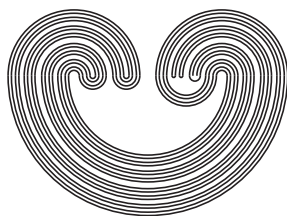

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SPACES HAVING SUBREGULAR BASES

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SPACES HAVING SUBREGULAR BASES

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A collection of subsets of a set X is said to have subinfinite rank provided that every infinite subcollection with nonempty intersection contains members related by set inclusion. Requiring open covers to have open refinements with subinfinite rank generalizes metacompactness and a base with subinfinite rank generalizes a point regular (uniform) base. This concept of a base of subinfinite rank was introduced in [GN]. In this paper we modify the concept of subinfinite rank to give a generalization of paracompactness (locally subinfinite refinable) and of a regular base (subregular base). The locally finite analogs of the following theorems hold.

Theorem [GN]. A topological space is metacompact if and only if every open cover has a Noetherian open refinement with subinfinite rank.

Theorem [FG]. A T_3 space is metacompact if and only if every open cover has an ω -Noetherian open refinement with subinfinite rank.

Theorem [FG]. A T_1 space having a base with subinfinite rank is hereditarily metacompact.

We will present properties of these generalizations of paracompactness and regular base which do not have subinfinite rank analogs and which are not shared by some of

the more standard generalizations of these concepts. For example, if X is a T_2 first countable locally subinfinite refinable space then X is T_3 . A T_2 space is metrizable if and only if it has a Noether subregular base.

Let us now introduce some definitions, terminology and results which will be used in this paper. We will use Z to denote the integers, N or Z^+ the natural numbers and ω the first countable ordinal. A collection \mathcal{W} of subsets of a set X is said to be well ordered by set inclusion provided there is an ordinal λ and a function $W: \lambda \rightarrow \mathcal{W}$ such that W is onto and if $\alpha < \beta < \lambda$ then $W(\alpha) \subset W(\beta)$. We say a collection of subsets of a set X is Noetherian (ω -Noetherian) provided every well ordered subcollection is finite (countable) [LN]. If \mathcal{V} is a Noetherian (ω -Noetherian) collection with subinfinite rank and nonempty intersection then \mathcal{V} contains a finite (countable) subcollection \mathcal{W} with $U\mathcal{W} = U\mathcal{V}$ [FG]. A collection \mathcal{U} of subsets of a set X is said to be directed provided if $H, K \in \mathcal{U}$ then there is a $W \in \mathcal{U}$ such that $H \cup K \subseteq W$. If \mathcal{V} is a collection of subsets of a set X with subinfinite rank and nonempty intersections then \mathcal{V} can be expressed as the union of finitely many directed subcollections [FG].

If \mathcal{U} is a collection of subsets of a set X and $x \in X$ then $st(x, \mathcal{U}) = \{U \in \mathcal{U} : x \in U\}$ and $St(x, \mathcal{U}) = Ust(x, \mathcal{U})$. If \mathcal{W} is a collection of subsets of a set X , then $M \subseteq U\mathcal{W}$ is maximally distinguished with respect to \mathcal{W} if every member of \mathcal{W} contains at most one element of M and $U\mathcal{W} = U\{St(x, \mathcal{W}) : x \in M\}$. For every collection \mathcal{U} of subsets of a set X

there is a maximally distinguished set with respect to \mathcal{U} [Au]. A collection \mathcal{V} of subsets of a set X is said to refine another collection \mathcal{U} of subsets of X ($\mathcal{V} < \mathcal{U}$) provided that for every $W \in \mathcal{V}$ there is a $U \in \mathcal{U}$ with $W \subseteq U$ and $U \cap \mathcal{U} = U \cap \mathcal{V}$.

1. Locally Subinfinite Refinable Spaces

A collection \mathcal{V} of subsets of a topological space X is said to be locally subinfinite provided for each $x \in X$ there is a neighborhood W of x such that every infinite subset of $\{V \in \mathcal{V} : V \cap W \neq \emptyset\}$ contains members related by set inclusion. For brevity we will call such spaces locally subinfinite refinable. Clearly, locally finite collections are locally subinfinite and locally subinfinite collections have subinfinite rank. All GO-spaces are locally subinfinite refinable (the open refinement constructed for LOTS in Lemma 3.1 in [Sc] is locally subinfinite and the same sort of construction works for GO-spaces) while the Moore plane and Bing's Example G are good examples of spaces which are not. The following lemmas provide a link between locally subinfinite collections and collections with subinfinite rank.

Lemma 1.1. A collection of subsets of a set X with nonempty intersection is locally subinfinite if and only if it has subinfinite rank.

Lemma 1.2. Suppose \mathcal{V} is a collection of subsets of a topological space X and $M \subseteq \mathcal{U}\mathcal{V}$ is maximally distinguished with respect to \mathcal{V} . If \mathcal{V} is locally subinfinite (has

subinfinite rank) then $\{St(x, V) : x \in M\}$ is locally finite (point finite).

Not surprisingly, many properties of paracompact spaces do not have locally subinfinite analogs. The countable ordinals ω_1 with the order topology is a countably compact locally subinfinite refinable space which is not compact. The product of a locally subinfinite refinable space and a compact space need not be locally subinfinite refinable. For example, $\omega_1 \times (\omega_1 + 1)$ is not locally subinfinite refinable. However some properties do remain. If \mathcal{V} is a locally subinfinite open cover of a topological space X then $\{cl(V) : V \in \mathcal{V}\}$ and $\{int(cl(V)) : V \in \mathcal{V}\}$ are locally subinfinite covers.

Theorem 1.3. If X is a T_2 first countable locally subinfinite refinable space then X is T_3 .

Proof. Let $x \in X$ and F be a closed subset of $X \setminus \{x\}$. Suppose if U is a neighborhood of x then $cl(U) \cap F \neq \emptyset$. Let $\{B(n) : n \in \mathbb{N}\}$ be a neighborhood base for x where if $m < n$ then $B(m) \supset B(n)$ (if x is isolated then the result is obvious).

Choose $n(0) \in \mathbb{N}$ such that $B(n(0)) \subseteq X \setminus F$ and a $z(1) \in cl(B(n(0))) \cap F$. Choose an open neighborhood $W(1)$ of $z(1)$ and an $n(1) \in \mathbb{N}$ with $n(1) > n(0)$ such that $B(n(1)) \cap W(1) = \emptyset$. Choose $z(2) \in cl(B(n(1))) \cap F$ (note that $z(2) \notin W(1)$), $W(2)$ an open neighborhood of $z(2)$ and $n(2) \in \mathbb{N}$ with $n(2) > n(1)$ such that $B(n(2)) \cap W(2) = \emptyset$ and $z(1) \notin W(2)$. Continue to choose points $z(1), z(2), \dots \in F$,

neighborhoods $W(1), W(2), \dots$ of $z(1), z(2), \dots$ respectively and integers $n(1) < n(2) < \dots$ such that

(1) if $i, j \in \mathbb{N}$ and $i \neq j$ then $z(i) \notin W(j)$

(2) if $i \in \mathbb{N}$ then $W(i) \cap B(n(i)) = \emptyset$.

Let $A = \{z(i) : i \in \mathbb{N}\}$. Since $A \subset F$, it is easy to show that A is closed. Let \mathcal{V} be an open locally subinfinite refinement of $\{X \setminus A\} \cup \{W(i) : i \in \mathbb{N}\}$. For each $i \in \mathbb{N}$ let $V(i) \in \mathcal{V}$ such that $z(i) \in V(i)$. Notice that if $i, j \in \mathbb{N}$ and $i \neq j$ then $z(i) \notin V(j)$. Let U be an open neighborhood of x . There is an $m \in \mathbb{N}$ such that $B(m) \subset U$. Let $S = \{i \in \mathbb{N} : n(i) \geq m\}$ and note, for every $i \in S$, $V(i) \cap U \neq \emptyset$. Therefore $\{V(i) : i \in S\}$ is an infinite incomparable subcollection of $\{W \in \mathcal{V} : W \cap U \neq \emptyset\}$. However \mathcal{V} is locally subinfinite. Thus there is a neighborhood U of x such that $\text{cl}(U) \cap F = \emptyset$ and so X is T_3 .

Notice that in Theorem 1.3 we only needed countably locally subinfinite refinable. We do need first countability as is demonstrated by the following example, based on Example 100 of [SS].

Example 1.4. Let $X = (\omega_1 \times \mathbb{Z}) \times \{0, 1\}$. We topologize X as follows. Points of the form (α, k) where $\alpha < \omega_1$ and $k \in \mathbb{Z} \setminus \{0\}$ are isolated. If $\alpha < \omega_1$ then basic open sets for $(\alpha, 0)$ are given by $B(\alpha, \gamma, m) = [(\gamma, \alpha] \times \{k \in \mathbb{Z} : |k| \geq m\} \cup [(\gamma, \alpha] \times \{0\}$ where $\gamma < \alpha$ and $m \in \mathbb{Z}^+$. Basic neighborhoods of 0 and 1 are given by

$$B(0, \beta) = \{0\} \cup [(\beta, \omega_1) \times \mathbb{Z}^+] \quad \text{for } 0,$$

$$B(1, \beta) = \{1\} \cup [(\beta, \omega_1) \times \mathbb{Z}^-] \quad \text{for } 1$$

where $\beta < \omega_1$. This space is T_2 but 0 and 1 cannot be separated by open sets whose closures miss (not $T_{2.5}$). The points 0 and 1 do not have countable neighborhood bases.

To show that X is locally subinfinite refinable let \mathcal{U} be an open cover of X . Let $\gamma_0, \gamma_1 < \omega_1$ such that $B(0, \gamma_0) \subseteq U$ and $B(1, \gamma_1) \subseteq U'$ some $U, U' \in \mathcal{U}$. For each $\beta < \omega_1$ let $\alpha(\beta) < \beta$ and $k(\beta) \in \mathbb{N}$ be chosen such that $B(\beta, \alpha(\beta), k(\beta)) \subseteq V$ some $V \in \mathcal{U}$. By the pressing-down lemma there is a $\xi < \omega_1$ and an uncountable set $A \subseteq \omega_1$ such that for all $\beta \in A$, $\alpha(\beta) = \xi$. Let $\xi^* = \max\{\xi, \gamma_0, \gamma_1\}$. Since $[0, \xi^*]$ is compact, there is a finite set $\{\sigma_1, \sigma_2, \dots, \sigma_n\} \subseteq [0, \xi^*]$ such that $[0, \xi^*] = \cup\{(\alpha(\sigma_i), \sigma_i] : i = 1, 2, \dots, n\}$. Since A is uncountable, there is an uncountable set $A' \subseteq A$ and a $k^* \in \mathbb{N}$ such that for all $\beta \in A'$, $k(\beta) = k^*$. The open cover $\{B(\beta, \xi^*, k^*) : \beta \in A'\} \cup \{B(\sigma_i, \alpha(\sigma_i), k(\sigma_i)) : i = 1, 2, \dots, n\} \cup \{B(0, \gamma_0), B(1, \gamma_1)\} \cup \{(\delta, k) : (\delta, k) \in X \text{ and is not in a previously defined member of the cover}\}$ is a locally subinfinite open refinement of \mathcal{U} .

If we let $Y = X \setminus \{1\}$ then Y with the subspace topology is $T_{2.5}$ but not T_3 locally subinfinite refinable space. Along the same lines as Example 1.4 we can construct a T_3 space which is not T_4 having this covering property. I do not know if every T_2 first countable locally subinfinite refinable space needs to be T_4 but this appears unlikely.

A space X is called discretely expandable if every discrete collection of subsets of X is expandable to a locally finite open collection [SK].

Theorem 1.5. A locally subinfinite refinable space X is discretely expandable.

Proof. Suppose $\mathcal{J} = \{F(a) : a \in A\}$ is a discrete collection of subsets of X where if $a, b \in A$ and $a \neq b$ then $F(a) \neq F(b)$. For every $x \in X$ let $U(x)$ be a neighborhood of x which meets at most one member of \mathcal{J} . Let \mathcal{V} be a locally subinfinite open refinement of $\{U(x) : x \in X\}$ and for each $a \in A$ let $G(a) = \cup\{V \in \mathcal{V} : V \cap F(a) \neq \emptyset\}$. Note that since each member of \mathcal{V} meets at most one member of \mathcal{J} , if $a, b \in A$ and $a \neq b$ then $G(a) \cap F(b) = \emptyset$.

Let $x \in X$ and W an open neighborhood of x such that incomparable subsets of $\{V \in \mathcal{V} : V \cap W \neq \emptyset\}$ are finite. Let $S = \{a \in A : G(a) \cap W \neq \emptyset\}$ and for each $a \in S$ let $V(a) \in \mathcal{V}$ such that $V(a) \cap F(a) \neq \emptyset$ and $V(a) \cap W \neq \emptyset$. Note that if $a, b \in S$ and $a \neq b$ then $V(a)$ and $V(b)$ are incomparable. Since $\{V(a) : a \in S\}$ is an incomparable subset of $\{V \in \mathcal{V} : V \cap W \neq \emptyset\}$, S is finite and therefore $\{G(a) : a \in A\}$ is locally finite.

Theorem 1.3 can be thought of as a corollary to Theorem 1.5, since every discretely expandable T_2 space is collectionwise Hausdorff, and every collectionwise Hausdorff, first countable space is regular. The latter is an unpublished result of Z. Nagy and S. Purisch, proven similarly to Theorem 1.3.

Corollary 1.6. A normal locally subinfinite refinable space is collectionwise normal. [SK; Corollary 2.3]

Corollary 1.7. A T_2 locally subinfinite refinable developable space is metrizable. [SK; Theorem 8.3]

A T_2 space is paracompact if and only if every open cover well ordered by set inclusion has a locally finite open refinement [Ma]. A collection of subsets of a topological space well ordered by set inclusion is locally subinfinite. In fact, it is such collections which stop locally subinfinite collections from being locally finite.

Lemma 1.8. A Noetherian locally subinfinite cover of a topological space contains a locally finite subcover.

Proof. If \mathcal{V} is a Noetherian locally subinfinite cover of a topological space X then the collection of all maximal elements of \mathcal{V} is a locally finite cover of X .

Theorem 1.9. The following are equivalent for a T_2 space X :

- (1) X is paracompact,
- (2) every open cover of X has a point finite locally subinfinite open refinement,
- (3) every open cover of X has a Noetherian locally subinfinite open refinement.

In the case of T_3 spaces we need only find σ -locally finite open refinements. We are, in this situation, able to have larger (countable) subcollections well ordered by set inclusion in the locally subinfinite refinements.

Theorem 1.10. The following are equivalent for a T_3 space X :

(1) *X is paracompact,*

(2) *every open cover of X has a point countable σ -locally subinfinite open refinement,*

(3) *every open cover of X has an ω -Noetherian σ -locally subinfinite open refinement.*

Proof. Clearly, (1) \rightarrow (2) \rightarrow (3). Let's show that (3) \rightarrow (1). Let \mathcal{U} be an open cover of X and $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ be an ω -Noetherian open refinement of \mathcal{U} such that V_n is locally subinfinite for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and M_n be a maximally distinguished subset of V_n . Let $x \in M_n$. Since $\{V \in \mathcal{V}_n : x \in V\}$ is ω -Noetherian and has subinfinite rank, it has a countable subset \mathcal{W} such that $U\mathcal{W} = U\{V \in \mathcal{V}_n : x \in V\}$, say $\mathcal{W} = \{W(x,n,i) : i \in \mathbb{N}\}$. By Lemma 1.2 we know that $\{W(x,n,i) : x \in M_n\}$ is locally finite for each $i \in \mathbb{N}$. Thus $\{W(x,n,i) : n \in \mathbb{N}, x \in M_n \text{ and } i \in \mathbb{N}\}$ is a σ -locally finite open refinement of \mathcal{U} .

For a topological space X, $c(X) = \omega * \sup\{|\mathcal{G}| : \mathcal{G} \text{ is a pairwise disjoint collection of open subsets of X}\}$. The following lemma is readily verified. Theorem 1.12 follows from the lemma.

Lemma 1.11. Let X be a topological space.

(i) *The collection $\{cl(U) : U \text{ is open in X}\}$ is $c(X)$ -Noetherian.*

(ii) *If \mathcal{U} is a collection of open subsets of X having subinfinite rank and nonempty intersections then $\{\text{int}(cl(U)) : U \in \mathcal{U}\}$ also has subinfinite rank.*

(iii) If V is a collection of regular open subsets of X having subinfinite rank and nonempty intersection then there is a $\mathcal{W} \subseteq V$ with $|\mathcal{W}| \leq c(X)$ and $\cup \mathcal{W} = \cup V$.

Theorem 1.12. A T_3 space having the Souslin property ($c(X) = \omega$) is Lindelöf if and only if every open cover of X has an open σ -locally subinfinite open refinement. (The metacompact version of this theorem is in [Gr₁].)

2. Subregular Bases

A base β for a topological space X is regular if for every $x \in X$ and every neighborhood U of x there is a neighborhood V of x with $V \subseteq U$ such that $\{B \in \beta: B \cap V \neq \emptyset \text{ and } B \not\subseteq U\}$ is finite [E].

Arhangel'skii Metrization Theorem. A topological space is metrizable if and only if it is a T_1 -space with a regular base.

A regular base for a topological space need not be locally subinfinite. In fact, if a T_2 space has a locally subinfinite base then it has the discrete topology. We need T_2 since a countably infinite set with the cofinite topology is a T_1 space having a locally subinfinite base.

A collection \mathcal{G} of subsets of a topological space X is said to be subregular provided for each $G \in \mathcal{G}$ and $x \in G$ there is an open neighborhood U of x with $U \subseteq G$ such that every incomparable subset of $\{G' \in \mathcal{G}: G' \cap U \neq \emptyset \text{ and } G' \not\subseteq G\}$ is finite. Clearly, locally subinfinite collections are subregular and a regular base is subregular. Although subregular collections need not be locally

subinfinite, subregular covers do contain locally subinfinite subcovers. To see this let \mathcal{U} be a subregular cover of a space X and $M \subseteq X$ be maximally distinguished with respect to \mathcal{U} . Then $\cup\{st(x, \mathcal{U}) : x \in M\}$ is a locally subinfinite subcover of \mathcal{U} .

Any discretization of a metric space such as the Michael line has a Noetherian subregular base but need not have a regular base. A subregular base appears to be the generalization of a regular base that corresponds to a base of subinfinite rank as a generalization of uniform base (point regular base). The next two theorems give a good indication of this.

Theorem 2.1. *Let X be a T_2 space having a subregular base β . Then X is hereditarily paracompact.*

Proof. Since having a subregular base is hereditary, we need only show that X is paracompact. Let \mathcal{W} be an open cover of X . [The following construction of a locally finite open refinement of \mathcal{W} is exactly the same as Forster's construction of the point finite open refinement in Theorem 3.3 of [FG].]

By induction choose a family $\{\beta_n : n < \omega\}$ of subsets of β and a family $\{M_n : n < \omega\}$ of subsets of X such that for all $n < \omega$:

- (1) $\beta_0 < \mathcal{W}$,
- (2) M_n is maximally distinguished with respect to β_n ,
- (3) $\cup\{M_j : j \leq n\}$ is closed in X ,
- (4) $\beta_{n+1} < \{W \setminus \cup\{M_j : j \leq n\} : W \in \mathcal{W}\}$.

For all $n < \omega$ and $x \in M_n$, we can decompose $\text{st}(x, \beta_n)$ into a finite family of directed subsets $\{\mathcal{G}(x, i) : i \leq k_x\}$. If $i \leq k_x$ and $\mathcal{G}(x, i) \subset W$ for some $W \in \mathcal{W}$ then let $V(x, i) = \cup \mathcal{G}(x, i)$; otherwise choose $V(x, i) \in \mathcal{G}(x, i)$.

Let $V_0 = \{V(x, i) : x \in M_0 \text{ and } i \leq k_x\}$. Assume that V_0, \dots, V_n have been defined and define $V_{n+1} = \{V(x, i) : x \in M_{n+1}, i \leq k_x \text{ and } V(x, i) \not\subseteq \cup V_j : j \leq n\}$. Let $V = \cup \{V_n : n < \omega\}$. The collection V is an open refinement of \mathcal{W} (see [FG]). To show that V is locally finite we make two observations.

(1) The collections $\{\cup V_k : k < \omega\}$ is locally finite.

(2) If $k < \omega$ and $x \in X$ then there is an open neighborhood of x meeting only finitely many members of V_k .

Clearly, these observations show that V is locally finite. The proof of observation 1 is similar to the proof that this collection is point finite in Theorem 3.3 of [FG]. To prove observation 2, let $x \in X$ and $k = \min\{n < \omega : x \in V_n\}$.

Let $y \in M_k$ and $i \leq k_y$ such that $x \in V(y, i)$. If $V(y, i) \in \beta$ then let $W = V(y, i)$. Otherwise let $W \in \mathcal{G}(y, i)$ such that $x \in W$. Choose an open neighborhood G of x with $G \subseteq W$ such that incomparable subcollections of $\{B \in \beta : B \cap G \neq \emptyset \text{ and } B \not\subseteq W\}$ are finite. It is straightforward to verify that G meets only finitely many members of V_m where $m \geq k$.

Suppose $0 < k$. The set $\cup \{M_1 : i \leq k - 1\}$ is closed, $\cup \{M_1 : i \leq k\} \subseteq \cup \{\cup V_1 : i \leq k\}$ and $x \notin \cup \{\cup V_1 : i \leq k - 1\}$. Choose a $W \in \beta$ such that $x \in W \subseteq X \setminus \cup \{M_1 : i \leq k - 1\}$ and an

open neighborhood G of x with $G \subseteq W$ such that incomparable subsets of $\{B \in \beta: B \cap G \neq \emptyset \text{ and } B \not\subseteq W\}$ are finite. Once again it is straightforward to verify that G meets only finitely many members of V_i for each $i \leq k - 1$.

The Sorgenfrey line, lexicographically ordered unit square and the Michael line are the standard examples of spaces having bases of subinfinite rank. All three are hereditarily paracompact GO-spaces.

Theorem 2.2. *A GO-space having a base with subinfinite rank has a subregular base consisting of intervals.*

Proof. Let X be a GO-space having a base with subinfinite rank β . For each $B \in \beta$ and $x \in B$ let $I(B,x) = \cup\{V \subseteq B: V \text{ is an interval, open in } X\}$. For each $B \in \beta$ let $\mathcal{I}(B) = \{I(B,x): x \in B\}$ and note that $\mathcal{I}(B)$ is a disjoint collection of intervals open in X with $B = \cup \mathcal{I}(B)$. Let $\beta' = \cup\{\mathcal{I}(B): B \in \beta\}$. For each $G \in \beta'$ let $B(G) \in \beta$ such that $G \in \mathcal{I}(B)$. Note that for each $G \in \beta'$ and $x \in G$, $G = I(B(G),x)$. Clearly, β' is a base for X . To see that β' has subinfinite rank we need only observe that if $H, H' \in \beta'$, $H \cap H' \neq \emptyset$ and $H \neq H'$ then $B(H) \neq B(H')$ and note that if $B, B' \in \beta$ and $B \subseteq B'$ then $I(B,x) \subseteq I(B',x)$ for every $x \in B$. We will now show that β' is subregular.

Let $x \in X$ and $B \in \beta'$ with $x \in B$. There are points $a, b \in B$ such that $x \in \text{int}[a,b] \cap [a,b] \subseteq B$. Suppose \mathcal{G} is an infinite subset of β' such that for all $G \in \mathcal{G}$, $G \cap \text{int}[a,b] \neq \emptyset$ and $G \not\subseteq B$. Since \mathcal{G} consists of intervals, for each $G \in \mathcal{G}$ either $a \in G$ or $b \in G$. Hence there is an

infinite $\mathcal{G}' \subseteq \mathcal{G}$ with $\cap \mathcal{G}' \neq \emptyset$. Thus, since \mathcal{G}' has subinfinite rank, there is a $G, G' \in \mathcal{G}'$ such that $G \subset G'$. Hence β' is subregular.

Note that if the base β is a Noetherian base with subinfinite rank then β' is also Noetherian.

Corollary 2.3. *A GO-space having a Noetherian base with subinfinite rank has a Noetherian subregular base consisting of intervals.*

I do not know of an hereditarily paracompact GO-space which has been shown to not have a subregular base. In particular, does (can) a Souslin line have a subregular base? It is not known if every hereditarily paracompact space with a base of subinfinite rank must have a subregular base.

A collection \mathcal{P} of subsets of a set X is said to be Noether provided for each $P \in \mathcal{P}$, $\{P' \in \mathcal{P} : P \subseteq P'\}$ is finite [Ar]. Noether collections are Noetherian but Noetherian collections need not be Noether. Unlike the case for spaces with Noetherian bases, many well known spaces do not have Noether bases. For example, any first countable T_1 space which does not have a point countable base can not have a Noether base (see Theorem 2.5 [Gr₂]). Some of the more surprising properties of spaces having Noether bases (Noether Spaces) can be found in [M].

Theorem 2.4. *If β is a Noether subregular base for a T_1 space X then X is σ -locally finite.*

Proof. Let β_0 be the set of all maximal members of β . Since β is Noetherian and subregular, β_0 is locally finite and if $B \in \beta \setminus \beta_0$ then there is a $B' \in \beta_0$ such that $B \subseteq B'$. The collection $\beta \setminus \beta_0$ is Noetherian and subregular, so let β_1 be the collection of all maximal members of $\beta \setminus \beta_0$. Note that if there is an $x \in X \setminus \beta_1$ then $\{x\}$ is open. Again β_1 is locally finite and if $B \in \beta \setminus (\beta_0 \cup \beta_1)$ then there is a $B' \in \beta_1$ such that $B \subset B'$. Continue in this manner to choose a locally finite collection $\beta_n \subseteq \beta \setminus \{\beta_k : k < n\}$ for each $n < \omega$ such that if $B \in \beta \setminus \{\beta_k : k \leq n\}$ then there is a $B' \in \beta_n$ such that $B \subset B'$.

Suppose that $B \in \beta \setminus \{\beta_n : n < \omega\}$. Then for each $n < \omega$ there is a $B_n \in \beta_n$ such that $B \subset B_n$. Note that if $n, m < \omega$ and $n \neq m$ then $B_n \neq B_m$. However, since β is Noether the collection $\{B' \in \beta : B \subset B'\}$ is finite. Hence $\beta = \{\beta_n : n < \omega\}$ where for each $n < \omega$, β_n is locally finite.

If a T_1 space has a σ -locally finite base then it has a σ -locally finite Noether base [Gr₂]. However, Example 1.57 of [E] is a T_2 space with a σ -locally finite base (in fact countable base) which is not even paracompact. On the other hand, the Michael line has a Noetherian point countable subregular base but not a σ -locally finite base. The following corollary to Theorem 2.4 follows from Theorem 2.1 and the Nagata-Smirnov Metrization Theorem.

Corollary 2.5. A T_2 space is metrizable if and only if it has a Noether subregular base.

Although a Noether subregular base need not be a regular base, does it contain one?

Some interesting results concerning subinfinite rank do not have subregular analogs. The finite product of spaces having Noetherian bases with subinfinite rank has a Noetherian base of subinfinite rank and is therefore meta-compact [GN]. However the Michel line and the irrationals with the usual subspace topology both have Noetherian subregular bases but their product is not even normal (see Example 5.1.31 of [E]). A base for a T_1 space is a uniform base if and only if it is a Noetherian base of countable order with subinfinite rank (see [Gr₂]). Although a Noetherian subregular base of countable order need not be a regular base it is a σ -locally finite uniform base and if the space is T_2 it is metrizable.

It is not known if a subregular base or a base with subinfinite rank is preserved by a perfect mapping. Is the image of a locally subinfinite refinable space by a closed mapping locally subinfinite refinable? I am not aware of any significant results concerning these questions.

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