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by

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Introduction

Let X be a space, and let $\mathcal{P}$ be a cover (not necessarily open or closed) of X. Recall that $\mathcal{P}$ is a k-*network* if whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. Such covers have played a role in $\aleph_0$-spaces [23] (i.e., spaces with a countable k-network) and $\aleph$-spaces [26] (i.e., spaces with a σ-locally finite k-network).

A cover $\mathcal{P}$ of X is called a cs-network [17], if whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is a nbd of x, then $\{x\} \cup \{x_m; m \geq n\} \subset P \subset U$ for some $n \in N$ and some $P \in \mathcal{P}$. Spaces with a countable cs-network, or cs-σ-spaces [18] (i.e., spaces with a σ-locally finite cs-network) were studied in [9], [12], [17], and [18], etc.

For a cover $\mathcal{P}$ of X, we shall consider the following modifications of k-networks or cs-networks.

(C_1) If $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is a nbd of x, then $P \subset U$, $P \ni x$, and P contains a subsequence of $\{x_n\}$ for some $P \in \mathcal{P}$.

(C_2) Same as (C_1), but without requiring $P \ni x$.

(C_3) If $K \subset X - \{x\}$ with K compact and $x \in X$, then $K \subset \cup\mathcal{P}' \subset X - \{x\}$ for some finite $\mathcal{P}' \subset \mathcal{P}$.

Every closed k-network $\mathcal{P}$ (i.e., k-network consisting of closed sets) satisfies (C_1), (C_1) implies (C_2), and every k-network $\mathcal{P}$ satisfies (C_2) and (C_3).

A cover satisfies (C_1) is called a cs*-network in [14]. Let $\mathcal{P}$ be a point-countable (i.e., every point is in at most countably many $P \in \mathcal{P}$) cover. Then $\mathcal{P}$ satisfies (C_2) if and only if $\mathcal{P}$ is a wcs-network in the sense of [25] (equivalently, Fcs-network in the sense of [12]). Spaces with a point-countable k-network, and spaces with a point-countable cover satisfying (C_3) were studied in [15], where the latter condition was labeled $(1.4)_p$.

In Section 1, we show that each of the following conditions implies that X has a point-countable k-network.

(1) X has a point-countable weak base.

(2) X is the closed image of an $\aleph$-space.

(3) X is the quotient, Lindelöf (i.e., each fiber is Lindelöf) image of a k-and-$\aleph$-space.

(4) X is dominated by $\aleph$-spaces.

In Section 2, we show that a sequential (resp. Frechet; first countable; Lašnev) space with a point-countable cover satisfying (C_1) is precisely the quotient (resp. pseudo-open; open; closed) s-image of a metric space.

In Section 3, we prove that a space (resp. completely regular space) is a Moore space if and only if it is a θ-refinable wΔ-space (resp. a strict p-space) with a point-countable cover satisfies (C_3). In particular, a strict p-space with a point-countable k-network is a Moore space.

The definitions of the spaces, maps etc. appearing in the above results will be respectively given in Sections dealing with these results.

We assume that all spaces are *regular*, T_1, and all maps are continuous and onto.

1. Point-Countable k-Networks

In view of [4; Theorem 3.5], we have

Lemma 1.1. *Every compact space with a point-countable cover satisfying* (C_3) *is metrizable.*

A collection $\{A_\alpha;\ \alpha \in A\}$ of subsets of X is called *hereditarily closure-preserving*, if $\overline{\cup\{B_\alpha;\ \alpha \in A'\}} = \cup\{\overline{B}_\alpha;\ \alpha \in A'\}$ for any $A' \subset A$ and $B_\alpha \subset A_\alpha$ for each $\alpha \in A'$.

Proposition 1.2. (1) *Let* $\mathcal{P}$ *be a point-countable cover of* X. *Then* $\mathcal{P}$ *is a* k*-network if and only if* $\mathcal{P}$ *satisfies* (C_2), *and each compact subset of* X *is sequentially compact.*

(2) *Let* $\mathcal{P}$ *be a* σ*-hereditarily closure-preserving cover of* X. *Then* $\mathcal{P}$ *is a* k*-network if and only if* $\mathcal{P}$ *satisfies* (C_2).

Proof. (1) By Lemma 1.1, every compact space with a point-countable k-network is metrizable, hence sequentially compact. Thus the "only if" part holds. To prove the "if" part, let K be compact and U be open in X with $K \subset U$. For each $x \in X$, let $\{P \in \mathcal{P};\ x \in P \subset U\} = \{P_n(x);\ n \in N\}$. Then K is covered by some finite $\mathcal{P}' \subset \{P_n(x);\ x \in X,\ n \in N\}$. Indeed, suppose not. Then there exists a sequence $\{x_n\}$ in K such that $x_n \notin P_i(x_j)$ for $i,j < n$. Since K is sequentially compact, there exists a subsequence S of $\{x_n\}$ converging to a point in K. Since $\mathcal{P}$ satisfies (C_2), there exists $P \in \mathcal{P}$ such that $P \subset U$ and P contains a subsequence of S. Then $P = P_i(x_j)$ for some i,j, and there exists $n > i,j$ such that $P_i(x_j) \ni x_n$. This is a contradiction. Hence $\mathcal{P}$ is a k-network.

(2) The "only if" part is obvious, so we prove the "if" part. Let $\mathcal{P} = \{\mathcal{P}_n; n \in N\}$ is a σ-hereditarily closure-preserving cover satisfying (C_2) with $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. Let K be compact and U be open with $K \subset U$. Let $\mathcal{P}'_n = \{P \in \mathcal{P}_n; P \subset U\}$ and $U_n = \cup\mathcal{P}'_n$ for each $n \in N$. Since each point of K is a G_δ in X (hence in K), the compact set K is first countable. Thus K is sequentially compact. Then, since $\mathcal{P}$ satisfies (C_2), it follows that $K \subset U_i$ for some $i \in N$. But, $\mathcal{P}'_i$ is hereditarily closure-preserving. Then the compact set K is covered by some finite $\mathcal{P}' \subset \mathcal{P}_i$. Thus $K \subset \cup\mathcal{P}' \subset U$. Then $\mathcal{P}$ is a k-network.

Remark 1.3. As a modification of (C_3), we define the following $(C_3)'$ weaker than (C_2).

$(C_3)'$ Same as (C_2) with $U = X - \{y\}$ for a point $y \in X - \{x\}$.

In view of the proof of Proposition 1.2(1), a point-countable cover $\mathcal{P}$ of X satisfies (C_3) if and only if $\mathcal{P}$ satisfies $(C_3)'$, and each compact subset of X is sequentially compact.

Definition 1.4. Let X be a space, and let $\mathcal{P}$ be a cover of X. Then X *is determined by* $\mathcal{P}$ [15], or X has the weak topology with respect to $\mathcal{P}$, if $A \subset X$ is closed in X if and only if $A \cap P$ is closed in P for every $P \in \mathcal{P}$. Here, we can replace "closed" by "open." Recall that a space X is *sequential*, if $A \subset X$ is closed in X if and only if no sequence in A converges to a point not in A. Then X is sequential if and only if it is determined by the cover of all compact metric subsets. If we replace "compact metric" by "compact," then such a space is called a *k-space*. First countable spaces are sequential, and sequential spaces are k-spaces. Let $\mathcal{P}$ be a closed cover of a space X. Then X *is dominated by* $\mathcal{P}$ [21], if the union of any $\mathcal{P}' \subset \mathcal{P}$ is closed in X, and the union is determined by $\mathcal{P}'$. As is well-known, every CW-complex is dominated by the cover of all finite subcomplexes (hence, compact metric subsets).

Corollary 1.5. *Let* X *be a sequential space, or a space in which every point is a* G_δ. *Then a point countable cover* $\mathcal{P}$ *of* X *is a* k-*network (resp. satisfies* (C_3)*) if and only if* $\mathcal{P}$ *satisfies* (C_2) *(resp.* $(C_3)'$*).*

Proof. It is easy to show that every compact subset of a sequential space is sequentially compact. Also, as is seen in the proof of Proposition 1.2(2), every compact subset of a space in which every point is a G_δ is sequentially compact. Thus the result follows from Proposition 1.2(1) and Remark 1.3.

Lemma 1.6. *Suppose that* $f: X \to Y$ *satisfies* (1) *or* (2) *below. Then for every sequence* $\{y_n\}$ *converging to* y *in* Y *with* $y \neq y_n$, *there exists a convergent sequence* $\{x_n\}$ *in* X *such that* $\{f(x_n)\}$ *is a subsequence of* $\{y_n\}$.

(1) f *is a quotient map such that* X *is a sequential space.*

(2) f *is a closed map such that each point of* X *is a* G_δ.

Proof. (1) Let $A = \{y_n;\ n \in N\}$. Since A is not closed in Y, $B = f^{-1}(A)$ is not closed in X. Thus, since X is sequential, there exists a sequence $\{x_n\}$ converging to a point not in B, hence $\{f(x_n)\}$ is a subsequence of $\{y_n\}$.

(2) Our proof is due to the proof of [14; Lemma 1]. Indeed, there exists a point $x \in f^{-1}(y)$ such that $x \in \overline{\cup\{f^{-1}(y_n);\ n \geq i\}}$ for any $i \in N$. Since $\{x\}$ is a G_δ in X, there exists a sequence $\{G_n;\ n \in N\}$ of nbds of x such that $\{G_n;\ n \in N\} = \{x\}$ and $\overline{G}_{n+1} \subset G_n$. Let $U_i = G_i \cap \cup\{f^{-1}(y_n);\ n \geq i\}$ for each $i \in N$. Since f is closed, any sequence $\{a_n\}$ with $a_n \in U_n$ accumulates to x. Hence, let $x_n \in U_n$ $(\neq \emptyset)$ for each $n \in N$. Then the sequence $\{x_n\}$ converges to x, and $\{f(x_n)\}$ is a subsequence of $\{y_n\}$.

Definition 1.7. Let X be a space. For each $x \in X$, let T_x be a finite multiplicative family of subsets of X containing x. The collection $\{T_x;\ x \in X\}$ is a *weak base* [1] for X, if $F \subset X$ is closed in X if and only if for each $x \notin F$, there exists $Q(x) \in T_x$ with $Q(x) \cap F = \emptyset$.

Proposition 1.8. *Each of the following conditions implies that* Y *has a point-countable* k*-network. (The result for* (3) *(resp.* (4)*) with* X *metric is due to* [10] *(resp.* [15]*).)*

(1) Y *has a point-countable weak base.*

(2) Y *has a* σ*-hereditarily closure-preserving* k-*network.*

(3) Y *is the closed image of an* $\aleph$*-space* X.

(4) Y *is the quotient, Lindelöf image of a* k-*and*-$\aleph$-*space* X.

(5) Y *is dominated by* $\aleph$*-spaces* X_α $(\alpha < \gamma)$.

Proof. (1) Let $\mathcal{P} = \{T_y; y \in Y\}$, where $T_y = \{Q_n(y); n \in N\}$, be a point-countable weak base for Y. Then it follows that for any $y \in Y$, any sequence $\{y_n\}$ with $y_n \in Q_n(y)$ converges to y. Conversely any sequence $\{y_n\}$ converging to y is eventually in $Q_n(y)$ for any $n \in N$. Indeed, suppose that $A = \{y_n; n \in N\} - Q_n(y)$ is infinite. Since $A \cup \{y\}$ is closed in Y, for each $p \notin A$ with $p \neq y$, $Q_m(p) \cap A = \emptyset$ for some $m \in N$. Since $Q_n(y) \cap A = \emptyset$, A is closed in Y. This is a contradiction. Therefore Y is a sequential space, and Y has a point-countable k-network $\mathcal{P}$ by Corollary 1.5.

(2) Let $\mathcal{P} = \cup\{\mathcal{P}_n; n \in N\}$ be a σ-hereditarily closure-preserving k-network for Y. For each $n \in N$, let $D_n = \{y \in Y; \mathcal{P}_n$ is not point-finite at $y\}$ and let $\mathcal{P}_n = \{P - D_n; P \in \mathcal{P}_n\} \cup \{\{y\}; y \in D_n\}$. Let $\mathcal{P}' = \cup\{\mathcal{P}_n; n \in N\}$. Then $\mathcal{P}'$ is a point-countable cover satisfying (C_2). While, each point of Y is a G_δ. Then, by Corollary 1.5, $\mathcal{P}'$ is a point-countable k-network for Y.

(3) Let $f: X \to Y$ be a closed map. Let $\mathcal{P}$ be a σ-locally finite k-network for X. Since each point of X is G_δ, by Proposition 1.2(2) and Lemma 1.6, $f(\mathcal{P})$ is a σ-hereditarily closure-preserving k-network for Y. Hence Y has a point-countable k-network by the result for (2).

(4) Let $f: X \to Y$ be a quotient, Lindelöf map. Let $\mathcal{P}$ be a σ-locally finite k-network for X. Since each point of X is a G_δ, a k-space X is sequential by [24; Theorem 7.3]. Thus, by Lemma 1.6, $f(\mathcal{P})$ is a point-countable cover satisfying (C_2). But, Y is sequential, for every quotient image of a sequential space is sequential. Thus, by Corollary 1.5, $f(\mathcal{P})$ is a point-countable k-network for Y.

(5) For each α, let $Y_\alpha = X_\alpha - \cup\{X_\beta;\ \beta < \alpha\}$. Let $\mathcal{P}_\alpha$ be a σ-locally finite k-network for $\overline{Y}_\alpha$, and let $\mathcal{P}'_\alpha = \mathcal{P}_\alpha \cap Y_\alpha$. Then $\mathcal{P} = \cup\{\mathcal{P}'_\alpha;\ \alpha\}$ is point-countable. To show that $\mathcal{P}$ is a k-network, let K be compact and U be open with $K \subset U$. Then K meets only finitely many Y_α's. Indeed, if there exists $D = \{x_n;\ n \in N\}$ such that $x_n \in K \cap Y_{\alpha_n}$ with $\alpha_n < \alpha_{n+1}$, then each $D \cap X_{\alpha_n}$ is finite. Hence D is closed discrete in X. This is a contradiction. Let $\{\alpha;\ Y_\alpha \cap K \neq \emptyset\} = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. For each $i = 1, 2, \cdots, n$, there exists a finite $\mathcal{F}_{\alpha_i} \subset \mathcal{P}_{\alpha_i}$ such that $\overline{Y}_{\alpha_i} \cap K \subset \cup\mathcal{F}_{\alpha_i} \subset U$. Hence, $K \subset \cup\{\mathcal{F}_{\alpha_i} \cap Y_{\alpha_i};\ i = 1, 2, \cdots, n\} \subset U$. Then $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$.

We shall give Remark to the previous proposition.

Remark 1.9. (i) In (2), we can not replace "σ-hereditarily closure preserving" with "σ-closure preserving." Indeed, there exists a non-metrizable, separable, first-countable space X with a σ-closure preserving base ([6; Example 9.2]). X has a σ-closure preserving k-network. But X does not have any point-countable k-network by [15; Theorem 5.2] and [22; Proposition 3.1] for X is separable, first-countable, but not metrizable.

(ii) In (4), we can not omit the k-ness of X; see [15; Example 9.6].

(iii) (3) implies (2) by the proof of (3). Unlike this, (5) (or (4)) need not imply (2). We shall give a counter-example. First, we need Lemmas.

Lemma A. *Let* X *be a sequential space with a* σ*-hereditarily closure-preserving* k*-network* $\mathcal{P} = \cup\{\mathcal{P}_n; n \in N\}$. *Then* $X = X_0 \cup X_1$, *where* X_0 *is an* $\aleph$*-space and* X_1 *is a countable union of closed, discrete subsets of* X.

Proof. Each $\overline{\mathcal{P}}_n$ is hereditarily closure-preserving (for, if not, there exist $\{A_\alpha; \alpha \in A'\}$ and $x \in X$ such that $A_\alpha \subset P_\alpha$, where $\mathcal{P}_n = \{P_\alpha; \alpha \in A\}$, $A' \subset A$, and $x \in \overline{\cup\{A_\alpha; \alpha \in A'\}} - \cup\{\overline{A}_\alpha; \alpha \in A'\}$. Let $U \supset A_\alpha$ be open with $\overline{U} \not\ni x$, and $B_\alpha = U \cap A_\alpha$ for $\alpha \in A'$. Then $x \in \overline{\cup\{B_\alpha; \alpha \in A'\}} - \cup\{\overline{B}_\alpha; \alpha \in A'\}$, a contradiction). Then we can assume $\mathcal{P}$ is a closed cover of X. For each $n \in N$, let $D_n = \{x \in X;$ $\mathcal{P}_n$ is not point-finite at $x\}$. Then each D_n is closed, discrete in X. Indeed, suppose D_n is not closed discrete in X. Since X is sequential, there exists a sequence $\{x_n\}$ in D_n converging to $x \in X$ with $x_n \neq x$. But there exists $\{P_n; n \in N\} \subset \mathcal{P}_n$ such that $x_n \in P_n$ and $P_m \neq P_n$ if $m \neq n$. Thus $x_n = x$ for some $n \in N$. This is a contradiction. Let $X_0 = X - \cup\{D_n; n \in N\}$. Then X_0 is an $\aleph$-space.

Lemma B. *Let* X *be dominated by metric subsets* X_α $(\alpha < \gamma)$, *and let* $Y \subset X$ *be an* $\aleph$*-space. Then* $Y = \cup\{Y_n; n \in N\}$, *where each* Y_n *is a metrizable, closed subset of* Y, *and each compact subset of* Y *is contained in some* Y_n.

Proof. Let $\mathcal{P} = \cup\{\mathcal{P}_n; n \in N\}$ be a σ-locally finite closed k-network for Y, where $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ and $\mathcal{P}_n$ is closed under finite intersections. Let $K \subset Y$ be compact, and let $\{Q_n; n \in N\}$ be the collection of finite unions of elements of $\{P \in \mathcal{P}; P \cap K \neq \emptyset\}$ with $Q_n \supset K$. Let $K_n = \cap\{Q_i; i \leq n\}$ for each $n \in N$. Since $\mathcal{P}$ is a k-network, any open subset of Y containing K contains some K_n. Thus any sequence $\{y_n\}$ with $y_n \in K_n$ has an accumulation point in Y. Hence some K_n is contained in a finite union of X_α's. Indeed, suppose not. Then there exists a sequence $\{a_n\}$ in Y such that $a_n \in K_n \cap X_{\alpha(n)} - \cup\{X_{\alpha(i)}; i < n\}$ for some $\{X_{\alpha(n)}; n \in N\} \subset \{X_\alpha; \alpha\}$. Since each $\{a_n; n \in N\} \cap X_{\alpha(n)}$ is finite, $\{a_n; n \in N\}$ is closed, discrete in Y. This is a contradiction. Then some K_n is metrizable, so is any K_m $(m \geq n)$. But any K_m can be expressed as a finite union of elements of $\mathcal{P}$. This implies that $\cup\{\mathcal{P}'_n; n \in N\}$, where $\mathcal{P}'_n = \{P \in \mathcal{P}_n;$ P is metrizable$\}$, is a k-network for Y. Thus $Y_n = \cup\mathcal{P}'_n$ $(n \in N)$ are the desired subsets of Y.

Example. Let I be the closed unit interval [0,1]. For each $\alpha \in I$, let S_α be a 2-sphere. Let S be the topological sum of $\{I, S_\alpha; \alpha \in I\}$. Let X be the quotient space obtained from S by identifying each $\alpha \in I$ with a point p_α of S_α. Then X is a CW-complex with cells $\{\{0\}, \{1\}, (0,1), S_\alpha - \{p_\alpha\}; \alpha \in I\}$, which is the quotient, finite-to-one image of a locally compact, metric space. But X does not have any σ-hereditarily closure-preserving k-network.

Proof. Suppose that X has a σ-hereditarily closure-preserving k-network. Let $L = \{0\} \times I$, and for each $\alpha \in I$, let $L_\alpha = I \times \{\alpha\}$. Let $\mathcal{L} = \{L\} \cup \{L_\alpha; \alpha \in I\}$ and $Y = \cup\mathcal{L}$. Let L and the L_α be subspaces of Euclidean 2-space E^2, and let us consider Y as a subspace of X. Then Y is determined by $\mathcal{L}$ consisting of compact metric subsets. Since Y is a sequential space with a σ-hereditarily closure-preserving k-network, by Lemma A, $Y = Y_0 \cup Y_1$, where Y_0 is an $\aleph$-space, and Y_1 is a countable union of closed discrete subsets of Y. By Lemma B, $Y_0 = \cup\{X_n; n \in N\}$, where each X_n is metrizable and closed in Y_0, and each compact subset of Y_0 is contained in some X_n. We note that $L \cap Y_1$ and $L_\alpha \cap Y_1$ are at most countable sets. For each $\alpha \in I$, let $\{\alpha_n\}$ be a sequence in $L_\alpha - Y_1$ converging to $(0,\alpha)$ $(\neq \alpha_n)$. For each $n \in N$, let $A_n = \{\alpha \in I; (0,\alpha) \notin Y_1$ and $X_n \supset \{\alpha_n; n \in N\}\}$. Then, by R. Baire's Category theorem, there exists $A = A_i \cap (\alpha,\beta)$ such that A is dense in $(\alpha,\beta) \cap I$ for some $i \in N$ and some open interval (α,β). Then there exists a sequence $\{x_n\}$ in A converging to a point $x \in (\alpha,\beta)$ with $x \neq x_n$ and $(0,x) \notin Y_1 \cap L$. For each $n \in N$, there exists a sequence $\{x_{nj}; j \in N\}$ in X_i converging to $(0,x_n)$ $(\neq x_{nj})$, where $x_{nj} \in L_{x_n} - Y_1$. Let $T = \{(0,x_n); n \in N\} \cup \{(0,x)\} \cup \{x_{nj}; n,j \in N\}$. Then $T \subset X_i$, hence T is metrizable. But, since T is closed in Y, T is determined by $\{T \cap L; L \in \mathcal{L}\}$. Thus T is not first countable (at x), hence T is not metrizable. This is a contradiction.

2. s-Images of Metric Spaces

In terms of various kinds of point-countable covers, some characterizations for the quotient (resp. open; closed) s-images of metric spaces are obtained by [15] or

[20] (resp. [27]; [13]). By means of point-countable covers satisfying (C_1) (i.e., point-countable cs*-networks), we will unify these characterizations.

Lemma 2.1. *Let* X *be a sequential space. Then the following* (1) *and* (2) *are equivalent.*

(1) X *has a point-countable cover* $\mathcal{P}$ *satisfying* (C_1).

(2) X *has a point-countable cover* $\mathcal{P}$ *such that each open* $U \subset X$ *is determined by* $\{P \in \mathcal{P};\ P \subset U\}$. *(This condition is labeled* (1.1) *in* [15].*)*

Proof. (1) $\to$ (2). Let U be open in X. Suppose $A \subset U$ is not open in U. Then X - A is not closed in X. Since X is sequential, there exists a sequence $\{x_n\}$ in X - A converging to a point $x \in A$. Since $\mathcal{P}$ satisfies (C_1), there exist $P \in \mathcal{P}$ and a subsequence S of $\{x_n\}$ such that $K = S \cup \{x\} \subset P \subset U$. Then $K \cap A$ is not open in K. Thus $P \cap A$ is not open in P. Hence U is determined by $\{P \in \mathcal{P};\ P \subset U\}$.

(2) $\to$ (1). Let $\{x_n\}$ be a sequence converging to x, and U be a nbd of x. We assume that $x_n \neq x$ and $x_n \in U$ for each $n \in N$. Since $\{x_n;\ n \in N\}$ is not closed in U, there exists $P \in \mathcal{P}$ such that $P \subset U$ and $P \cap \{x_n;\ n \in N\}$ is not closed in P. Then P contains x and a subsequence of $\{x_n\}$. Hence $\mathcal{P}$ satisfies (C_1).

A map f: X $\to$ Y is called *pseudo-open* if for any $y \in Y$ and any open set U containing $f^{-1}(y)$, $y \in \text{int } f(U)$. Every open map, or closed map is pseudo-open.

Recall that a space X is *Fréchet* if whenever $x \in \overline{A}$, then there exists a sequence in A converging to x.

Lemma 2.2. (1) [15]. X *is the quotient (resp. pseudo-open)* s-*image of a metric space if and only if* X *is a sequential (resp. Fréchet) space satisfying* (2) *in Lemma* 2.1.

(2) [27]. X *is the open* s-*image of a metric space if and only if* X *has a point-countable base.*

(3) [13]. X *is the closed* s-*image of a metric space if and only if* X *is a Fréchet space with a* σ-*closure preserving point-countable closed* k-*network (equivalently, Frechet* $\aleph_0$-*space).*

Theorem 2.3. X *is the quotient (resp. pseudo-open; open)* s-*image of a metric space if and only if* X *is a sequential (resp. Fréchet; first countable) space with a point-countable cover satisfying* (C_1).

Proof. This follows from Lemmas 2.1 and 2.2 (1) and (2) and the result that every first countable space which is the quotient s-image of a metric space has a point-countable base ([8]).

Lemma 2.4. *Let* S_{ω_1} *be the quotient space obtained from the disjoint union of* ω_1 *convergent sequences* L_α $(\alpha < \omega_1)$ *by identifying all the limit points to a single point* ∞. *Then any point-countable cover of* S_{ω_1} *does not satisfy* (C_1).

Proof. Suppose that X has a point-countable cover $\mathcal{P}$ satisfies (C_1). Let $\mathcal{P}' = \{P \in \mathcal{P}; P \ni \infty\}$, and $\mathcal{P}_1 = \{P \in \mathcal{P}; P - \{\infty\}$ meets infinitely many L_α's$\}$. Let $\mathcal{P}_1 = \{P_n; n \in N\}$. We can choose a sequence $\{x_n\}$ such that $x_n \in P_n - \{\infty\}$, and

$x_n \notin L_\alpha$ for any L_α containing $x_1, x_2, \cdots, x_{n-1}$. Let $\mathcal{P}_2 = \mathcal{P}' - \mathcal{P}_1$, and $S = \cup\{P \in \mathcal{P}_2\}$. Then $S - \{\infty\}$ meets at most countably many L_α's. Hence, there exists L_β such that $L_\beta \cap S = \{\infty\}$, $L_\beta \not\ni x_n$ $(n \in N)$. Since $L_\beta \subset X - \{x_n; n \in N\}$, there exists $P \in \mathcal{P}_2$ with $P \cap (L_\beta - \{\infty\}) \neq \emptyset$. This is a contradiction.

The following proposition shows that, among point-countable covers, there exist essential gaps between "k-network" and "closed k-network," and also between "(C_1)" and "(C_2)" by Proposition 1.8.

Proposition 2.5. *Let* $f: X \to Y$ *be a closed map such that* X *is a paracompact* k-*and*-$\aleph$-*space. Then the following are equivalent. (The equivalence* (1) *and* (2)*, where* X *is metric, is due to* [30].*)*

(1) *Every* $\partial f^{-1}(y)$ *is Lindelöf.*

(2) Y *has a point-countable closed* k-*network.*

(3) Y *has a point-countable cover satisfying* (C_1).

Proof. (1) → (2). As in the proof of [22; Corollary 1.2], we can assume that every $f^{-1}(y)$ is Lindelöf. Thus, the Proof of Proposition 1.8(3) implies that (1) → (2) holds.

(2) → (3). This is trivial.

(3) → (1). By Lemma 2.4, Y does not contain a copy of S_{ω_1}. Thus, by [31; Lemma 1.5], each $\partial f^{-1}(y)$ is ω_1-compact (i.e., every subset of $\partial f^{-1}(y)$ of cardinality ω_1 has an accumulation point). But, $\partial f^{-1}(y)$ is an $\aleph$-space (or paracompact space). Hence each $\partial f^{-1}(y)$ is Lindelöf.

A space is called *Lašnev* if it is the closed image of a metric space.

L. Foged [10] proved that a space is Lašnev if and only if it is a Fréchet space with a σ-hereditarily closure-preserving k-network.

Theorem 2.6. *The following are equivalent.*

(1) X *is the closed* s-*image of a metric space.*

(2) X *is a Lašnev space with a point-countable cover satisfying* (C_1).

(3) X *is a Fréchet space with a* σ-*closure-preserving point-countable closed cover satisfying* (C_1).

Proof. (1) → (3). This follows from Lemma 2.2 (3).

(3) → (2). Since X is sequential, by Corollary 1.5, every point-countable cover of X satisfying (C_1) is a k-network. Then (3) → (2) follows from Lemma 2.2 (3).

(2) → (1). This follows from Proposition 2.5.

Remark 2.7. In (3) of the previous theorem, we can not omit the closedness of the cover, because any Lašnev space has a σ-hereditarily closure-preserving point-countable k-network by the proof of Proposition 1.8. We can replace "σ-closure-preserving point-countable closed cover" by "σ-locally countable closure-preserving cover" in (3).

3. Moore Spaces

By means of point-countable covers satisfying (C_3), we will obtain general conditions under which a wΔ-space (or a strict p-space) is a Moore space.

A cover $\mathcal{P}$ of X is said to be *separating* if for any points x, y $\in$ X with $x \neq y$, there exists $P \in \mathcal{P}$ with $x \in P \subset X - \{y\}$. Recall that a space X is *subparacompact* if every open cover of X has a σ-locally finite closed refinement.

Proposition 3.1. *Each of the following conditions implies that* X *has a point-countable cover satisfying* (C_3).

(1) X *has a point-countable separating open cover.*

(2) X *has a point-countable* k-*network.*

(3) X *is a subparacompact (or metacompact) space having a* G_δ*-diagonal. In particular,* X *is a* σ*-space.*

Proof. The result for (1) and (2) is obvious.

(3) Since X has a G_δ-diagonal, by [6; Lemma 5.4] there exists a sequence $\{\mathcal{U}_n;\ n \in N\}$ of open covers of X such that for any points x, y $\in$ X with $x \neq y$, $y \notin St(x, \mathcal{U}_n)$ for some $n \in N$. If X is subparacompact (resp. metacompact), each $\mathcal{U}_n$ has a σ-locally finite closed refinement (resp. a point-finite open refinement). Thus if X is metacompact, X has a point-countable cover satisfying (C_3). If X is subparacompact, X has also a point-countable cover satisfying (C_3) by the same way as in the proof of [4; Theorem 5.2], where X is a σ-space.

Definition 3.2. A space X is called a wΔ-*space* if there exists a sequence $\{\mathcal{U}_n\}$ of open covers of X such that if $x \in X$ and $x_n \in St(x, \mathcal{U}_n)$ for each $n \in N$, then the sequence $\{x_n\}$ has an accumulation point y in X. When

$y = x$, such a space is called a *Moore space*, or a *developable space*.

A completely regular space X is called a p-*space* if, in the Stone-Čech compactification $\beta(X)$, there exists a sequence $\{\mathcal{U}_n\}$ of families of open subsets of $\beta(X)$ such that each $\mathcal{U}_n$ covers X, and $\cap\{St(x,\mathcal{U}_n); n \in N\} \subset X$ for each $x \in X$. If we also have an additional property that for any $x \in X$ and any $i \in N$, $\overline{St(x,\mathcal{U}_n)} \subset St(x,\mathcal{U}_i)$ for some $n \in N$, then X is called a *strict* p-*space*. Every locally compact space, more generally every Čech complete space (i.e., space which is a G_δ-set in its Stone-Čech compactification) is a p-space. It is well-known that every completely regular, Moore space is a strict p-space, and every strict p-space is a $w\Delta$-space (e.g., see [16; p. 443]).

A space X is called θ-*refinable* (= submetacompact) if for every open cover $\mathcal{U}$ of X, there exists a sequence $\{\mathcal{U}_n\}$ of open refinements of $\mathcal{U}$ such that each point of X is in at most finite number of elements of some $\mathcal{U}_n$. Such a sequence is called a θ-*refinement* of $\mathcal{U}$. As is well-known, metacompact spaces, and σ-spaces, more generally subparacompact spaces are θ-refinable (e.g., see [5; p. 360]). We note that, among completely regular θ-refinable spaces, $w\Delta$-spaces, p-spaces, and strict p-spaces are equivalent ([2]).

R. E. Hodel [19; Theorem 3.6] proved that every θ-refinable, $w\Delta$-space with a point-countable separating open cover is a Moore space.

We will use the techniques in the proof of the Hodel's result to obtain a more general result.

Lemmas 3.3 and 3.4 below are respectively due to [15; Corollary 3.5] and [3; Lemma 3.2].

Lemma 3.3. *Let* X *be first countable. Then every point-countable cover* $\mathcal{P}$ *satisfying* (C_3) *satisfies the following* (*).

(*) *If* $y \in X - \{x\}$, *then there exists a finite* $\mathcal{P}' \subset \mathcal{P}$ *such that* $y \in \text{int } \cup\mathcal{P}' \subset \cup\mathcal{P}' \subset X - \{x\}$.

Lemma 3.4. *Let* X *be first countable, and* $A \subset X$. *If* $\mathcal{P}$ *is a point-countable collection of subsets of* X, *then there exists at most countably many minimal finite* $\mathcal{F} \subset \mathcal{P}$ *such that* $A \subset \text{int } \cup\mathcal{F}$, *where minimal means that* $A \not\subset \text{int } \cup\mathcal{F}'$ *if* $\mathcal{F}' \subsetneq \mathcal{F}$.

Theorem 3.5. *A space* X *is a Moore space if and only if* X *is a* θ*-refinable* $w\Delta$*-space with a point-countable cover satisfying* (C_3).

Proof. "Only if." Every Moore space is a $w\Delta$-space, and a σ-space (e.g., [16; p. 447]). Hence, X is a θ-refinable space with a point-countable cover satisfying (C_3) by [4; Theorem 5.2] (or Proposition 3.1).

"If." Since X is a θ-refinable $w\Delta$-space, by [2; Remark 1.9] there exists a sequence $\{\mathcal{G}_n\}$ of open covers of X such that for each $x \in X$, $C_x = \cap\{St(x,\mathcal{G}_n); n \in N\}$ is compact, and $\{St(x,\mathcal{G}_n); n \in N\}$ is a base for C_x; that is, any open set U with $U \supset C_x$ contains some $St(x,\mathcal{G}_n)$. We note that each C_x is metrizable by Lemma 1.1. Hence X is first countable. Then, $\mathcal{P}$ satisfies (*) in Lemma 3.3. Hence X^2 is a first countable space with a point-countable cover $\mathcal{P}^2$ satisfying (*).

Let $\mathcal{G}(i,j) = \mathcal{G}_i \times \mathcal{G}_j$ for each $i,j \in N$. Then, for any $p = (x,y) \in X^2$, $\{St(p,\mathcal{G}(i,j));\ i,j \in N\}$ is a base for a compact set C_p. For each $i \in N$, let $\{\mathcal{H}(i,k);\ k \in N\}$ be a θ-refinement of $\mathcal{G}_i$, and let $\mathcal{H}(i,j,k,\ell) = \mathcal{H}(i,k) \times \mathcal{H}(j,\ell)$.

We show that every closed subset of X^2 is a G_δ. Let F be a closed subset of X^2. For each $H \in \mathcal{H}(i,j,k,\ell)$, by Lemma 3.4, the collection $\mathcal{H}$ of all finite minimal subcollections $\mathcal{F} \subset \mathcal{P}^2$ with $H \cap F \subset \text{int} \cup \mathcal{F}$ is at most countable, so let $\mathcal{H} = \{H(i,j,k,\ell,m);\ m \in N\}$. Let $H^*(i,j,k,\ell,m) = \text{int}(\cup H(i,j,k,\ell,m))$ and $W(i,j,k,\ell,m) = \cup\{H \cap (\cap\{H^*(i,j,k,\ell,n);\ n \le m\});\ H \in \mathcal{H}(i,j,k,\ell)\}$. Then $F = \cap\{W(i,j,k,\ell,m);\ i,j,k,\ell,m \in N\}$. Indeed, let $p = (x,y) \notin F$. Since $C_p \cap F$ is compact, there exists a finite $\mathcal{F} \subset \mathcal{P}^2$ such that $C_p \cap F \subset \text{int} \cup \mathcal{F} \subset \cup \mathcal{F} \subset X^2 - \{p\}$. Hence $C_p \subset St(p,\mathcal{G}(i,j)) \subset \text{int} \cup \mathcal{F} \cup (X^2 - F)$ for some $i,j \in N$. While, for some $k, \ell \in N$, $\{H \in \mathcal{H}(k,j,k,\ell);\ H \ni p\} = \{H_1,H_2,\cdots,H_t\}$. For each $n = 1,2,\cdots,t$, since $H_n \cap F \subset \text{int} \cup \mathcal{F}$, select from $\mathcal{F}$ a minimal subcollection which covers $H_n \cap F$, and label it $H_n(i,j,k,\ell,m_n)$. Let $m_0 = \text{Max}\{m_1,m_2,\cdots,m_t\}$. Then, $p \notin W(i,j,k,\ell,m_0)$. This implies that F is a G_δ-set in X^2. Thus X has a G_δ-diagonal. Then, since X is θ-refinable, X is a Moore space by [19; Corollary 2.6] (or [28]).

By Theorem 3.5 and Propositions 1.8 and 3.1, we have

Corollary 3.6. (1) [19]. *A* θ*-refinable* wΔ*-space with a point-countable separating open cover is a Moore space.*

(2) [28]. *A* θ*-refinable* wΔ*-space with a point-countable weak base is a Moore space.*

Corollary 3.7. [15]. *A paracompact* p*-space (= paracompact* M*-space) with a point-countable cover satisfying* (C_3) *is metrizable.*

Proof. We recall that every paracompact p-space is strict p-space, and every paracompact Moore space is metrizable. Thus the result follows from Theorem 3.5.

Quite recently, S. Jiang [29] proved the following interesting result.

Lemma 3.8. *Every strict* p*-space is* θ*-refinable.*

Recall that every completely regular Moore space is a strict p-space. Thus, combining Theorem 3.5 with Lemma 3.8, we have

Theorem 3.9. *A completely regular space is a Moore space if and only if it is a strict* p*-space with a point-countable cover satisfying* (C_3).

Corollary 3.10. *A strict* p*-space (or a* θ*-refinable* wΔ*-space) with a point-countable* k*-network is a Moore space.*

Remark 3.11. In the previous corollary, we can not replace "strict p-space" with "p-space" or "wΔ-space" by (i) and (ii) below. Also, by (ii) we can not omit the θ-refinability in the parenthetic part.

(i) S. W. Davis [7] constructed a non-developable, Čech-complete space (hence, p-space) with a point-countable base.

(ii) Z. Frolik [11] constructed an infinite, completely regular, countably compact space, each of whose compact set is finite. This space is a non-developable, wΔ-space with a point-finite k-network.

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