal prime closed filter}\}.$$

These subspaces of the prime Wallman compactification $w(X)$ are homeomorphic to the absolute of X , the hyperabsolute X , and, if X is a Hausdorff space, the Fomin H -closed extension of X , respectively.

Making use of the fact that the hyperabsolute of X is homeomorphic to a subspace of wX , namely $w\theta(X)$, we will establish a one-to-one correspondence between the nonempty closed subsets of the hyperabsolute of X and a special class of closed filters.

4.4 Definition. A closed filter \mathcal{F} is called a minimal balanced closed filter provided it is equal to the intersection of the family of minimal prime closed filters that contain it.

A closed filter \mathcal{F} is called open generated if there exists an open filter \mathcal{O} such that $\mathcal{F} = \mathcal{G}(\mathcal{O})$.

4.5 Theorem. Carlson [9]. *Let \mathcal{F} be a closed filter in X . The following statements are equivalent.*

- (1) \mathcal{F} is a minimal balanced closed filter.
- (2) \mathcal{F} is open generated.

We are now in a position to characterize the nonempty closed subsets of the hyperabsolute of X .

4.6 Theorem. *Let X be a T_1 topological space. Let $\emptyset \neq F' \subset w\theta(X)$. F' is closed in $w\theta(X)$ if and only if there exists a minimal balanced closed filter \mathcal{F} such that $F' = \{H \in w\theta(X) : \mathcal{F} \subset H\}$.*

Proof. Suppose F' is a nonempty closed subset of $w\mathcal{O}(X)$. Let $\mathcal{G} = \cap\{H: H \in F'\}$. Since $F' \neq \phi$, \mathcal{G} is a closed filter. Let $F'' = \{H \in w\mathcal{O}(X): \mathcal{G} \subset H\}$.

Since F' is closed, there exists a closed set \hat{F} in wX and a closed filter \mathcal{J} such that $F' = \hat{F} \cap w\mathcal{O}(X)$ and $\hat{F} = \{H \in wX: \mathcal{J} \subset H\}$. Easily, $\mathcal{J} \subset \mathcal{G}$.

Now $F' \subset F''$ and if $H \in F''$ we have that $\mathcal{J} \subset \mathcal{G} \subset H$ and $H \in \hat{F}$ and thus $H \in F'$. Therefore, $F' = F''$ and \mathcal{G} is a minimal balanced closed filter and $F' = \{H \in w\mathcal{O}(X): \mathcal{G} \subset H\}$.

To complete the proof, let \mathcal{G} be a minimal balanced closed filter and $F' = \{H \in w\mathcal{O}(X): \mathcal{G} \subset H\}$. Now $F' \neq \phi$, and $\hat{F} = \{H \in wX: \mathcal{G} \subset H\}$ is a closed set in wX by Theorem 2.10. Moreover, $F' = \hat{F} \cap w\mathcal{O}(X)$ and thus F' is closed.

4.7 Corollary. *There exists a one-to-one correspondence between the nonempty closed subsets of $w\mathcal{O}(X)$ and the minimal balanced closed filters.*

Proof. The correspondence established by Theorem 4.6 is easily one-to-one.

4.8 Corollary. *There exists a one-to-one correspondence between the nonempty closed subsets of the hyper-absolute of X and the open generated closed filters.*

Proof. Follows immediately by Corollary 4.7 and Theorem 4.5.

4.9 Definition. An open grill is called a balanced open grill provided it is the union of all of the open ultrafilters it contains.

Using the homeomorphism between wX and $v(X)$ given by $\mathcal{F} \rightarrow 0(\mathcal{F})$ and Corollary 4.7 we have the following theorem.

4.10 *Theorem.* *There exists a one-to-one correspondence between the nonempty closed subsets of the hyperabsolute of X and the balanced open grills on X .*

4.11 *Definition.* A filter is called fixed if it has a nonempty adherence. A closed filter is called a fixed minimal balanced closed filter provided it is equal to the intersection of the family of fixed minimal prime closed filters that contain it. An open grill is called a fixed balanced open grill provided it is equal to the union of the family of fixed open ultrafilters it contains.

Using precisely the same techniques that were employed to prove the above theorems we can establish the following result.

4.12 *Theorem.* *Let X be a T_1 topological space and $\phi \neq F' \subset wE(X)$. F' is closed if and only if there exists a fixed minimal balanced closed filter such that $F' = \{\mathcal{H} \in wE(X) : \mathcal{F} \subset \mathcal{H}\}$.*

4.13 *Theorem.* *There exists a one-to-one correspondence between the nonempty closed subsets of the absolute of X and the fixed minimal prime balanced closed filters on X .*

4.14 *Theorem.* *There exists a one-to-one correspondence between the nonempty closed subsets of the absolute of X and the fixed balanced open grills on X .*

5. Special Open and Closed Sets in the Prime Wallman Compactification

5.1 *Notation.* Let X be a T_1 topological space with F closed in X and O open in X . Set:

$$F^* = \{K \in w(X) : F \in K\} \text{ and } F' = \{\mathcal{P} \in v(X) : F \in \mathcal{F}(\mathcal{P})\}$$

$$O^* = \{K \in w(X) : O \in O(K)\} \text{ and } O' = \{\mathcal{P} \in v(X) : O \in \mathcal{P}\}.$$

Recall: $\{F^* : F \text{ closed in } X\}$ is a base for the closed sets in $w(X)$.

$\{O^* : O \text{ open in } X\}$ is a base for the open sets in $w(X)$.

$\{F' : F \text{ closed in } X\}$ is a base for the closed sets in $v(X)$.

$\{O' : O \text{ open in } X\}$ is a base for the open sets in $v(X)$.

Recall that $f: w(X) \rightarrow v(X)$ given by $f(K) = O(K)$ is a homeomorphism and $f^{-1}(\mathcal{P}) = \mathcal{F}(\mathcal{P})$. Let $x \in X$. Set $\mathcal{M}_x = \{F : F \text{ closed and } x \in F\}$ and $O_x = \{O : O \text{ open and } x \in O\}$. Then \mathcal{M}_x is the unique closed ultrafilter converging to x and O_x is the open neighborhood filter of x , and since X is T_1 , O_x is a minimal prime open filter.

Let O be open in X . Set

$$\tilde{O} = \{\mathcal{M}_x : x \in O\} \text{ and } \tilde{O}' = \{O_x : x \in O\}$$

Let \hat{O} and \hat{P}' be open sets in wX and $v(X)$, respectively.

Set

$$O = \{x : \mathcal{M}_x \in \hat{O}\} \text{ and } P = \{x : O_x \in \hat{O}'\}$$

Now $\tilde{X} = \{\mathcal{M}_x : x \in X\}$ and $\tilde{X}' = \{O_x : x \in X\}$. If \hat{O} is an open set in wX then $\tilde{X} \cap \hat{O} = \tilde{O}$; if \hat{O}' is an open set in $v(X)$ then $\tilde{X}' \cap \hat{O}' = \tilde{O}'$. Moreover, $f(\tilde{O}) = \{O(\mathcal{M}_x) : \mathcal{M}_x \in \hat{O}\} = \{O_x : O_x \in \hat{O}'\} = \tilde{O}'$, where $f(K) = O(K)$ for $K \in wX$. If O

is open in X then $\tilde{O} = \tilde{X} \cap O^*$, and hence \tilde{O} is open in \tilde{X} . Similarly, if \hat{O} is open in wX then $\tilde{O} = \tilde{X} \cap \hat{O}$ is open in \tilde{X} .

5.2 *Theorem.*

- (1) X , \tilde{X} , and \tilde{X}' are homeomorphic.
- (2) \tilde{X} is dense in wX .
- (3) \tilde{X}' is dense in $v(X)$.

Proof. Let $h: X \rightarrow \tilde{X}$ be defined by $h(x) = \mathcal{M}_x$. h is a homeomorphism and X and \tilde{X}' are homeomorphic by what has been shown previously. To see that \tilde{X} is dense in wX , let \hat{O} be any nonempty open set in wX . Then there exists $K \in \hat{O}$ and an open set S in X with $K \in S^* \subset \hat{O}$. Now $K \in S^*$ if and only if $X-S \not\subset K$. Hence $S \neq \phi$. Let $x \in S$. Then $X-S \not\subset \mathcal{M}_x$ and $\mathcal{M}_x \in S^* \subset \hat{O}$. Hence $\hat{O} \cap \tilde{X} \neq \phi$ and we have that \tilde{X} is dense in wX . Similarly, X and \tilde{X}' are homeomorphic; a corresponding proof establishes statement (3).

5.3 *Theorem.* Let O be open and F closed in X . Then:

- (1) $O \subset F$ iff $O^* \subset F^*$ iff $O' \subset F'$.
- (2) If F is closed in wX , $K \in \hat{F}$ and $K \subset L \in wX$ then $L \in \hat{F}$.
- (3) If \hat{O} is open in wX and $K \in \hat{O}$ and $L \in wX$ with $L \subset K$ then $L \in \hat{O}$.
- (4) If \hat{F}' is closed in $v(X)$, $\mathcal{P} \in \hat{F}'$ and $\mathcal{P}' \in v(X)$ with $\mathcal{P}' \subset \mathcal{P}$ then $\mathcal{P}' \in \hat{F}'$.
- (5) If \hat{O}' is open in $v(X)$ and $\mathcal{P} \in \hat{O}'$ with $\mathcal{P} \subset \mathcal{P}' \in wX$ then $\mathcal{P}' \in \hat{O}'$.
- (6) Let O and Q be open in X . Then $O \cap Q = \phi$ iff $O^* \cap Q^* = \phi$.

(7) Let \hat{Q} be open in wX and $\hat{F} = wX - \hat{Q} = \cap \{F^* : F \in \mathcal{F}\}$ and $\tilde{Q} = \{\mathcal{M}_x : \mathcal{M}_x \in \hat{Q}\}$.

Then: (A) $\tilde{Q} \subset Q^*$

(B) If $\mathcal{M}_x \in \tilde{Q}$ and K is a prime closed filter with $K \subset \mathcal{M}_x$ then $K \in Q^*$.

(C) Let $F \in \mathcal{F}$ and $O = X - F$. Then $O \subset Q$.

(8) Let \mathcal{F} be a closed filter with empty adherence.

Set $\hat{F} = \{K \in wX : \mathcal{F} \subset K\}$ and $\hat{Q} = wX - \hat{F}$. Then $\tilde{Q} = \hat{Q} \cap \tilde{X} = \tilde{X}$. Moreover, $Q^* = wX$.

5.4 Theorem. Let \hat{O} be open in wX and $\hat{F} = cl_{wX}(\hat{O})$.

Then:

$$\tilde{O} \subset \hat{O} \subset O^* \subset cl_{wX}(\hat{O}).$$

Proof. Easily $\tilde{O} = \{\mathcal{M}_x : \mathcal{M}_x \in \hat{O}\} \subset \hat{O}$. Suppose $K \in \hat{O}$. Then there exists an open set S in X with $K \in S^* \subset \hat{O}$. Now $K \in S^*$ implies $X - S \notin K$ and $S \neq \emptyset$.

If $K \notin O^*$ then $X - O \in K$. Now $S \subset O$ and $X - S \supset X - O$ but $X - O \in K$ and $X - S \notin K$ which is impossible. Therefore, $K \in O^*$ and we have that $\hat{O} \subset O^*$.

$O^* \subset cl_{wX}(\hat{O})$. Let $\hat{F} = cl_{wX}(\hat{O})$. Then there exists a closed filter \mathcal{F} such that $\hat{F} = \cap \{F^* : F \in \mathcal{F}\}$. Let $x \in O$ and $F \in \mathcal{F}$. Then $\mathcal{M}_x \in \hat{O} \subset cl_{wX}(\hat{O}) \subset \hat{F} = \cap \{F^* : F \in \mathcal{F}\}$ and $\mathcal{M}_x \in F^*$. Hence $F \in \mathcal{M}_x$ and $x \in F$. Thus, for each $F \in \mathcal{F}$ we have $O \subset F$, and by Theorem 5.3, $O^* \subset F^*$. Therefore, $O^* \subset \cap \{F^* : F \in \mathcal{F}\} = cl_{wX}(\hat{O})$.

5.5 Corollary. Let \hat{O}' be open in $v(X)$ and

$\hat{F}' = cl_{v(X)}(\hat{O}')$. Then

$$\tilde{O}' \subset \hat{O}' \subset O' \subset cl_{v(X)}(\hat{O}') = \hat{F}'.$$

5.6 Corollary. Let O be a nonempty open set in wX and $\mathcal{F} = \{F: F \text{ is closed and } F \supset O\}$. Then $\text{cl}_{wX}(\hat{O}) = \{K: \mathcal{F} \subset K\} = \cap\{F^*: F \in \mathcal{F}\}$.

5.7 Corollary. Let O be a nonempty open set in X and $\mathcal{F} = \{F: F \text{ is closed and } F \supset O\}$. Then $\text{cl}_{wX}(O^*) = \{K: \mathcal{F} \subset K\} = \cap\{F^*: F \in \mathcal{F}\}$.

Recall that a closed set F is called regularly closed provided $F = \text{cl}(\text{int}(F))$. The following theorem characterizes the closed filters that give rise to the nonempty regularly closed subsets in wX .

5.8 Theorem. Let \hat{F} be a nonempty closed subset in wX and \mathcal{F} the corresponding closed filter on X . Then \hat{F} is regularly closed in wX if and only if there exists an open set O in X such that $\mathcal{F} = \{F: F \text{ is closed and } F \supset O\}$.

Proof. We are given that $\hat{F} = \{K: \mathcal{F} \subset K\} = \cap\{F^*: F \in \mathcal{F}\}$. Suppose \hat{F} is regularly closed. Let $\hat{O} = \text{int}_{wX}(\hat{F})$. Then $\hat{F} = \text{cl}_{wX}(\hat{O})$. Let $\tilde{O} = \{M_x: M_x \in \hat{O}\}$ and $O = \{x: M_x \in \tilde{O}\}$. By Corollary 5.6, $\mathcal{F} = \{F: F \text{ is closed and } F \supset O\}$.

Conversely, suppose that O is a nonempty open set in X and $\mathcal{F} = \{F: F \text{ is closed and } F \supset O\}$ and $\hat{F} = \{K: \mathcal{F} \subset K\} = \cap\{F^*: F \in \mathcal{F}\}$. Always, $\text{cl}_{wX}(\text{int}_{wX}(\hat{F})) \subset \hat{F}$.

Since $F \supset O$ we have $F^* \supset O^*$ for each $F \in \mathcal{F}$ by Theorem 5.3. Hence $O^* \subset \text{int}_{wX}(\hat{F})$. By Corollary 5.7, $\hat{F} = \cap\{F^*: F \in \mathcal{F}\} = \text{cl}_{wX}(O^*) \subset \text{cl}_{wX}(\text{int}_{wX}(\hat{F})) \subset \hat{F}$. Therefore, $\hat{F} = \text{cl}_{wX}(\text{int}_{wX}(\hat{F}))$, and \hat{F} is regularly closed.

5.9 Lemma. Let O_1 and O_2 be open sets with $\bar{O}_1 = \bar{O}_2 \neq \emptyset$. Set $\mathcal{F}_i = \{F: F \text{ is closed and } F \supset O_i\}$ for $i = 1, 2$. Then $\mathcal{F}_1 = \mathcal{F}_2$.

Let O be a nonempty open set in X and $\mathcal{F} = \{F: F \text{ is closed and } F \supset O\}$. Then $O(\mathcal{F}) = \{Q: Q \text{ is open and } X-Q \notin \mathcal{F}\} = \{Q: X-Q \not\supset O\} = \{Q: Q \cap O \neq \emptyset\}$. Thus, we have the following characterization of the nonempty regularly closed subsets of $v(X)$, given by Corollary 5.11.

5.10 Theorem. Let \mathcal{S} be the collection of equivalence classes of the nonempty open sets with the equivalence relation $O_1 \sim O_2$ provided $\bar{O}_1 = \bar{O}_2$. Then there exists a one-to-one correspondence between \mathcal{S} and the nonempty regularly closed subsets of wX .

5.11 Corollary. Let F' be a nonempty closed subset of $v(X)$. F' is regularly closed in $v(X)$ if and only if there exists a nonempty open set O in X and an open grill \mathcal{G} defined by $\mathcal{G} = \{Q: Q \text{ open in } X \text{ and } O \cap Q \neq \emptyset\}$ such that $F' = \{\mathcal{P} \in v(X): \mathcal{P} \subset \mathcal{G}\} = \{\mathcal{P} \in v(X): P \cap O \neq \emptyset \text{ for each } P \in \mathcal{P}\}$.

5.12 Theorem. Let $\emptyset \neq M \neq X$, with M both open and closed. Let $\mathcal{F} = \{F: F \text{ is closed and } F \supset M\}$. Set $\hat{F} = \{K \in wX: \mathcal{F} \subset K\} = \bigcap \{F^*: F \in \mathcal{F}\}$. Then $\emptyset \neq \hat{F} \neq wX$ and \hat{F} is both open and closed in wX .

Proof. Easily $\emptyset \neq \hat{F} \neq wX$ and \hat{F} is closed. By Corollary 5.7, $\hat{F} = \text{cl}_{wX}(M^*)$, where, since M is open, $M^* = \{K \in wX: M \in O(K)\} = \{K \in wX: X-M \notin K\}$ and since M

is closed, $M^* = \{K \in wX: M \in K\}$ which is closed in wX . Hence $M^* = \text{cl}_{wX}(M^*)$. Thus, F is both open and closed.

5.13 *Theorem.* Let \hat{F} be both open and closed in wX with $\phi \neq \hat{F} \neq wX$. Let $\tilde{F} = \{M_x: M_x \in \hat{F}\}$ and $F = \{x: M_x \in \tilde{F}\}$. Then $\phi \neq F \neq X$ and F is both open and closed.

Proof. Since \hat{F} is a nonempty open set it follows that F is a nonempty open set. Since $\hat{F} \neq wX$, we have that $F \neq X$. Since \tilde{X} is dense in wX it follows that $F \neq \phi$. Let $\hat{G} = wX - \hat{F}$. Then G is open in X and $F = X - G$ is closed.

5.14 *Theorem.* X is connected if and only if wX is connected.

By Theorem 5.3, we have that if O is open and F is closed with $O \subset F$ then $O^* \subset F^*$. Using this result, we are able to characterize the nowhere dense subsets of wX .

5.15 *Theorem.* Let \hat{F} be a nonempty closed subset of wX and \mathcal{J} the closed filter on X such that $\hat{F} = \{K \in wX: \mathcal{J} \subset K\} = \cap \{F^*: F \in \mathcal{J}\}$. If there exists a nonempty open set O in X with $O \subset F$ for each $F \in \mathcal{J}$ then $O^* \subset \hat{F}$.

5.16 *Definition.* A closed filter \mathcal{J} on X will be called nowhere dense if for each nonempty open set O there exists $F \in \mathcal{J}$ with $O \not\subset F$.

5.17 *Theorem.* Let \hat{F} be a nonempty closed subset of wX and \mathcal{J} the generating closed filter. \hat{F} is nowhere dense if and only if \mathcal{J} is nowhere dense.

5.18 *Theorem.* wX is second category if and only if given any countable family $\{\mathcal{F}_i: i \in \mathbb{N}\}$ of nowhere dense closed filters there exists a prime closed filter K such that $\mathcal{F}_i \not\subseteq K$ for each $i \in \mathbb{N}$.

5.19 *Theorem.* wX is a Baire space if and only if for each countable family $\{\mathcal{F}_i: i \in \mathbb{N}\}$ of nowhere dense closed filters there exists a closed filter \mathcal{J} such that $\{K \in wX: \mathcal{J} \not\subseteq K\} \subset \cup\{\mathcal{F}_i: i \in \mathbb{N}\}$.

5.20 *Theorem.* Let \hat{F} be a nonempty closed subset of wX generated by the closed filter \mathcal{J} . F is a G_δ closed set if and only if there exists a countable family $\{\mathcal{F}_i: i \in \mathbb{N}\}$ of closed filters such that for each $K \in wX$ we have $\mathcal{J} \subset K$ if and only if $\mathcal{F}_i \not\subseteq K$ for each $i \in \mathbb{N}$.

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