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DISPERSION POINTS AND FIXED POINTS OF SUBSETS OF THE PLANE

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During the Spring Topology Conference in 1986 Hidefumi Katsuura asked whether there is a connected subset X of the plane with the dispersion point p such that for some non-constant function f from X into itself the point p is not the fixed point of f. He also asked whether the function f can be onto. We answer both of these questions in affirmative.

Definition. A point p in a connected topological space X is said to be a *dispersion point* of X if each component of $X \setminus \{p\}$ consists of a single element, i.e. if $X \setminus \{p\}$ is totally disconnected.

Definition. If f is a continuous function from a space X into itself then a point x of X is said to be a *fixed point* of f if $f(x) = x$.

Connected spaces with dispersion points were first defined by Knaster and Kuratowski in [K.K], and were extensively studied by Duda in [D]. In [C.V.] Cobb and Voxman asked whether the dispersion point was a fixed point of any non-constant function f defined on a connected space with a dispersion point. In [K] Katsuura described a space X with a dispersion point p and a continuous non-constant mapping f on X such that p is not a fixed point.

We modify Katsuura's construction to obtain such an example in the plane. We show that function f may be onto. In the construction we use the following theorem by Katsuura:

Theorem [K]. *Suppose* X *is a totally disconnected space, and* {Y(i): i ∈ I} *the collection of all quasi-components of* X. *Let* F *be a proper closed subset of* X *that has a point in common with every quasi-component. Let* q *be the quotient map from* X *onto* X/F. *Then* X/F *is a connected space with the dispersion point* q(F).

Example 1. Let Q denote the set of rational numbers, let R denote the set of real numbers. Let C be the Cantor ternary set in the interval [0,1], i.e. $C = \{\sum_{n=1}^{\infty} \frac{a_n}{3^n}: a_n = 0,2 \text{ and } n = 1,2,3,\cdots\}$. If A is a subset of R and b is a real number, then $b + A = \{b + a: a \in A\}$ and $b * A = \{b \cdot a: a \in A\}$. If A is a subset of the plane and (x,y) is a pair of numbers then $(x,y) + A = \{(x + a,y + b): (a,b) \in A\}$ and $(x,y) * A = \{(xa,yb): (a,b) \in A\}$.

Let d be a real number and let $D = \{(c,d): c \in C\}$. For any point (u,d) in the plane and (c,d) in D let

$$S^+((u,d);(c,d)) = \{(c + |c - u|\cos t,\ d + |c - u|\sin t): 0 \le t \le \pi \text{ and } t = c + q \text{ for some } q \text{ in } Q\}$$ and

$$S^-((u,d);(c,d)) = \{(c + |c - u|\cos t,\ d + |c - u|\sin t): -\pi \le t \le 0 \text{ and } t = c + q \text{ for some } q \text{ in } Q\}.$$

We put $S^+((u,d);D) = \cup\{S^+(u,d);(c,d)): (c,d) \in D\}$ and

$$S^-((u,d);D) = \cup\{S^-(u,d);(c,d)): (c,d) \in D\}.$$

For any real number d let [a,b](d) denote the set $[a,b] \cap Q$ if d is a rational number, and $[a,b]\setminus Q$ if d is an irrational number. Put $S(0) = \cup\{\{c\} \times [0,1](c): c \in C\} \cup C \times \{0\} \cup C \times \{1\}$. Let $C(1,i) = \frac{8+2i}{27} + \frac{1}{27} * C$ and let $S(1,i) = \cup\{\{c\} \times [\frac{1}{2},3](c): c \in C(1,i)\}$, where $i = 1,2,3,4$.

Let $S(1) = S(1,1) \cup S(1,2) \cup S(1,3) \cup S(1,4) \cup S^{+}((\frac{15}{54},3); C(1,1) \times \{3\}) \cup S^{-}((\frac{23}{54},\frac{1}{2}); C(1,1) \times \{\frac{1}{2}\}) \cup S^{-}((\frac{31}{54},\frac{1}{2}); C(1,4) \times \{\frac{1}{2}\}) \cup S^{+}((\frac{39}{54},3); C(1,4) \times \{3\})$, (see figure 1).

For convenience we write $u(1,i) = \frac{7+8i}{54}$, $i = 1,2,3,4$. In order to obtain $S(2)$ we repeat the construction of $S(1)$ for the sets $C \cap [0,3^{-n}]$ and $C \cap [\frac{2}{3},1]$ and replace in that construction the segments $[2^{-1},3]$ by the segments $[2^{-2},3]$. The figure 2 shows the set $S(0) \cup S(1) \cup S(2)$.

Formal description of $S(n)$, $n > 1$, is as follows. Put

$$C(n,i) = 3^{-n} * C(n-1,i) \text{ if } i = 1,2,\cdots,2^{n}, \text{ and}$$

$$c(n,i) = \frac{2}{3} + 3^{-n} * C(n-1,i-2^{n}) \text{ if } i = 2^{n}+1,\cdots,2^{n+1}.$$

Let $S(n,i) = \cup\{\{c\} \times [2^{-n},3](c): c \in C(n,i)\}$, where $i = 1,2,\cdots,2^{n+1}$.

Let $u(n,i) = 3^{-n} \cdot u(n-1,i)$ if $i = 1,2,\cdots,2^{n}$, and $u(n,i) = \frac{2}{3} + 3^{-n} u(n-1,i)$ if $i = 2^{n}+1,\cdots,2^{n+1}$.

Let $S(n) = \cup\{S(n,i): i = 1,2,\cdots,2^{n+1}\} \cup S^{+}((u(n,1),3); C(n,1) \times \{3\}) \cup S^{-}((u(n,2),2^{-n}); C(n,1) \times \{2^{-n}\}) \cup S^{+}((u(n,3),2^{-n}); C(n,4) \times \{3\}) \cup \cdots \cup S^{+}((u(n,2^{n+1}),3); C(n,2^{n+1}) \times \{3\})$.

Let $X = \cup\{S(n): n = 0,1,2,\cdots\}$. Observe that any quasi-component $K(c)$ of X is the union of a segment-like set $\{c\} \times I(c)$ and $\sin(\frac{1}{x})$-like curve emerging from $(\frac{4}{9} + \frac{1}{9} c,3)$, where $c \in C$. By the theorem of Katsuura the quotient $Y = X/C \times \{0\}$ is a connected space and $q(C \times \{0\})$

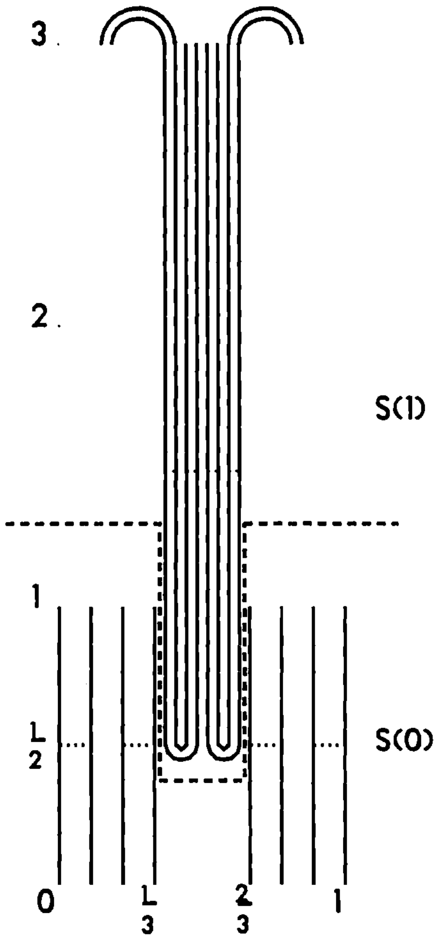

FIGURE 1

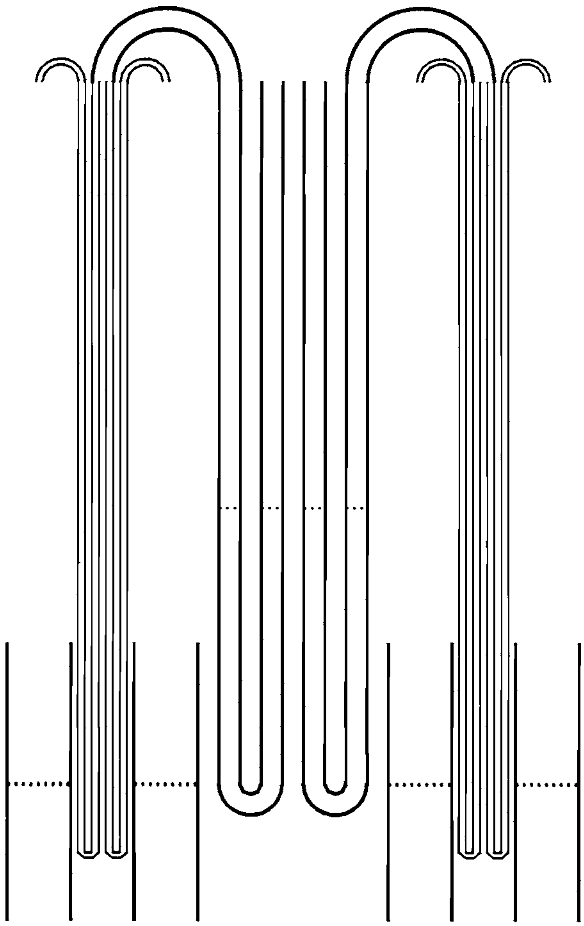

FIGURE 2

is the dispersion point. By q we denote the quotient map from X onto Y.

Let g be a linear and order-preserving mapping from $C(n,i)$ onto $[0,3^{-n}] \cap C$ if $i \equiv 2(\text{mod}4)$, and onto $[\frac{2}{3^n},\frac{3}{3^n}] \cap C$ if $i \equiv 3(\text{mod}4)$, and let g be a linear and order-reversing mapping from $C(n,i)$ onto $[0,3^{-n}] \cap C$ if $i \equiv 1(\text{mod}4)$, and onto $[\frac{2}{3^n},\frac{3}{3^n}] \cap C$ if $i \equiv 0(\text{mod}4)$. Let the map f from X into itself be defined as follows:

$$f(x) = (0,1) \text{ if } x \in S(0),$$
$$f(a,b) = (0,0) \text{ if } b \geq \tfrac{5}{2},$$
$$f(a\cdot b) = (g(a),\tfrac{5}{2} - b) \text{ if } \tfrac{3}{2} < b < \tfrac{5}{2},$$
$$f(a,b) = (g(a),1) \text{ if } (a,b) \in S(n,i) \text{ and } b \leq \tfrac{3}{2},$$
$$f(x) = (g(c),1) \text{ if } x \in S^{-}((u(n,i),2^{-n});(c,2^{-n})) \text{ for some } c \text{ in } C(n,i).$$

Let f_q denote a map from Y into itself induced by f. The map f_q is a continuous and non-constant function, and the dispersion point is not a fixed point of the map. The proof of continuity is straightforward but tedious.

Example 2. We modify the example 1 to obtain a mapping onto. Let f and X have the same meaning as in the example 1. For any point c in the Cantor set C let $D(c)$ denote the set of all the points on the segment joining $(\frac{4}{9} + \frac{1}{9}c,4)$ and $(c,5)$ the second coordinate of which is rational if c is rational, and irrational if c is likewise. Let

$$X(0) = X \cup \cup\{D(c): c \in C\} \cup \cup\{\{c\} \times [3,4](c): c \in C(1,2) \cup C(1,3)\} \text{ (see figure 3).}$$

Let $X(n) = (0,5) + X(n-1)$ for $n = 1,2,3,\cdots$. Put $X(\infty) = \cup\{X(n): n = 0,1,2,\cdots\}$. Let F be a mapping from $X(\infty)$

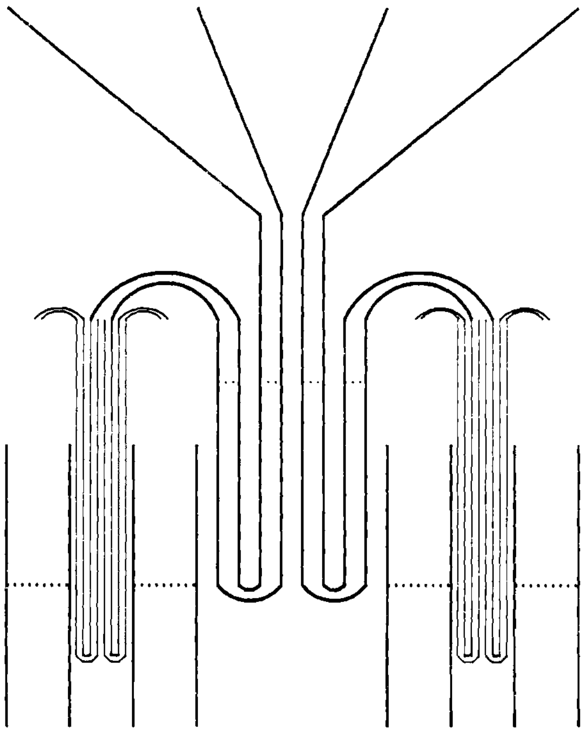

FIGURE 3

into itself defined by

$$F|X = f$$

$$F(x) = (0,0) \text{ if } x \in X(0) \setminus X$$

$$F(x) = x - (0,5) \text{ if } x \in X(n),\ n = 1,2,3,\cdots$$

It is easy to see that F is onto.

Let Z be the quotient space $X(\infty)/C \times \{0\}$, let q be the quotient map from $X(\infty)$ onto Z and let F_q be the function on Z induced by F. Observe that Z is a connected subset of the plane with the dispersion point $q(C \times \{0\})$, F_q is a continuous function from Z onto itself, and the dispersion point is not a fixed point of F_q.

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