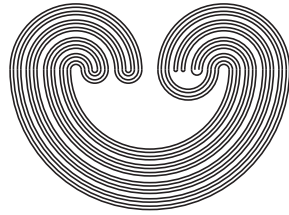

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QUASICOMPONENTS AND SHAPE THEORY

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K. Borsuk [Bo] (p. 214) constructed a functor $\Lambda: SH_{CM} \rightarrow TOP$ from the shape category of compacta to the topological category such that $\Lambda(X)$ is the space of components of X . Moreover, for every fundamental sequence $f = \{f_k, X, Y\}$ from X to Y and for every component C of X , $\{f_k, C, \Lambda(C)\}$ is a fundamental sequence from C to $\Lambda[f](C)$. Several authors (see [Ba_{1,2,3}], [G] and [S]) tried to generalize this result to non-compact spaces (locally compact or metrizable). In the present paper we show that the correct setting for possible generalizations is the shape analogue of quasicomponents.

Given a space X let ΔX be the set of all quasicomponents of X with the quotient topology. $p_X: X \rightarrow \Delta X$ is the projection and $S: HTOP \rightarrow Sh$ denotes the shape functor (see [D-S]) from the homotopy category to the shape category. For any map $a: X \rightarrow Y$ and $X_0 \in \Delta X$ there is a unique element of ΔY containing $a(X_0)$. That means the map $p_Y a: X \rightarrow \Delta Y$ factors through ΔX , so there is a continuous map $\Delta(a): \Delta X \rightarrow \Delta Y$ such that $\Delta(a)p_X = p_Y a$. Moreover, a homotopic to b implies $\Delta(a) = \Delta(b)$. Thus we have a functor $\Delta: HTOP \rightarrow TOP$ from the homotopy category to the topological category. If one wants to get a functor $\tilde{\Delta}: SH \rightarrow TOP$ from the shape category, the natural way is

to take the Čech system $\{X_U, p_{UV}, COV\}$ of X (X_U are the nerves of numerable coverings U of X) and define $\check{\Delta}(X)$ as the inverse limit $\varprojlim \{\Delta X_U, \Delta(p_{UV}), COV\}$ (with the inverse limit topology) of the system $\{\Delta X_U, \Delta(p_{UV}), COV\}$.

Theorem 1. *If every open cover of ΔX admits an open refinement consisting of mutually disjoint sets, then the natural map $\Delta X \rightarrow \check{\Delta}(X)$ is a homeomorphism.*

Proof. The map $\Delta X \rightarrow \check{\Delta}(X)$ is always one-to-one. Indeed, if F and G are two different quasicomponents of X , then there is an open-closed set U in X with U containing F and $X-U$ containing G . The covering $u = \{U, X-U\}$ determines X_U such that F and G are sent to two different points of $\Delta X_U = X_U$.

Claim. *If C is a closed set in ΔX , then the image of C in $\check{\Delta}(X)$ is equal to $\Pi\{\Delta p_U(C) : u \in COV\} \cap \check{\Delta}(X)$.*

Proof of Claim. Obviously the image of C in $\check{\Delta}(X)$ is contained in $\Pi\{\Delta p_U(C) : u \in COV\} \cap \check{\Delta}(X)$. Suppose $\{C_U, u \in COV\} \in \Pi\{\Delta p_U(C) : u \in COV\} \cap \check{\Delta}(X)$. Then $\{p_U^{-1}(C_U), u \in COV\}$ is a system of open-closed sets in X . If its intersection has a mutual point with C , we are done. So let us assume $\cap \{p_U^{-1}(C_U), u \in COV\} \subset X - C$. Notice that the sets $X - p_U^{-1}(C_U), u \in COV$, are open-closed in X . Thus there is a refinement ω of $\{X - C\} \cup \{X - p_U^{-1}(C_U), u \in COV\}$ consisting of mutually disjoint open-closed sets. By the definition of the Čech system of X , $\omega \in COV$, X_ω is the nerve of ω and $p_\omega : X \rightarrow X_\omega$ is an enumeration of ω .

In our case X_ω is a 0-dimensional complex and $p_\omega: X \rightarrow X_\omega$ maps each element U of ω onto the vertex U of X . Observe that $\Delta X_\omega = X_\omega$ and C_ω must be a vertex U_0 of X_ω . On the other hand U_0 is contained in $X - p_V^{-1}(C_V)$ for some V . Thus $p_V(C_V)$ is disjoint with C_V , a contradiction.

Notice that for each X_U the set $\Delta p_U(C)$ is closed in the discrete space ΔX_U (here $p_U: X \rightarrow X_U$ are the projections). Since $\Pi\{\Delta p_U(C): U \in COV\}$ is closed, we infer $\Delta X \rightarrow \check{\Delta}(X)$ is closed. Taking $C = X$ we get that $\Delta X \rightarrow \check{\Delta}(X)$ is onto.

Corollary 1. The natural map $\Delta X \rightarrow \check{\Delta}(X)$ is a homeomorphism if one of the following conditions is satisfied:

- a. ΔX is paracompact and $\dim \Delta X = 0$,
- b. X is locally compact metrizable and each component of X is compact.

Proof. Case a) follows from Theorem 3 in [E] (p. 278). In case b) X is a topological sum of compact metrizable spaces (see [Ba₂], p. 258).

Remark. In [M] (proposition 1.4) it is shown that each quasicomponent of X is connected provided X is normal, $p_X: X \rightarrow \Delta X$ is closed and $\text{ind}(\Delta X) = 0$.

Theorem 2. Suppose $f: X \rightarrow Y$ is a shape morphism, Y is paracompact, $\dim \Delta Y = 0$ and $p_Y: Y \rightarrow \Delta Y$ is closed. If B is a closed subset of ΔY and $A \subset \Delta X \cap \check{\Delta}(f)^{-1}(B)$, then there exists a unique shape morphism

$f_0: p_X^{-1}(A) \rightarrow p_Y^{-1}(B)$ such that $S(j)f_0 = f \cdot S(i)$, where $i: p_X^{-1}(A) \rightarrow X$ and $j: p_Y^{-1}(B) \rightarrow Y$ are inclusions.

Theorem 2 is a simple consequence of the following:

Lemma 1. Suppose B is a subset of a paracompact space Y such that each neighborhood of B in Y contains an open-closed neighborhood of B in Y . Let $\{Y_U, p_{UU}, COV\}$ be the Čech system of Y . For each covering $U \in COV$ consider the subcomplex $N(U|B)$ (equal to the nerve of U restricted to B) of Y_U and let B_U be the union of all components of points of $N(U|B)$ in Y_U . Then the natural pro-homotopy map $B \rightarrow \{B_U, U \in COV\}$ satisfies the continuity condition.

Proof. COV is the set of all numerable coverings of Y , Y_U is the nerve $N(U)$ of U and $p_U: Y \rightarrow Y_U$ is an enumeration of U (see [D-S], pp. 20-22). The induced maps $B \rightarrow B_U$ will be denoted by q_U . We need to show two properties:

- a. For each map $f: B \rightarrow K \in ANR(M)$ there is $U \in COV$ and $f_U: B_U \rightarrow K$ with f homotopic to $f_U q_U$.
- b. If $f_U, h_U: B_U \rightarrow K$ are maps such that $f_U q_U$ is homotopic to $h_U q_U$, then there is $V \geq U$ with $f_U q_U$ is homotopic to $h_U q_{VU}$, where $q_{VU}: B_V \rightarrow B_U$.

Since we can change K up to homotopy type we may assume K is a complete metric space, and therefore it is an ANE for paracompact spaces (see [D-K]).

If $f: B \rightarrow K$, we extend f to $f': V \rightarrow K$, where V is an open-closed set containing B , and then we extend f' to $f'': Y \rightarrow K$. Since $Y = \{Y_U, U \in A\}$ satisfies the continuity condition, there is $U \in \mathcal{C}OV$ and $g_U: Y_U \rightarrow K$ with f'' homotopic to $g_U \circ q_U$. Now take $f_U = g_U|_B$.

If $f_U, h_U: B_U \rightarrow K$ are maps such that $f_U \circ q_U$ is homotopic to $h_U \circ q_U$, choose an open-closed neighborhood V of B in Y with $q_U(V)$ contained in B_U . By extending the homotopy between $f_U \circ q_U$ and $h_U \circ q_U$ we may assume that those two maps are homotopic as maps from V to K . Let $W = (U|V) \cup (U|Y - V)$. Define a map $s: Y_W \rightarrow K$ ($t: Y_W \rightarrow K$) as $f_U \circ p_{WU}$ on $N(U|V)$ ($g_U \circ p_{WU}$ on $N(U|V)$) and constant on $N(U|Y - V)$. Then sp_W is homotopic to tp_W , so there is $V \supseteq W$ with sp_{VW} is homotopic to tp_{VW} . Since $N(U|V) = B_W$, we are done.

A simple consequence of Theorem 2 is the following:

Corollary. Suppose $f: X \rightarrow Y$ is a shape isomorphism of paracompact spaces. If both maps $p_X: X \rightarrow \Delta X$, $p_Y: Y \rightarrow \Delta Y$ are closed and $\dim \Delta X = \dim \Delta Y = 0$, then for every $X_0 \in \Delta X$ there is a shape equivalence $f_0: X_0 \rightarrow Y_0 = \check{\Delta}(f)(X_0)$ such that $S(j)f_0 = fS(i)$, where $i: X_0 \hookrightarrow X$ and $j: Y_0 \hookrightarrow Y$ are inclusions.

Here is a partial converse to the above Corollary:

Theorem 3. Suppose $f: X \rightarrow Y$ is a closed map of paracompact spaces such that for every $X_0 \in \Delta X$ the

restriction $f_0: X_0 \rightarrow Y_0 = \Delta(X_0)$ of f is a shape equivalence and $\Delta(f): \Delta(X) \rightarrow \Delta(Y)$ is a bijection. If $p_Y: Y \rightarrow \Delta Y$ is closed and $\dim \Delta Y = 0$, then f is a shape equivalence.

Proof. Replacing Y by the mapping cylinder of f we may assume that f is the inclusion of a closed set X into Y . We need to show that for each map $a: X \rightarrow K \in \text{ANR}(M)$ there is an extension $a': Y \rightarrow K$. Since we can change K up to homotopy type we may assume K is a complete metric space, and therefore it is an ANE for paracompact spaces (see [D-K]). For each $X_0 \in \Delta X$ we can extend $a|_{X_0}$ to $Y_0 \in \Delta Y$ containing X_0 . Subsequently we extend a from X to an open-closed neighborhood V of Y_0 in Y . Now, choose a refinement of $\{V\}$ consisting of mutually disjoint open sets and it is clear how to define an extension of a .

If $a, b: Y \rightarrow K$ are two maps such that $a|_X$ and $b|_X$ are homotopic, we proceed similarly as above to show that a and b are homotopic.

Remark. Theorem 3 is a generalization of Theorem 4.5.5 in [D-S] (p. 62).

Theorem 4. Suppose X is a paracompact space such that $p_X: X \rightarrow \Delta X$ is closed and $\dim \Delta X = 0$. If for every $X_0 \in \Delta X$ the deformation dimension $\text{def-dim}(X_0)$ is less than or equal to n , then $\text{def-dim}(X) \leq n$.

Proof. It suffices to show that for every simplicial complex K (with the metric topology) any map $f: X \rightarrow K$ is homotopic to a map g with $g(X) \subset K^{(n)}$.

Claim. If $\text{def-dim}(Y) \leq n$, then any map $g: Y \rightarrow \bigcup_{p=1}^{\infty} K^{(p)} \times [p, \infty)$ is homotopic to a map which values lie in $\bigcup_{p=1}^n K^{(p)} \times [p, \infty)$.

Proof of Claim. Since the identity map

$$\bigcup_{p=1}^{\infty} K_w^{(p)} \times [p, \infty) \rightarrow \bigcup_{p=1}^{\infty} K^{(p)} \times [p, \infty)$$

is a homotopy euqivalence (here K_w means K with the weak topology) we can factor g up to homotopy through

$$L = \bigcup_{p=1}^{\infty} K_w^{(p)} \times [p, \infty)$$

(L is considered with the natural CW structure) and using the fact that $\text{def-dim}(Y) \leq n$ we

can push g into $L^{(n)}$ which is contained in

$$L_n = \bigcup_{p=1}^n K^{(p)} \times [p, \infty).$$

Suppose $g: X \rightarrow K$. Since the projection

$$\pi: L' = \bigcup_{p=1}^{\infty} K^{(p)} \times [p, \infty) \rightarrow K$$

is a homotopy equivalence (π induces isomorphisms of homotopy groups at each point),

there is $g': X \rightarrow L'$ with $g' \approx \pi g$. Given any quasi-

component X_0 of X , $g'|_{X_0}$ is homotopic to g_0 with $g_0(X_0)$

contained in L_n . Since L' is an ANE for paracompact

spaces (see [H], p. 63), we can find an open closed

neighborhood V_0 of X_0 in X such that $g'|_{V_0}$ is homotopic to

a map with values in L_n . Finally we can find a covering

\mathcal{V} of X consisting of mutually disjoint open sets such that

$g'|_V$ is homotopic to a map with values in L_n for every V

in \mathcal{V} . Now, it is clear that g' is homotopic to a map g''

with values in L_n . Then $g''\pi$ is homotopic to g and its

values are in $K^{(n)}$.

Remark. In the compact metrizable case Theorem 4 was proved by S. Nowak [N].

Theorem 5. Suppose X is a paracompact space such that $p_X: X \rightarrow \Delta X$ is closed and $\dim \Delta X = 0$. If F is a closed covering of X such that $p_X^{-1}(p_X(F)) = F$ and every $B \in F$ is movable (uniformly movable), then X is movable (uniformly movable).

Proof. We will prove only the uniformly movable case. The proof of the movable case is similar. Let $\{X_U, p_{UV}, COV\}$ be the Čech system of X . By Lemma 1, the natural pro-homotopy map $B \rightarrow \{B_U, U \in COV\}$ satisfies the continuity condition for each $B \in F$. Thus for each covering $U \in COV$ and for each $B \in F$ there is a covering $V(B) \in COV$, $V(B) \geq U$, and a shape morphism $g_B: B_{V(B)} \rightarrow X$ such that $S[p_U]g_B = S[p_{V(B)U}]$. Choose an open refinement \mathcal{W} of $\{(p_{V(B)})^{-1}(B_{V(B)}): B \in F\}$ consisting of mutually disjoint open sets. Now, we define an open refinement \mathcal{V} of U as follows: given W in \mathcal{W} we choose $B \in F$ with $W \subset (p_{V(B)})^{-1}(B_{V(B)})$ and declare the elements of \mathcal{V} intersecting W to be precisely $W \cap V(B)$. Notice that $N(W \cap V)$ is contained in $B_{V(B)}$, so that we can piece together g_B 's to get $g: N(\mathcal{V}) = X_V \rightarrow X$ with $S[p_U]g = S[p_{VU}]$.

Remarks. Theorem 5 is related to Theorem 3.1 of [M]. In [D-S-S] there is an example of a non-movable metrizable space X such that each of its quasicomponents is movable. The same example shows that one cannot drop both of the hypotheses (p_X closed and $\dim \Delta X = 0$) in Theorem 4.

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