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by

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1. Introduction

In recent years ordered sets and graphs have been considered as generalized metric spaces where, instead of real numbers, the values of the distance belong to an ordered semigroup with some additional properties; and some results on ordered sets, graphs and classical metric spaces have been unified under this frame. In this paper we show that topological spaces and uniform spaces can also be represented as this kind of generalized metric spaces.

Let V be a partially ordered set with a least element 0. This partially ordered set is equipped with a binary operation + which is associative and compatible with the ordering, i.e.

(1) $r \leq s$ and $t \leq l$ imply $r + t \leq s + l$ for any $r,s,t,l \in V$.

It is also endowed with an involution* (i.e. $r^{**} = r$ for any $r \in V$) which is an order automorphism and reverses the semigroup operation, i.e.

(2) $(r + s)^* = s^* + r^*$ for any $r,s \in V$.

Let E be underlying set and d be a function of $E \times E$ to V satisfying the following conditions: For any $x,y,z \in E$,

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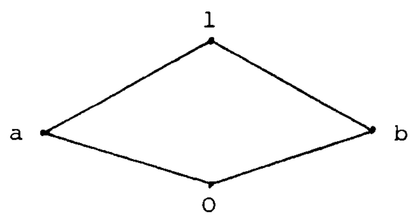
[1]Editorial note. After this paper was received and while it was in the reviewing process a paper by Ralph Kopperman (Reference 2) appeared whose results overlap with those of the present paper.

(3) $x = y$ iff $d(x,y) = 0$;

(4) $d(x,y)^* = d(y,x)$;

(5) $d(x,y) \leq d(x,z) + d(z,y)$.

(E,d) is called a generalized metric space over the ordered semigroup V, or in brief a V-metric space. Obviously, usual metric spaces are obtained if V, the range of distance functions, is taken as the nonnegative real numbers with the usual ordering, additive operation and identity involution. Some discrete structures can be also represented as this kind of generalized metric space. For example, let V be the complete lattice of four elements $\{0,a,b,1\}$ with a incomparable to b:



The semigroup operation is the join, and the involution * exchanges a and b while it fixes 0 and 1. On every generalized metric space (E,d) over the above V we can define the following ordering: $x \leq y$ iff $d(x,y) \leq a$ for any $x,y \in E$. Inversely, given an ordered set E, the map d defined by $d(x,y) = 0$ if $x = y$, $d(x,y) = a$ if $x < y$, $d(x,y) = b$ if $y < x$, $d(x,y) = 1$ if x and y are incomparable, is a V-metric on E which generates the original ordering on E in the above way. Meanwhile, nonexpansive mappings (i.e. mappings f such that $d(f(x),f(y)) \leq d(x,y)$) correspond to order-preserving mappings. Graphs can be treated in a similar way. Thus, generalized metric spaces over ordered semigroups provide a general framework in which to unify different mathematical structures, both continuous and discrete. In recent years, some classical concepts and results of metric spaces, ordered sets and graphs, such as retracts, fixed points, have been unified under this frame. Readers may refer to the survey paper [1] on this subject for details, including an extensive list of references.

In this paper topology and uniformity are considered. In Section 2 we shall establish topologies generated by V-metrics and show that every T_1-topology can be generated in this way. In other words every T_1-topology is metrizable in a generalized sense. In Section 3 we shall give an additional condition to V-metric spaces under which uniformities can be defined and show that each T_1-uniform space can be generated by a V-metric space with the additional condition. In Section 4 we shall define products of V-metric spaces and show that the topology generated by the product of a family of V-metric space coincides with the product topology of topologies generated by V-metrics on coordinate spaces. Similar results on uniformities will be given as well.

2. T_1-Topological Spaces and V-Metric Spaces

Let (E,d) be a V-metric space. We can define a topology $\mathcal{J}$ on E, called the V-metric topology generated by d, in the following way.

For any set of finitely many points r_k in V, where $1 \leq k \leq n$ and each $r_k > 0$, the subset of V,

$H = V - \cup\{[r_k): 1 \leq k \leq n\}$,

where $[r) = \{s \in V: s \geq r\}$ for any $r \in V$,

is called a radius. Obviously, the family of radii is closed with respect to finite intersections, i.e., the intersection of finitely many radii is still a radius; and H^* is a radius whenever H is a radius, where A^* denotes the set $\{r^*: r \in A\}$ for any subset A of V. The family of radii forms a base of the neighborhood system of 0 with respect to the interval topology on V. For any $x \in E$ and radius H, subsets of E,

$G(x,H) = \{y \in E: d(x,y) \in H\}$ and

$G(H,x) = \{y \in E: d(y,x) \in H\}$

are called the right ball and the left ball with the center x and radius H, respectively. Since every left ball with a center x and a radius H is just the right ball with the same center and the radius H^*, we only use right balls when the V-metric topology is defined. A subset G of E is said to be open with respect to the V-metric topology if for any $x \in G$ there exists a radius H such that $G(x,H) \subseteq G$. It is easy to see that $\mathcal{J}$ is a topology on E. And, the V-metric topology $\mathcal{J}$ must be a T_1-topology by the condition (3).

More generally, for any family of some initial segments in V which is closed with respect to finite intersections we can also define a topology on E in the same way. But, we prefer to choose the family of radii because all T_1-topologies can be represented by V-metric topologies.

We should note that in the general case, perhaps, a right (or left) ball itself may not be open with respect to the V-metric topology. If every right ball is open with respect to the V-metric topology (then, so is every left ball), we say that the V-metric space (E,d) is compatible with its V-metric topology. Our main result is the following:

Theorem 1. *For any* T_1*-topological space* (E,τ)*, there exist an ordered semigroup* V *and* V*-metric* d *on* E *such that the* V*-metric space* (E,d) *is compatible with its* V*-metric topology and its* V*-metric topology coincides with the original one.*

Proof. Let $\mathcal{U}$ be the family of all open neighborhoods of Δ, the diagonal, with respect to the product topology, and let $V = \mathcal{P}(\mathcal{U})$, the power set of $\mathcal{U}$. In other words, each $r \in V$ is a family of some open neighborhoods of Δ. The ordering on V is the inclusion relation of sets, that is,

$$r \leq s \text{ iff } s \subseteq r \text{ for any } s,r \in V.$$

Thus, $(V,\leq)$ forms a complete Boolean algebra with the least element $0 = \mathcal{U}$ and the greatest element $1 = \phi$.

Since the composition of two open neighborhoods of Δ is still an open neighborhood of Δ, we may define

$$r + s = \{u \circ v: v \in r \text{ and } u \in s\} \text{ for any } r,s \in V.$$

For any $r \in V$ we define that

$$r^* = \{u^{-1}: u \in r\},\ u^{-1} = \{(x,y) \in E \times E: (y,x) \in u\} \text{ for any } u \in \mathcal{U}.$$

It is easy to see that the operation + is associative and compatible with the ordering $\leq$, and the involution * is an order automorphism and reverses the operation +.

For any x and y in E, define

$$d(x,y) = \{u \in \mathcal{U}: (x,y) \in u\}.$$

It is not difficult to check that conditions (3), (4) and (5) hold. We only write down the proof of (3) in detail. Suppose $x \neq y$. Since τ is an T_1-topology, the set

$$((E - \{x\}) \times (E - \{x\})) \cup ((E - \{y\}) \times (E - \{y\}))$$

is an open neighborhood of Δ. (x,y) does not belong to it. That means, there exists an open neighborhood of Δ which does not belong to d(x,y), so $d(x,y) \neq 0$.

The V-metric topology generated by d is denoted by $\mathcal{J}$. Firstly, we assert that the original topology τ is contained by the V-metric topology $\mathcal{J}$. Suppose G is an open set with respect to τ and $x \in G$. We should find a right ball with the center x and some radius H contained in G.

Let

$$r = \{u \in \mathcal{U}: u(x) \not\subseteq G\}, \text{ where } u(x) = \{y \in E: (x,y) \in u\}$$

and

$$H = V - [r).$$

To show that H is a radius indeed, we need to prove $r \neq 0$. Since τ is a T_1-topology, $E - \{x\}$ is an open set with respect to τ, then

$$u = (G \times G) \cup ((E - \{x\}) \times (E - \{x\}))$$

is an open neighborhood of Δ. Obviously, $u(x) = G$, that means, there is $u \in \mathcal{U}$ but $u \notin r$, namely, $r \neq 0$.

For any $y \in G(x,H)$, $d(x,y) \nleq r$. Then there exists $u \in d(x,y)$ but $u \not\in r$, that is, there exists $u \in \mathcal{U}$ such that

$$(x,y) \in u \text{ and } u(x) \subseteq G.$$

Therefore,

$$y \in u(x) \subseteq G, \text{ namely, } G(x,H) \subseteq G.$$

Thus, each set open with respect to τ is open with respect to $\mathcal{J}$.

Secondly, let us note that for any $x \in E$ and any radius $H = V - \cup\{[r_k): 1 \leq k \leq n\}$, the right ball

$$G(x,H) = \cup\{u_1(x) \cap u_2(x) \cap \cdots \cap u_n(x): u_1 \not\in r_1, \cdots, u_n \not\in r_n\}.$$

Suppose $y \in G(x,H)$, or $d(x,y) \in H$. For each $1 \leq k \leq n$, $d(x,y) \nleq r_k$, then there is $u_k \in d(x,y)$ but $u_k \not\in r_k$. By the definition of d, $(x,y) \in u_k$, or $y \in u_k(x)$. Therefore, y belongs to the right of the equality. Conversely, suppose that y belongs to the right. Then, for each $1 \leq k \leq n$, there is $u_k \not\in r_k$ such that

$$y \in \cap\{u_k(x): 1 \leq k \leq n\}.$$

So, y belongs to every $u_k(x)$, or $(x,y) \in u_k$. Therefore,

$$d(x,y) \nleq r_k \text{ for each } 1 \leq k \leq n.$$

Consequently, $y \in G(x,H)$ and the equality is proved.

Immediately from the equality, each right ball $G(x,H)$ is open with respect to τ since $u_k(x)$ $(1 \leq k \leq n)$ are open. This fact implies that $\mathcal{J} = \tau$ and (E,d) is compatible with its V-metric topology.

We may also prove Theorem 1 in a slightly different way. Let V be the set in which every element is a family of some open covers. For any $r,s \in V$, $r < s$ iff $r \supseteq s$. This V with the ordering forms a complete lattice. The family of all open covers is its least element 0 and ϕ is its greatest element 1. Define the involution * to be the identity map. For two open covers $\mathcal{U}$ and $\mathcal{V}$, denote

$$\mathcal{U} + \mathcal{V} = \{u \cup v\colon u \in \mathcal{U} \text{ and } v \in \mathcal{V}\},$$

and for any $r,s \in V$, define

$$r + s = \{\mathcal{U} + \mathcal{V}\colon \mathcal{U} \in r \text{ and } \mathcal{V} \in s\},$$

and for any $x,y \in E$, define

$$d(x,y) = \{\mathcal{U}\colon \mathcal{U} \text{ is an open cover and there is } G \in \mathcal{U} \text{ such that } x \in G \text{ and } y \in G\}.$$

This (E,d) will be also a V-metric space to make Theorem 1 true.

3. Uniform Spaces and V-Metric Spaces

In this section we consider the problem as to how to represent uniform spaces with V-metric spaces. To generate uniformity from a V-metric space, we should introduce the following condition:

(6) In V for any $r > 0$, there exists $s > 0$ such that $m + n \not\geq r$ provided that $m \not\geq s$ and $n \not\geq s$ for any $m,n \in V$.

Now, assume (E,d) to be a V-metric space with the additional condition (6). We shall define a uniformity $\mathcal{U}$ on E, called the V-metric uniformity generated by d, in the following way. For any radius H denote

$$u(H) = \{(x,y) \in E \times E\colon d(x,y) \in H\}.$$

We shall show that the family of subsets of $E \times E$,

$$\mathcal{B} = \{u(H)\colon H \text{ is a radius in } V\},$$

satisfies the conditions to be the base of a uniformity, then take it as the base of V-metric uniformity generated by d. In fact, condition (3) implies that every u(H) contains the diagonal Δ. $u(H)^{-1} = u(H^*)$ for each radius H and

$$u(H_1) \cap u(H_2) = u(H_1 \cap H_2)$$

for any two radii H_1 and H_2. So, both $u(H)^{-1}$ and $u(H_1) \cap u(H_2)$ belong to $\mathcal{B}$, since the family of radii is closed under * and finite intersections. Now, we should show that for any radius H there exists another radius T such that $u(T) \circ u(T) \subseteq u(H)$.

Let

$$H = V - \cup\{[r_k): 1 \leq k \leq n\}, \text{ each } r_k > 0.$$

By condition (6), for every $1 \leq k \leq n$, there is s_k satisfying the property shown in (6). Choose

$$T = V - \cup\{[s_k): 1 \leq k \leq n\}.$$

For $(x,z) \in u(T) \circ u(T)$ there is $y \in E$ such that

$$(x,y) \in u(T) \text{ and } (y,z) \in u(T), \text{ or}$$

$$d(x,y) \in T \text{ and } d(y,z) \in T,$$

that means,

$$d(x,y) \not\geq s_k \text{ and } d(y,z) \not\geq s_k \text{ for any } 1 \leq k \leq n.$$

By condition (6),

$$d(x,y) + d(y,z) \not\geq r_k \text{ for any } 1 \leq k \leq n.$$

But, by condition (5),

$$d(x,z) \leq d(x,y) + d(y,z).$$

Therefore,

$$d(x,z) \not\geq r_k \text{ for } 1 \leq k \leq n, \text{ or } d(x,z) \in H, \text{ or}$$

$$(x,z) \in u(H).$$

This completes the proof that $\mathcal{B}$ is qualified to be the base of the V-metric uniformity $\mathcal{U}$.

We should notice that the uniform topology of the V-metric uniformity generated by d coincides with the V-metric topology generated by d, because, for any radius H, u(H)(x) coincides with the right ball with the center x and the radius H. Furthermore, a more important fact is the following.

Theorem 2. For any uniform space $(E,\mathcal{V})$ *with* T_1-*topology, there exist an ordered semigroup* V *and a* V-*metric* d *on* E *such that the* V-*metric space* (E,d) *satisfies condition* (6) *and the* V-*metric uniformity* $\mathcal{U}$ *generated by* d *coincides with the original one* $\mathcal{V}$.

Proof. Let Q be the family of all uniformly continuous metrics with respect to $\mathcal{V}$. It is well known that Q generates the uniformity $\mathcal{V}$, i.e., all

$$V_{q,r} = \{(x,y) \in E \times E: q(x,y) < r\}, \text{ where } q \in Q \text{ and } r > 0 \text{ is real,}$$

form the subbase of $\mathcal{V}$. R^+ denotes nonnegative real numbers. Let $V = (R^+)^Q$, the set of all functions of Q to R^+. Define an ordering $\leq$ on V: for any $f,g \in V$,

$$f \leq g \text{ iff } f(q) \leq g(q) \text{ for any } q \in Q.$$

$(V,\leq)$ is a partially ordered set with the least element 0, the function with value 0 for any $q \in Q$. Define a semi-group operation + on V: for any $f,g,h \in V$,

$$f + g = h \text{ iff } f(q) + g(q) = h(q) \text{ for any } q \in Q.$$

Define the identity map as the involution *. It is easy to check conditions (1) and (2).

For any nonzero function $f \in V$ there is $q_o \in Q$ such that $f(q_o) \neq 0$. Define

$$g(q) = \begin{cases} \frac{1}{2} f(q_o) & \text{if } q = q_o, \\ 0 & \text{otherwise.} \end{cases}$$

Of course, $g \neq 0$. For any $m, n \in V$, if $m \not\geq g$ and $n \not\geq g$, we have that

$$m(q_o) < \frac{1}{2} f(q_o) \text{ and } n(q_o) < \frac{1}{2} f(q_o).$$

Therefore

$$(m + n)(q_o) = m(q_o) + n(q_o) < f(q_o),$$

that means, $m + n \not\geq f$. Thus, condition (6) holds.

Define the V-metric d on E: for any $x, y \in E$, $d(x,y)$ is the function of Q to R^+ such that

$$d(x,y)(q) = q(x,y) \text{ for any } q \in Q.$$

It is easy to see that the V-metric d satisfies conditions (3), (4) and (5). We only show that $d(x,y) \neq 0$ for different $x, y \in E$. Since the uniform topology of $\mathcal{V}$ is T_1, there are a metric $q_o \in Q$ and a positive real number r such that

$$y \notin V_{q_o, r}(x), \text{ or } q_o(x,y) \geq r > 0.$$

Then

$$d(x,y)(q_o) = q_o(x,y) > 0, \text{ namely } d(x,y) \neq 0.$$

Now, let us show that the V-metric uniformity $\mathcal{U}$ generated by d coincides with the original one $\mathcal{V}$.

To prove $\mathcal{V} \subseteq \mathcal{U}$, it is enough to show that for any real number $r > 0$ and any $q_o \in Q$ there exists a radius H in V such that

$$u(H) = \{(x,y) \in E \times E: d(x,y) \in H\} \subseteq V_{q_o, r} =$$
$$\{(x,y) \in E \times E: q_o(x,y) < r\}.$$

We define a function $f \in V$ as follows:

$$f(q) = \begin{cases} r, & \text{if } q = q_0; \\ 0, & \text{otherwise.} \end{cases}$$

Then, the radius $H = V - [f)$ is as desired. In fact, if $d(x,y) \in H$, that means $d(x,y) \not\geq f$, then $d(x,y)(q_0) < f(q_0) = r$, because $d(x,y)(q) \geq f(q) = 0$ whenever $q \neq q_0$. Therefore,

$$q_0(x,y) = d(x,y)(q_0) < r, \text{ i.e., } u(H) \subseteq V_{q_0,r}.$$

To show $\mathcal{U} \subseteq \mathcal{V}$, for any radius $H = V - \cup\{[f_k): 1 \leq k \leq n\}$, we should find an element $U \in \mathcal{V}$ such that $U \subseteq u(H)$. For each $1 \leq k \leq n$, since $f_k \neq 0$, there is $q_k \in Q$ such that $r_k = f_k(q_k) > 0$. Then

$$U = \cap\{V_{q_k,r_k}: 1 \leq k \leq n\}$$

will suffice. If $(x,y) \in U$, then, for each $1 \leq k \leq n$,

$$(x,y) \in V_{q_k,r_k}, \text{ or } d(x,y)(q_k) = q_k(x,y) < r_k = f_k(q_k).$$

Therefore

$$d(x,y) \not\geq f_k, \text{ namely } (x,y) \in u(H).$$

Thus, $U \subseteq u(H)$.

4. Products of V-Metric Spaces

Suppose I is an index set, for each $i \in I$, V_i is an ordered semigroup and (E_i,d_i) is a generalized metric space over V_i. Denote

$$E = \Pi\{E_i: i \in I\} \text{ and } V = \Pi\{V_i: i \in I\}.$$

In the following, x,y,z represent elements in V and x_i,y_i,z_i represent their i-th coordinates in E_i, respectively. A similar convention is used for elements $r,s,t,0,r^k$ in V.

In the product V and different coordinate spaces V_i we use the same symbols $\leq$, + and * without any confusion. For example, in the symbolism $r \leq s$, $\leq$ is considered as the ordering in V, but, in the symbolism $r_i \leq s_i$, $\leq$ is considdred as the ordering in V_i. We define

$r \leq s$ iff $r_i \leq s_i$ for each $i \in I$;

$r + s = t$ iff $r_i + s_i = t_i$ for each $i \in I$;

$r^* = s$ iff $r_i^* = s_i$ for each $i \in I$;

$d(x,y) = r$ iff $d_i(x_i,y_i) = r_i$ for each $i \in I$.

It is clear that (E,d) is a general metric space over the ordered semigroup V, which is called the product of V_i-metric spaces (E_i,d_i). What is the relationship between the V-metric topology on E generated by d and the Cartesian product of V_i-metric topologies on coordinate spaces?

The answer to the question is based on the following fact.

Lemma 3. *For any* $x \in E$ *and any radius* H *in* V *there exist finitely many indices* $i_k \in I$ $(1 \leq k \leq n)$ *and, correspondingly, radii* H_{i_k} *in* V_{i_k} *such that*

$$G(x,H) \supseteq \cap\{p_{i_k}^{-1}(G(x_{i_k},H_{i_k})): 1 \leq k \leq n\},$$

where p_i *represents the projection to the i-th coordinate space* E_i. *And inversely, for any finitely many indices* $i_k \in I$ $(1 \leq k \leq n)$ *and radii* H_{i_k} *in* V_{i_k} *there exists a radius* T *in* V *such that*

$$\cap\{p_{i_k}^{-1}(G(x_{i_k},H_{i_k})): 1 \leq k \leq n\} \supseteq G(x,T).$$

Proof. Suppose

$$H = V - \cup\{[r^k): 1 \le k \le n\}.$$

for every $1 \le k \le n$, there is an index i_k such that $r^k_{i_k} \ne 0_{i_k}$ by the definition of radii. Let

$$H_{i_k} = V_{i_k} - [r^k_{i_k}).$$

We shall prove the first conclusion. Suppose

$$y \in \cap\{p^{-1}_{i_k}(G(x_{i_k}, H_{i_k})): 1 \le k \le n\}.$$

Then for any $1 \le k \le n$, $y_{i_k} \in G(x_{i_k}, H_{i_k})$, i.e.

$$d_{i_k}(x_{i_k}, y_{i_k}) \in H_{i_k}, \text{ or } d_{i_k}(x_{i_k}, y_{i_k}) \not\geq r^k_{i_k}, \text{ or}$$
$$d(x,y) \not\geq r^k.$$

The fact holds for every $1 \le k \le n$, therefore

$$d(x,y) \in H, \text{ namely } y \in G(x,H).$$

Because the finite intersection of radii is a radius, it is enough to prove the second conclusion, that for any index $i_0 \in I$ and any radius H_{i_0} in V_{i_0} with the special form $H_{i_0} = V_{i_0} - [r_{i_0})$, where $r_{i_0} > 0_{i_0}$, there exists a radius T in V such that

$$p^{-1}_{i_0}(G(x_{i_0}, H_{i_0})) \supseteq G(x,T).$$

We define an element r in V in the following way:

$$r_i = \begin{cases} r_{i_0}, & \text{if } i = i_0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $r \ne 0$ and $T = V - [r)$ is a radius in V. For any $y \in G(x,T)$,

$$d(x,y) \not\geq r.$$

But for any $i \ne i_0$, the i-th coordinate of d(x,y),

$$d(x,y)_i = d_i(x_i, y_i) \ge r_i = 0_i.$$

So, we have

$$d_{i_o}(x_{i_o}, y_{i_o}) \not\geq r_{i_o}, \text{ or } y_{i_o} \in G(x_{i_o}, H_{i_o}).$$

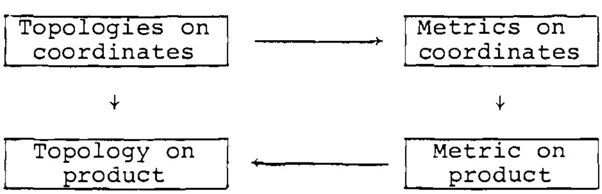
In other words,

$$y \in p_{i_o}^{-1}(G(x_{i_o}, H_{i_o})).$$

Using the lemma, we get the following result immediately.

Theorem 4. *Without any additional condition, for any* $i \in I$, *the projection* p_i *from* E *to* E_i *is both continuous and open with respect to the* V-*metric topology on* E *generated by* d *and the* V_i-*metric topology on* E_i *generated by* d_i. *Moreover, if for each* $i \in I$, *the* V_i-*metric space* (E_i, d_i) *is compatible with its* V_i-*metric topology, then the* V-*metric topology on* E *coincides with the Cartesian product of* V_i-*metric topologies on coordinate spaces.*

The theorem shows that if we start from the V_i-metric on coordinate spaces E_i, perhaps the V-metric topology on E is larger than the Cartesian product of V_i-metric topologies on E_i, but if we start from topologies on E_i, that is, we fix a topology τ_i on each E_i first, then we can find a V_i-metric space (E_i, d_i) with V_i-metric topology equal to τ_i by Theorem 1, and we get the product (E,d) of these V_i-metric spaces (E_i, d_i), and the product topology coincides with the V-metric topology generated by d.



Now, we consider uniform spaces. A similar situation occurs. For every index $i \in I$ given a V_i-metric space (E_i, d_i) satisfying condition (6). $\mathcal{U}_i$ denote the V_i-metric uniformity generated by d_i on E_i. These uniformities produce the Cartesian product $\mathcal{U}$ on E. On the other hand, all V_i-metric spaces (E_i, d_i) produce their product (E,d) in the sense defined at the beginning of this section. This (E,d) satisfies condition (6), because for any nonzero $r \in V$ there is an index $i_o \in I$ such that $r_{i_o} \neq 0$, so we can find a nonzero element s_{i_o} in V_{i_o} satisfying the property in (6), and the nonzero element s in V defined by $s_i = s_{i_o}$ if $i = i_o$ and $s_i = 0_i$ otherwise will have the property in (6), too. Therefore, the second uniformity, the V-metric uniformity generated by d, is equipped on E. Are these two uniformities the same? The answer is yes.

To show it, we should prove the following lemma. Its argument is similar to that of Lemma 3 and is omitted.

Lemma 5. *For every* $i \in I$, $p_{i,2}$ *represents the binary projection to* E_i, *i.e., the function of* $E \times E$ *to* $E_i \times E_i$ *such that*

$$p_{i,2}((x,y)) = (x_i, y_i) \text{ for any } x,y \in E.$$

Then, for any radius H *in* V *there exist finitely many indices* i_k $(1 \leq k \leq n)$ *and, correspondingly, radii* H_{i_k} *in* V_{i_k} *such that*

$$u(H) \supseteq \cap\{p^{-1}_{i_k,2}(u(H_{i_k})) : 1 \leq k \leq n\}.$$

And on the other hand, for any finitely many indices $i_k \in I$ $(1 \leq k \leq n)$ *and radii* H_{i_k} *in* V_{i_k}, *there exists a radius* T *in* V *such that*

$$\cap\{p_{i_k,2}^{-1}(u(H_{i_k})): 1 \leq k \leq n\} \supseteq u(T).$$

Since all subsets of E × E with forms $\cap\{p_{i_k,2}^{-1}(u(H_{i_k})):$ $1 \leq k \leq n\}$ are the base of the Cartesian product of $\mathcal{U}_i$, therefore we have

Theorem 6. *If every coordinate space* (E_i,d_i) *satisfies condition* (6), *then so does their product* (E,d). *Furthermore, the* V-*metric uniformity on* E *generated by* d *coincides with the Cartesian product of* V_i-*metric uniformities on* E_i *generated by* d_i.

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