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K. Borsuk [Bo] (p. 214) constructed a functor $\Lambda\colon SH_{CM} \to TOP$ from the shape category of compacta to the topological category such that $\Lambda(X)$ is the space of components of X. Moreover, for every fundamental sequence $f = \{f_k, X, Y\}$ from X to Y and for every component C of X, $\{f_k, C, \Lambda(C)\}$ is a fundamental sequence from C to $\Lambda[f](C)$. Several authors (see [$Ba_{1,2,3}$], [G] and [S]) tried to generalize this result to non-compact spaces (locally compact or metrizable). In the present paper we show that the correct setting for possible generalizations is the shape analogue of quasicomponents.

Given a space X let ΔX be the set of all quasi-components of X with the quotient topology. $p_X\colon X \to \Delta X$ is the projection and $S\colon HTOP \to Sh$ denotes the shape functor (see [D-S]) from the homotopy category to the shape category. For any map $a\colon X \to Y$ and $X_0 \in \Delta X$ there is a unique element of ΔY containing $a(X_0)$. That means the map $p_Y a\colon X \to \Delta Y$ factors through ΔX, so there is a continuous map $\Delta(a)\colon \Delta X \to \Delta Y$ such that $\Delta(a)p_X = p_Y a$. Moreover, a homotopic to b implies $\Delta(a) = \Delta(b)$. Thus we have a functor $\Delta\colon HTOP \to TOP$ from the homotopy category to the topological category. If one wants to get a functor $\check{\Delta}\colon SH \to TOP$ from the shape category, the natural way is to take the Čech system $\{X_{\mathcal{U}}, p_{\mathcal{U}\mathcal{V}}, COV\}$ of X ($X_{\mathcal{U}}$ are the nerves of numerable coverings $\mathcal{U}$ of X) and define $\check{\Delta}(X)$ as the inverse limit $\lim\limits_{\leftarrow}\{\Delta X_{\mathcal{U}}, \Delta(p_{\mathcal{U}\mathcal{V}}), COV\}$ (with the inverse limit topology) of the system $\{\Delta X_{\mathcal{U}}, \Delta(p_{\mathcal{U}\mathcal{V}}), COV\}$.

Theorem 1. *If every open cover of* ΔX *admits an open refinement consisting of mutually disjoint sets, then the natural map* $\Delta X \to \check{\Delta}(X)$ *is a homeomorphism.*

Proof. The map $\Delta X \to \check{\Delta}(X)$ is always one-to-one. Indeed, if F and G are two different quasicomponents of X, then there is an open-closed set U in X with U containing F and X-U containing G. The covering $\mathcal{U} = \{U, X-U\}$ determines $X_{\mathcal{U}}$ such that F and G are sent to two different points of $\Delta X_{\mathcal{U}} = X_{\mathcal{U}}$.

Claim. If C *is a closed set in* ΔX, *then the image of* C *in* $\check{\Delta}(X)$ *is equal to* $\Pi\{\Delta p_{\mathcal{U}}(C) : \mathcal{U} \in COV\} \cap \check{\Delta}(X)$.

Proof of Claim. Obviously the image of C in $\check{\Delta}(X)$ is contained in $\Pi\{\Delta p_{\mathcal{U}}(C) : \mathcal{U} \in COV\} \cap \check{\Delta}(X)$. Suppose $\{C_{\mathcal{U}}, \mathcal{U} \in COV\} \in \Pi\{\Delta p_{\mathcal{U}}(C) : \mathcal{U} \in COV\} \cap \check{\Delta}(X)$. Then $\{p_{\mathcal{U}}^{-1}(C_{\mathcal{U}}), \mathcal{U} \in COV\}$ is a system of open-closed sets in X. If its intersection has a mutual point with C, we are done. So let us assume $\cap\{p_{\mathcal{U}}^{-1}(C_{\mathcal{U}}), \mathcal{U} \in COV\} \subset X - C$. Notice that the sets $X-p_{\mathcal{U}}^{-1}(C_{\mathcal{U}})$, $\mathcal{U} \in COV$, are open-closed in X. Thus there is a refinement $\mathcal{W}$ of $\{X - C\} \cup \{X - p_{\mathcal{U}}^{-1}(C_{\mathcal{U}}), \mathcal{U} \in COV\}$ consisting of mutually disjoint open-closed sets. By the definition of the Čech system of X, $\mathcal{W} \in COV$, $X_{\mathcal{W}}$ is the nerve of $\mathcal{W}$ and $p_{\mathcal{W}}: X \to X_{\mathcal{W}}$ is an enumeration of $\mathcal{W}$.

In our case $X_{\mathcal{W}}$ is a 0-dimensional complex and $p_{\mathcal{W}}: X \to X_{\mathcal{W}}$ maps each element U of $\mathcal{W}$ onto the vertex U of X . Observe that $\Delta X_{\mathcal{W}} = X_{\mathcal{W}}$ and $C_{\mathcal{W}}$ must be a vertex U_0 of $X_{\mathcal{W}}$. On the other hand U_0 is contained in $X - p_{\mathcal{V}}^{-1}(C_{\mathcal{V}})$ for some $\mathcal{V}$. Thus $p_{\mathcal{V}}(C_{\mathcal{W}})$ is disjoint with $C_{\mathcal{V}}$, a contradiction.

Notice that for each $X_{\mathcal{U}}$ the set $\Delta p_{\mathcal{U}}(C)$ is closed in the discrete space $\Delta X_{\mathcal{U}}$ (here $p_{\mathcal{U}}: X \to X_{\mathcal{U}}$ are the projections). Since $\Pi\{\Delta p_{\mathcal{U}}(C): \mathcal{U} \in COV\}$ is closed, we infer $\Delta X \to \check{\Delta}(X)$ is closed. Taking C = X we get that $\Delta X \to \check{\Delta}(X)$ is onto.

Corollary 1. *The natural map* $\Delta X \to \check{\Delta}(X)$ *is a homeomorphism if one of the following conditions is satisfied:*

a. ΔX *is paracompact and* $\dim \Delta X = 0$,

b. X *is locally compact metrizable and each component of* X *is compact.*

Proof. Case a) follows from Theorem 3 in [E] (p. 278). In case b) X is a topological sum of compact metrizable spaces (see [Ba_2], p. 258).

Remark. In [M] (proposition 1.4) it is shown that each quasicomponent of X is connected provided X is normal, $p_X: X \to \Delta X$ is closed and $\text{ind}(\Delta X) = 0$.

Theorem 2. *Suppose* f: X → Y *is a shape morphism,* Y *is paracompact,* $\dim \Delta Y = 0$ *and* $p_Y: Y \to \Delta Y$ *is closed. If* B *is a closed subset of* ΔY *and* $A \subset \Delta X \cap \check{\Delta}(f)^{-1}(B)$, *then there exists a unique shape morphism*

$f_0: p_X^{-1}(A) \to p_Y^{-1}(B)$ *such that* $S(j)f_0 = f \cdot S(i)$, *where* $i: p_X^{-1}(A) \to X$ *and* $j: p_Y^{-1}(B) \to Y$ *are inclusions.*

Theorem 2 is a simple consequence of the following:

Lemma 1. *Suppose* B *is a subset of a paracompact space* Y *such that each neighborhood of* B *in* Y *contains an open-closed neighborhood of* B *in* Y. *Let* $\{Y_{\mathcal{U}}, p_{\mathcal{U}\mathcal{V}}, COV\}$ *be the Čech system of* Y. *For each covering* $\mathcal{U} \in COV$ *consider the subcomplex* $N(\mathcal{U}|B)$ (equal to the nerve of $\mathcal{U}$ restricted to B) *of* $Y_{\mathcal{U}}$ *and let* $B_{\mathcal{U}}$ *be the union of all components of points of* $N(\mathcal{U}|B)$ *in* $Y_{\mathcal{U}}$. *Then the natural pro-homotopy map* $B \to \{B_{\mathcal{U}}, \mathcal{U} \in COV\}$ *satisfies the continuity condition.*

Proof. COV is the set of all numerable coverings of Y, $Y_{\mathcal{U}}$ is the nerve $N(\mathcal{U})$ of $\mathcal{U}$ and $p_{\mathcal{U}}: Y \to Y_{\mathcal{U}}$ is an enumeration of $\mathcal{U}$ (see [D-S], pp. 20-22). The induced maps $B \to B_{\mathcal{U}}$ will be denoted by $q_{\mathcal{U}}$. We need to show two properties:

a. For each map $f: B \to K \in ANR(\mathcal{M})$ there is $\mathcal{U} \in COV$ and $f_{\mathcal{U}}: B_{\mathcal{U}} \to K$ with f homotopic to $f_{\mathcal{U}}q_{\mathcal{U}}$.

b. If $f_{\mathcal{U}}, h_{\mathcal{U}}: B_{\mathcal{U}} \to K$ are maps such that $f_{\mathcal{U}}q_{\mathcal{U}}$ is homotopic to $h_{\mathcal{U}}q_{\mathcal{U}}$, then there is $\mathcal{V} \geq \mathcal{U}$ with $f_{\mathcal{U}}q_{\mathcal{V}\mathcal{U}}$ is homotopic to $h_{\mathcal{U}}q_{\mathcal{V}\mathcal{U}}$, where $q_{\mathcal{V}\mathcal{U}}: B_{\mathcal{V}} \to B_{\mathcal{U}}$.

Since we can change K up to homotopy type we may assume K is a complete metric space, and therefore it is an ANE for paracompact spaces (see [D-K]).

If $f: B \to K$, we extend f to $f': V \to K$, where V is an open-closed set containing B, and then we extend f' to $f'': Y \to K$. Since $Y \to \{Y_{\mathcal{U}}, \mathcal{U} \in A\}$ satisfies the continuity condition, there is $\mathcal{U} \in \mathcal{COV}$ and $g_{\mathcal{U}}: Y_{\mathcal{U}} \to K$ with f'' homotopic to $g_{\mathcal{U}}q_{\mathcal{U}}$. Now take $f_{\mathcal{U}} = g_{\mathcal{U}}|B$.

If $f_{\mathcal{U}}, h_{\mathcal{U}}: B_{\mathcal{U}} \to K$ are maps such that $f_{\mathcal{U}}q_{\mathcal{U}}$ is homotopic to $h_{\mathcal{U}}q_{\mathcal{U}}$, choose an open-closed neighborhood V of B in Y with $q_{\mathcal{U}}(V)$ contained in $B_{\mathcal{U}}$. By extending the homotopy between $f_{\mathcal{U}}q_{\mathcal{U}}$ and $h_{\mathcal{U}}q_{\mathcal{U}}$ we may assume that those two maps are homotopic as maps from V to K. Let $\mathcal{W} = (\mathcal{U}|V) \cup (\mathcal{U}|Y - V)$. Define a map $s: Y_{\mathcal{W}} \to K$ ($t: Y_{\mathcal{W}} \to K$) as $f_{\mathcal{U}}p_{\mathcal{WU}}$ on $N(\mathcal{U}|V)$ ($g_{\mathcal{U}}p_{\mathcal{WU}}$ on $N(\mathcal{U}|V)$) and constant on $N(\mathcal{U}|Y - V)$. Then $sp_{\mathcal{W}}$ is homotopic to $tp_{\mathcal{W}}$, so there is $\mathcal{V} \geq \mathcal{W}$ with $sp_{\mathcal{VW}}$ is homotopic to $tp_{\mathcal{VW}}$. Since $N(\mathcal{U}|V) = B_{\mathcal{W}}$, we are done.

A simple consequence of Theorem 2 is the following:

Corollary. Suppose $f: X \to Y$ *is a shape isomorphism of paracompact spaces. If both maps* $p_X: X \to \Delta X$, $p_Y: Y \to \Delta Y$ *are closed and* $\dim\Delta X = \dim\Delta Y = 0$, *then for every* $X_0 \in \Delta X$ *there is a shape equivalence* $f_0: X_0 \to Y_0 = \check{\Delta}(f)(X_0)$ *such that* $S(j)f_0 = fS(i)$, *where* $i: X_0 \hookrightarrow X$ *and* $j: Y_0 \hookrightarrow Y$ *are inclusions.*

Here is a partial converse to the above Corollary:

Theorem 3. *Suppose* $f: X \to Y$ *is a closed map of paracompact spaces such that for every* $X_0 \in \Delta X$ *the* *restriction* $f_0: X_0 \to Y_0 = \Delta(X_0)$ *of* f *is a shape equivalence and* $\Delta(f): \Delta(X) \to \Delta(Y)$ *is a bijection. If* $p_Y: Y \to \Delta Y$ *is closed and* $\dim\Delta Y = 0$, *then* f *is a shape equivalence.*

Proof. Replacing Y by the mapping cylinder of f we may assume that f is the inclusion of a closed set X into Y. We need to show that for each map $a: X \to K \in ANR(M)$ there is an extension $a': Y \to K$. Since we can change K up to homotopy type we may assume K is a complete metric space, and therefore it is an ANE for paracompact spaces (see [D-K]). For each $X_0 \in \Delta X$ we can extend $a|X_0$ to $Y_0 \in \Delta Y$ containing X_0. Subsequently we extend a from X to an open-closed neighborhood V of Y_0 in Y. Now, choose a refinement of {V} consisting of mutually disjoint open sets and it is clear how to define an extension of a.

If $a,b: Y \to K$ are two maps such that $a|X$ and $b|X$ are homotopic, we proceed similarly as above to show that a and b are homotopic.

Remark. Theorem 3 is a generalization of Theorem 4.5.5 in [D-S] (p. 62).

Theorem 4. *Suppose* X *is a paracompact space such that* $p_X: X \to \Delta X$ *is closed and* $\dim\Delta X = 0$. *If for every* $X_0 \in \Delta X$ *the deformation dimension* def-dim(X_0) *is less than or equal to* n, *then* def-dim(X) $\leq$ n.

Proof. It suffices to show that for every simplicial complex K (with the metric topology) any map $f: X \to K$ is homotopic to a map g with $g(X) \subset K^{(n)}$.

Claim. If def-dim(Y) $\leq$ n, *then any map* $g\colon Y \to \bigcup_{p=1}^{\infty} K^{(p)} \times [p,\infty)$ *is homotopic to a map which values lie in* $\bigcup_{p=1}^{n} K^{(p)} \times [p,\infty)$.

Proof of Claim. Since the identity map

$$\bigcup_{p=1}^{\infty} K_w^{(p)} \times [p,\infty) \to \bigcup_{p=1}^{\infty} K^{(p)} \times [p,\infty)$$

is a homotopy euqivalence (here K_w means K with the weak topology) we can factor g up to homotopy through $L = \bigcup_{p=1}^{\infty} K_w^{(p)} \times [p,\infty)$ (L is considered with the natural CW structure) and using the fact that def-dim(Y) $\leq$ n we can push g into $L^{(n)}$ which is contained in $L_n = \bigcup_{p=1}^{n} K^{(p)} \times [p,\infty)$.

Suppose $g\colon X \to K$. Since the projection $\pi\colon L' = \bigcup_{p=1}^{\infty} K^{(p)} \times [p,\infty) \to K$ is a homotopy equivalence (π induces isomorphisms of homotopy groups at each point), there is $g'\colon X \to L'$ with $g' \approx \pi g$. Given any quasi-component X_0 of X, $g'|X_0$ is homotopic to g_0 with $g_0(X_0)$ contained in L_n. Since L' is an ANE for paracompact spaces (see [H], p. 63), we can find an open closed neighborhood V_0 of X_0 in X such that $g'|V_0$ is homotopic to a map with values in L_n. Finally we can find a covering $\mathcal{V}$ of X consisting of mutually disjoint open sets such that $g'|V$ is homotopic to a map with values in L_n for every V in $\mathcal{V}$. Now, it is clear that g' is homotopic to a map g" with values in L_n. Then $g''\pi$ is homotopic to g and its values are in $K^{(n)}$.

Remark. In the compact metrizable case Theorem 4 was proved by S. Nowak [N].

Theorem 5. *Suppose* X *is a paracompact space such that* p_X: X → ΔX *is closed and* dimΔX = 0. *If* $\mathcal{F}$ *is a closed covering of* X *such that* $p_X^{-1}(p_X(\mathcal{F})) = \mathcal{F}$ *and every* B ∈ $\mathcal{F}$ *is movable* (uniformly movable), *then* X *is movable* (uniformly movable).

Proof. We will prove only the uniformly movable case. The proof of the movable case is similar. Let $\{X_{\mathcal{U}}, p_{\mathcal{U}\mathcal{V}}, COV\}$ be the Čech system of X. By Lemma 1, the natural pro-homotopy map B → $\{B_{\mathcal{U}}, \mathcal{U} \in COV\}$ satisfies the continuity condition for each B ∈ $\mathcal{F}$. Thus for each covering $\mathcal{U} \in COV$ and for each B ∈ $\mathcal{F}$ there is a covering $\mathcal{V}(B) \in COV$, $\mathcal{V}(B) \geq \mathcal{U}$, and a shape morphism g_B: $B_{\mathcal{V}(B)} \to X$ such that $S[p_{\mathcal{U}}]g_B = S[p_{\mathcal{V}(B)\mathcal{U}}]$. Choose an open refinement $\mathcal{W}$ of $\{(p_{\mathcal{V}(B)})^{-1}(B_{\mathcal{V}(B)})$: B ∈ $\mathcal{F}\}$ consisting of mutually disjoint open sets. Now, we define an open refinement $\mathcal{V}$ of $\mathcal{U}$ as follows: given W in $\mathcal{W}$ we choose B ∈ $\mathcal{F}$ with $W \subset (p_{\mathcal{V}(B)})^{-1}(B_{\mathcal{V}(B)})$ and declare the elements of $\mathcal{V}$ intersecting W to be precisely W ∩ $\mathcal{V}$(B). Notice that N(W ∩ $\mathcal{V}$) is contained in $B_{\mathcal{V}(B)}$, so that we can piece together g_B's to get g: $N(\mathcal{V}) = X_{\mathcal{V}} \to X$ with $S[p_{\mathcal{U}}]g = S[p_{\mathcal{V}\mathcal{U}}]$.

Remarks. Theorem 5 is related to Theorem 3.1 of [M]. In [D-S-S] there is an example of a non-movable metrizable space X such that each of its quasicomponents is movable. The same example shows that one cannot drop both of the hypotheses (p_X closed and dimΔX = 0) in Theorem 4.

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